Polynomial System Satisfying a Special Functional Equation

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A system of polynomials $\{P_n(x)\}$ is called an Appell set, if

$$(1) \qquad P_n'(x) = P_{n-1}(x) \qquad (n = 0, 1, 2, \cdots) .$$

Nielsen [2] considered a remarkable subset of the Appell set by considering a set of polynomials which satisfy the two functional equations,

$$(2) \qquad P_n'(x) = P_{n-1}(x) \quad \text{and} \quad P_n(-x-1) = (-1)^n P_n(x)$$

for $n = 0, 1, 2, \cdots$; he has shown their importance in the theory of Bernoulli and Euler numbers. Later Ward [4] generalized the set by considering a set of polynomials $\{Y_n(x)\}$ for which

$$(3) \qquad Y_n'(x) = Y_{n-1}(x) \quad \text{and} \quad Y_n(ax+b) = \tau_n Y_n(x)$$

for $n = 0, 1, 2, \cdots$ and a and b are any complex numbers. More recently Sharma and Chak* [3] studied a class of polynomials $\{H_n(x)\}$ in x, such that for $n = 0, 1, 2, \cdots$

$$(4) \qquad D_q\{H_n(x)\} = H_{n-1}(x) \quad \text{where} \quad D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} .$$

Immediately after, a paper by Carlitz [1] appeared studying various polynomials related to Theta functions. It was this paper of Carlitz which suggested, as a natural study, the systems of polynomials which satisfy a functional equation of the form

$$(5) \qquad D_q\{H_n(x)\} = H_{n-1}(q^k x) \quad \text{where } k \text{ is a real number,}$$

and $n = 0, 1, 2, \cdots$; and also to examine some subsets of this class of polynomials which have properties analogous to regular and cyclic sets of Nielsen [2] and of Ward [4].

1. (q, k)**-Harmonic Sequences.** If $H_n(x) = \sum_{i=0}^n a_i x^{n-i}$, we put

$$H_n[ax+b] = \sum_{i=0}^n a_i (ax+b)_{n-i} \quad \text{where} \quad (ax+b)_n = \prod_{r=0}^{n-1} (ax + bq^r) ;$$

$$H_n^*[ax+b] = \sum_{i=0}^n a_i (ax+b)^*_{n-i} \quad \text{where} \quad (ax+b)_n^* q^{n(n-1)/2} = (b+ax)_n ;$$

$$H_n[x] = \sum_{i=0}^n a_i q^{(n-i)(n-i-1)/2} x^{n-i} ; \qquad H_n^*[x] = \sum_{i=0}^n a_i q^{-(n-i)(n-i-1)/2} x^{n-i} .$$

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* R. P. Boas and R. C. Buck, (*Polynomial Expansions of Analytic Functions*, Ergebnisse der Math., Vol. 19, 1958, pp. 44–45) call them Brenke polynomials and Waleed A. Al-Salam (a paper in press) calls them q-Appell polynomials and discusses their algebraic structure also.

We shall use the following notations:

$$[x] = (q^x - 1)/(q - 1)\,, \qquad [x]_s = [x][x - 1]\cdots[x - s + 1]\,;$$

$$[x]! = [x][x - 1]\cdots[1]\,, \qquad [0]! = 1\,; \qquad \begin{bmatrix} x \\ s \end{bmatrix} = [x]_s/[s]!\,, \qquad \begin{bmatrix} x \\ 0 \end{bmatrix} = 1\,.$$

In analogy with Nielsen [2] we shall call the set of polynomials $\{H_n(x, q)\} \equiv \{H_n(x)\}$ which satisfy (5) to be "(q, k)-harmonic". If $[H_n(x), h_n]$ and $[K_n(x), k_n]$ are two (q, k)-harmonic sequences, then it is easy to prove that

(a) there exists a sequence of constants $\{h_n\}$ such that

$$H_n(x) = \sum_{i=0}^{n} h_i q^{k(n-i)(n-i-1)/2} \frac{x^{n-i}}{[n - i]!}\,, \qquad H_n(0) = h_n\,;$$

(b)
$$D_q\{H_n[x]\} = H_{n-1}[q^{k+1}x]\,, \qquad (n = 0, 1, 2, \cdots)\,;$$
$$D_q\{H_n{}^*[x]\} = H^*_{n-1}[q^{k-1}x]\,,$$

(c) $\sum_{n=0}^{\infty} H_n(x)t^n = e_{q,k}(xt)h(t)$ where

$$e_{q,k}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} q^{kn(n-1)/2} \quad \text{and} \quad h(t) = \sum_{n=0}^{\infty} h_n t^n\,;$$

(d)
$$H_n[x + b] = \sum_{i=0}^{n} \frac{x^i}{[i]!} q^{ki(i-1)/2} H_{n-i}[q^{ki}b]\,,$$
$$H_n{}^*[x + b] = \sum_{i=0}^{n} \frac{x^i}{[i]!} q^{-ki(i-1)/2} H^*_{n-i}[q^{ki}b]\,,$$
$$H_n[b + x] = \sum_{i=0}^{n} \frac{x^i}{[i]!} q^{(k+1)i(i-1)/2} H_{n-i}(q^{ki}b)\,,$$
$$H_n{}^*[b + x] = \sum_{i=0}^{n} \frac{x^i}{[i]!} q^{(k-1)i(i-1)/2} H_{n-i}(q^{(k-1)i}b)\,;$$

(e) there exists a unique sequence $\{\alpha_n\}$ such that for all n

$$K_n(x) = \alpha_0 H_n(x) + \alpha_1 H_{n-1}(x) + \cdots + \alpha_n H_0(x)\,;$$

(f) $D_q H_n{}^*(x) = H_{n-1}(q^{1-k}x)$ where $H_n{}^*(x) = H_n{}^*(x, q) = H_n(x, 1/q)$;

(g) the expression

$$A_n{}^k = \sum_{s=0}^{n} (-1)^s H_{n-s}(q^{ks}x) K_s[q^{(n-s)k}x]\, q^{-ks(n-s)}$$

is a constant, while the polynomials

$$G_n{}^k(x) = \prod_{r=1}^{n} (1 + q^{r-k})^{-1} \sum_{s=0}^{n} H_{n-s}(q^{(k-1)s}x) K_s[q^{(n-s)k}x]$$

form a new (q, k)-harmonic sequence;

(h) if

$$H_n[x] - H_n[-1 + x] = K_{n-1}(x)\,,$$

then if only one of these (q, k)-harmonic sets is given the second is completely determined;

(i) if

$$H_n[x] + H_n[-1 + x] = K_n(x) ,$$

then also if one of these is given the other can be completely determined.

As particular cases of these results we can get all the results obtained by Sharma and Chak [3] by just taking $k = 0$ and those of Nielsen [2] by further taking the limiting case of the difference operator D_q with $q \to 0$ i.e. the differential operator $D \equiv d/dx$.

2. (q, k)**-(I) and** (q, k)**-(II)-Regular Sequences (when "a" is Not a Root of Unity).** If the set of polynomials $\{H_n(x)\}$ satisfies the two functional relations $(n = 0, 1, 2, \cdots)$

$$D_q\{H_n(x)\} = H_{n-1}(xq^k) \quad \text{and} \quad H_n[ax + b/q^{kn}] = \tau_n H_n(x) , \tag{2.1}$$

then we shall call them (q, k)-(I)-regular sequences. If "a" is not 0 or 1 it is easy to prove that:

The necessary and sufficient condition for a (q, k)-harmonic sequence $\{H_n(x)\}$ to be (q, k)-(I)-regular is

$$H_n[b/q^{kn}] = a^n H_n(0) ; \qquad n = 0, 1, 2, \cdots . \tag{2.2}$$

If we expand the left-hand side of (2.2) and equate, we easily get

$$h_n = \frac{h_0 b^n \Delta_n^{\ k}(a, q)}{(a - 1)(a^2 - 1)\cdots(a^n - 1)} \tag{2.3}$$

where

$$\Delta_n^{\ k}(a, q) = \begin{vmatrix} \dfrac{1}{q^k[1]!} & (1 - a) & 0 & 0 & \cdots & 0 \\ \dfrac{q^{k+1}}{(q^{2k})^2[2]!} & \dfrac{1}{q^{2k}[1]!} & (1 - a^2) & 0 & \cdots & 0 \\ \dfrac{q^{3(k+1)}}{(q^{3k})^3[3]!} & \dfrac{q^{k+1}}{(q^{3k})^2[2]!} & \dfrac{1}{q^{3k}[1]!} & (1 - a^3) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \dfrac{q^{(k+1)n(n-1)/2}}{(q^{nk})^n[n]!} & \dfrac{q^{(k+1)(n-1)(n-2)/2}}{(q^{nk})^{n-1}[n - 1]!} & \cdot & \cdot & \cdots & \dfrac{1}{q^{nk}[1]!} \end{vmatrix}$$

We can also define another set of polynomials $\{H_n(x)\}$ satisfying, for $n = 0, 1, 2, \cdots$,

$$D_q\{H_n(x)\} = H_{n-1}(xq^k) \quad \text{and} \quad H_n[b/q^{kn} + ax] = \tau_n H_n[x] \tag{2.4}$$

and call them (q, k)-(II)-regular sequences. If "a" is not a root of unity we can easily get the necessary and sufficient condition [3] for this set also.

3. (q, k)**-(T)-Regular Sequences and the Case when "a" is a Root of Unity.** For both (q,k)-(I)-regular and (q,k)-(II)-regular sequences we see that $h_1, h_2, \cdots$

are finite only if "a" is not a root of unity. If, however, $a^r = 1$ [$r \equiv 0 \pmod p$], then it is easy to see that an infinite sequence of polynomials $\{H_n(x)\}$ does not exist which satisfies (2.1) or (2.4). In order to be able to discuss the case when "a" is a root of unity we take the two functional equations as given in (3.1) below (instead of (2.1) and (2.4)) introducing special triangular matrices T and T' of nonzero numbers similar to the ones introduced in [3] and get all the results obtained in [3], for (q, k)-(T)-regular sequences defined by $(n = 0, 1, 2, \cdots)$

$$D_q\{H_n(x)\} = H_{n-1}(q^k x) \quad \text{and} \quad H_n{}^T(ax + b/q^{kn}) = \tau_n H_n(x) \,. \tag{3.1}$$

Following [3] we can easily get the results for these more general cases. If we take $k = 0$ we get the results given in [3].

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