Solutions of the Diophantine Equation $x^2 - Dy^4 = k$

By Mohan Lal and James Dawe

Introduction. The problem of determining the positive integral solutions of the equation

$$x^2 - Dy^4 = k \tag{1}$$

appears to be difficult even though it can readily be transformed to the general Pell's equation. For $|k| = 1$ and for a few small values of D, Eq. (1) has been dealt with in the literature [1]–[4]. For $k = 1$, Ljunggren [2] proved that Eq. (1) has at most two solutions in positive integers and Mordell [3] found that when D is an odd prime, $D \equiv 5, 9$ or $13 \pmod{16}$ except for $D = 5$, Eq. (1) has no solutions. In a subsequent paper, Ljunggren [4] showed that the same holds good for all primes $p \equiv 1 \pmod 4$. For $k = -1$, Ljunggren [1] proved that under certain restrictions, Eq. (1) has at most two solutions in positive integers, and he showed that for $k = -1$ and $D = 2$, $(x, y) = (1, 1)$ and $(239, 13)$ are the only two solutions and $(x, y) = (2, 1)$ is the only solution for $k = -1$ and $D = 5$.

For $|k| > 1$ and $D > 5$, Eq. (1) has not been treated so far and the investigations [1], [2] related to finding the upper bound for the number of solutions provide means that are hardly practical for determining these solutions. It was therefore felt desirable to obtain solutions of Eq. (1) by means of a numerical search.

The search was conducted for 37 values of D, in the range $2 \leqq D \leqq 43$, where D is not a perfect square. The parameters y and k were arbitrarily chosen to be $y \leqq 2 \times 10^5$ and $|k| \leqq 999$.

We anticipate that the results of such an extensive search will be useful for checking some of the conjectures concerning this equation and also provide further insight into the methods of finding complete sets of solutions.

Method. Rewrite (1) as

$$x^2 = Dy^4 + k\ ;$$

then, for a given y, x is obtained by

$$x_{\min} < x < (Dy^4 + 999)^{1/2}, \qquad x_{\min} = (Dy^4 - 999)^{1/2}.$$

Thus, if y is large, the possible values of x in this search are severely limited.

The starting values of x for $y < 9999$ were obtained by taking the square root directly and for $y > 10^4$ by a special subroutine.

Results. The integral solutions of the equation $x^2 - Dy^4 = k$ for 37 values of D and for $|k| \leqq 999$ have been tabulated. These tables contain all solutions found, for a given D, in the ascending order of $|k|$. Out of 6275 solutions found, 1453 are imprimitive, i.e., these solutions have a factor in common. These solutions have been retained in the tables and they are indicated by an asterisk following y. How-

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ever, they are not counted in preparing the present summary. For example, we found that $(x, y) = (16, 2), (44, 4), (96, 6)$ and $(1524, 24)$ are imprimitive solutions for $D, k = 7, 144$ as they can be derived from the 4 primitive solutions for $D, k = 7, 9$. Similarly, 3 solutions listed for $D, k = 27, -423$ are based upon the corresponding solutions for $D, k = 3, -47$. But the solutions $(x, y) = (2, 1), (58, 3), (2858, 21)$ and $(74198, 107)$ for $D, k = 42, -38$ are primitive. A copy of the tables of these solutions has been submitted to the Unpublished Mathematical Tables file of this journal; a review of the tables appears on page 692 of this issue.

Some of our results may be summarized as follows:

1. In the entire search, no solution was found for the range

$$9999 \leqq y \leqq 2 \times 10^5.$$

2. For $|k| = 1$, except for

$$239^2 - 2 \times 13^4 = -1, \qquad 161^2 - 20 \times 6^4 = +1$$

the only solutions found were those given by

$$n^2 - (n^2 \pm 1)1^4 = \mp 1, \qquad (8n \pm 1)^2 - (4n^2 \pm n)2^4 = 1.$$

3. The number of sets of D, k having 3, 4, 5 and 6 solutions is 120, 22, 6 and 2 respectively.

4. The values of D and k permitting more than 2 primitive solutions are given in Table 1.

TABLE 1

Values of D and k with No. of Solutions > 2

D	k	No. of Solutions	D	k	No. of Solutions
2	−953	3	7	57	5
2	−721	3	7	249	3
2	−508	3	7	274	3
2	−161	4	7	522	3
2	−126	3	7	954	4
2	−41	3	8	−959	5
2	−28	3	8	−127	3
2	34	3	8	−7	3
2	164	3	8	41	3
2	194	3	8	113	3
2	322	3	8	161	4
2	452	3	8	721	3
2	644	4	10	279	3
2	959	5	10	954	4
3	−767	3	11	−791	4
3	−407	4	11	−95	3
3	−207	3	11	−7	3
3	−194	3	11	185	3
3	−143	3	11	553	3
3	−47	3	11	665	4
3	−44	3	11	873	3
3	−39	3	11	889	3

TABLE 1 (*Continued*)

Values of D and k with No. of Solutions > 2

3	33	3	12	−11	3
3	118	3	13	−207	3
3	193	4	13	36	3
3	241	3	13	828	3
3	321	3	14	−693	3
3	481	5	14	−650	3
3	913	6	14	137	3
5	−380	3	14	260	3
5	−341	3	14	562	3
5	−305	3	15	−231	3
5	−209	3	15	−159	3
5	−149	3	15	−119	3
5	−44	3	15	49	3
5	220	3	15	385	5
5	545	3	15	721	3
5	596	3	17	559	3
5	649	3	17	824	3
5	745	3	18	−434	3
5	820	3	18	−252	3
5	836	4	18	−14	3
6	−482	4	18	658	4
6	−386	3	19	−639	3
6	190	3	19	−15	3
6	580	3	19	177	3
6	865	4	19	225	3
7	−423	5	19	657	6
7	−111	3	21	−332	3
7	9	4	21	100	3
21	148	4	35	65	4
21	340	4	35	865	3
21	564	3	37	−583	3
22	−18	3	37	−396	3
22	609	3	37	−33	3
22	819	3	37	−21	3
23	41	3	37	84	3
23	818	3	38	−319	3
24	−263	3	38	−34	4
24	145	3	38	481	3
24	265	3	39	−455	3
24	457	4	39	−350	3
27	−396	3	39	217	3
27	−23	3	40	−639	3
27	313	4	40	321	3
27	937	3	41	−695	3
28	−279	3	41	−655	3
28	333	3	41	−295	3
29	52	3	42	−636	3
29	676	3	42	−143	3
29	900	3	42	−38	4
30	−476	3	42	319	3
31	−207	3	42	697	3
31	90	3	43	441	4
31	225	3	43	873	3

5. Total number of primitive solutions with positive and negative k are 2672 and 2150, respectively.

Department of Mathematics
Memorial University of Newfoundland
St. John's, Newfoundland, Canada

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