Rigorous Machine Bounds for the Eigensystem of a General Complex Matrix*

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Introduction. We are concerned here with giving rigorous error bounds for the eigensystem of a general complex $n \times n$ matrix A, given an approximate eigensystem such as is furnished by [2]. In Section 1, we outline the technique in general terms and show that the bounds can be found in terms of computed quantities if $\|E\|_\infty = \|I - XY\|_\infty < 1$, where X is the matrix of approximate eigenvectors and Y is an approximate inverse for X. Then in Sections 2, 3, and 4 we give the specific roundoff error bounds for these general error terms, which include all the rounding errors made during the computation. An Algol program using the method is given in the microfiche section, and the results for the matrix example given in [2] are presented in Section 5, using the results of [2] as the initial approximation.

1. Theoretical Bounds for the Eigensystem. We assume we have a complex matrix A° of order n represented for our calculation by the matrix A, with

$$A = A^\circ + \Delta, \qquad |\Delta_{ij}| \leqq \delta \cdot \max_{1 \leqq i,j \leqq n} |A_{ij}|$$

and δ specified. We further assume a complete approximate eigensystem has been given for A, that is, a diagonal complex matrix Λ of eigenvalue approximations and a complex matrix X whose columns are approximations to the corresponding column eigenvectors of A, normalized in some way so that all components are less than or equal to 1.0 in modulus. We wish to give rigorous bounds for the true eigensystem of A°. In this section we outline the technique used, which follows Wilkinson [4, Chapter 9]. In Section 4, we will give the actual bounds used, which include bounds on the roundoff errors committed in the calculations.

We first perform a similarity transformation on A with X, assuming X can be inverted, giving

$$X^{-1}A^\circ X = \Lambda + P + Q\,,$$

where P is known exactly, and a bound for the modulus of each element of Q is known. To perform this similarity, let Y be a calculated approximate inverse for X, and define the following matrices:

$$F = AX - X\Lambda + Q_1\,, \qquad P = YF + Q_2\,, \qquad E = I - XY\,,$$

where the elements of Q_1 and Q_2 are small. In the actual computation, we form

$$F = fl_2(AX - X\Lambda) \qquad \text{and} \qquad P = fl_2(YF)\,,$$

using double-precision accumulation of inner products. Then we use Q_1 and Q_2 to

Received August 25, 1967. Revised January 24, 1968.

* This work was supported in part by the Office of Naval Research under Contract No. Nonr-225(37) (NR-044-211).

denote the errors made in these machine computations. We will show that we can obtain bounds for $|Q_{ij}|$ assuming only that

$$\|E\|_\infty = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|E_{ij}|\right) < 1 .$$

Using these matrices above, we have

$$\begin{aligned} P &= YF + Q_2 \\ &= X^{-1}F + (Y - X^{-1})F + Q_2 \\ &= X^{-1}AX - \Lambda + X^{-1}Q_1 + (Y - X^{-1})F + Q_2 , \end{aligned}$$

so that

$$X^{-1}A^{\circ}X = \Lambda + P + Q , \quad \text{with} \quad Q = -[X^{-1}\Delta X + X^{-1}Q_1 + (Y - X^{-1})F + Q_2] .$$

To bound Q, we note that if XY is invertible,

$$\begin{aligned} Y - X^{-1} &= Y(I - (XY)^{-1}) \\ &= Y(I - (I - E)^{-1}) \\ &= -YE(I - E)^{-1} . \end{aligned}$$

Also

$$\begin{aligned} X^{-1} &= (Y^{-1} - EY^{-1})^{-1} \\ &= Y(I - E)^{-1} , \end{aligned}$$

so that

$$Q = -[Y(I - E)^{-1}(\Delta X + Q_1) - YE(I - E)^{-1}F + Q_2] .$$

Now, for $i = 1, 2, \cdots, n$, let

$$\begin{aligned} \alpha_i &= \sum_{j=1}^{n}|Y_{ij}| , \quad \beta_i = \sum_{j=1}^{n}|X_{ji}| , \\ \gamma_i &= \max_{1\leq j\leq n}|F_{ji}| , \quad \sigma_i = \max_{1\leq j\leq n}|(Q_1)_{ji}| , \\ \tau_i &= \sum_{j=1}^{n}|(YE)_{ij}| ; \end{aligned}$$

and let $A_{\max} = \max_{1\leq i,j\leq n}|A_{ij}|$. Then we have

$$|(\Delta X + Q_1)_{ij}| \leqq \delta\cdot A_{\max}\cdot\beta_j + \sigma_j ,$$

so that

$$|[(I - E)^{-1}(\Delta X + Q_1)]_{ij}| \leqq \|(I - E)^{-1}\|_\infty\cdot(\delta\cdot A_{\max}\cdot\beta_j + \sigma_j) .$$

Also,

$$|[YE(I - E)^{-1}F]_{ij}| \leqq \|(I - E)^{-1}\|_\infty\cdot\tau_i\cdot\gamma_j .$$

Thus, assuming $\|E\|_\infty < 1$,

$$|Q_{ij}| \leqq \frac{\alpha_i(\delta\cdot A_{\max}\cdot\beta_j + \sigma_j) + \tau_i\cdot\gamma_j}{1 - \|E\|_\infty} + |(Q_2)_{ij}| .$$

We bound the eigenvalues of A° using Gerschgorin's theorems, which we now state for reference. For a given matrix B, define the Gerschgorin disks

$$G_i = \left\{\lambda: |\lambda - b_{ii}| \leqq \sum_{j \neq i} |b_{ij}|\right\}, \qquad i = 1, \cdots, n .$$

Then the first theorem of Gerschgorin states that all the eigenvalues of B are contained in the union of the n disks $\{G_i\}$. The second theorem states that if k of the disks are isolated from the others, then there are precisely k eigenvalues of B in the union of those k disks. For proofs of these theorems, see Marcus and Minc [1, p. 146].

We apply these theorems to the matrix $B = X^{-1}A^\circ X = \Lambda + P + Q$. If the elements of P and Q are small, and no other eigenvalue approximation Λ_{jj} is too close to Λ_{ii}, the ith disk will be isolated from the others, so that there is only one eigenvalue of B in the disk. For the ith disk to be isolated, we must have for all $k \neq i$,

$$|b_{kk} - b_{ii}| > \sum_{j \neq i} |b_{ij}| + \sum_{j \neq k} |b_{kj}| .$$

For the ith eigenvalue, we can usually obtain a better bound by applying Gerschgorin's theorem to the matrix B modified by multiplying the ith row by β^{-m} and the ith column by β^m, where β is the number base of the machine used and m is a nonnegative integer, chosen as large as possible under the restriction that the ith Gerschgorin disk of this modified B matrix remain isolated. For such an m, the disk is defined by

$$|\lambda - (\Lambda_{ii} + P_{ii} + Q_{ii})| \leqq \beta^{-m} \cdot \sum_{j \neq i} |P_{ij} + Q_{ij}| ,$$

so that one eigenvalue λ_i of A° satisfies the inequality

$$|\lambda_i - (\Lambda_{ii} + P_{ii})| \leqq r_i = |Q_{ii}| + \beta^{-m} \sum_{j \neq i} (|P_{ij}| + |Q_{ij}|) .$$

The ith disk will be isolated if for all $k \neq i$,

$$|(\Lambda_{ii} + P_{ii} + Q_{ii}) - (\Lambda_{kk} + P_{kk} + Q_{kk})| > \beta^{-m} \sum_{j \neq i} |P_{ij} + Q_{ij}| + \beta^m |P_{ki} + Q_{ki}| + \sum_{j \neq k, i} |P_{kj} + Q_{kj}| ,$$

which holds if

$$|(\Lambda_{ii} + P_{ii}) - (\Lambda_{kk} + P_{kk})| > |Q_{ii}| + |Q_{kk}| + \beta^{-m} \sum_{j \neq i} (|P_{ij}| + |Q_{ij}|) + \beta^m (|P_{ki}| + |Q_{ki}|) + \sum_{j \neq k, i} (|P_{kj}| + |Q_{kj}|) .$$

If such a bound can be obtained for the ith eigenvalue, we can also bound the corresponding eigenvector of A°. We first bound the corresponding eigenvector u of $B = X^{-1}A^\circ X$. Since B is nearly diagonal, u is close to the unit vector e_i. Hence we can set $u_i = 1.0$ and bound the other components of u by using the relation $Bu = \lambda_i u$ and the bound for λ_i obtained above. In fact, the kth equation of $Bu = \lambda_i u$ gives

$$\begin{aligned} &[(\Lambda_{ii} + P_{ii}) - (\Lambda_{kk} + P_{kk})]u_k \\ &\qquad = P_{ki} + Q_{ki} + (Q_{kk} + \theta_1 r_i)u_k + \sum_{j \neq k, i} (P_{kj} + Q_{kj})u_j \end{aligned} \tag{1}$$

where $|\theta_1| \leqq 1$. We set $\rho_k = (\Lambda_{ii} + P_{ii}) - (\Lambda_{kk} + P_{kk})$ for convenience. We first obtain a crude bound for u_k by assuming $|u_j| \leqq 1$ for $j \neq i$. This gives

$$|u_k| \leqq s_k^{(1)} = \left[|Q_{kk}| + r_i + \sum_{j\neq k} (|P_{kj}| + |Q_{kj}|)\right]/|\rho_k| , \qquad k = 1, \cdots, n .$$

Now we use this bound for all $|u_k|$ in (1), obtaining the more precise bounds

$$\left|u_k - \frac{P_{ki}}{\rho_k}\right| \leqq \left[|Q_{ki}| + (|Q_{kk}| + r_i)s_k^{(1)} + \sum_{j\neq k, i} (|P_{kj}| + |Q_{kj}|)s_j^{(1)}\right]/|\rho_k| ,$$

or, using $U_k = P_{ki}/\rho_k$ as the approximation to u_k,

$$|u_k - U_k| \leqq s_k^{(2)} , \qquad k \neq i \qquad (\text{and } u_i = 1.0) .$$

This bounds the eigenvector u of $B = X^{-1}A^\circ X$. To bound the corresponding eigenvector $v = Xu$ of A°, we must transform the estimate and bound for u by multiplying by X. Thus $|v_k - (XU)_k| \leqq \sum_{j=1}^n |X_{kj}|s_j^{(2)}$, $k = 1, \cdots, n$. Finally, we can normalize the estimate so that its largest component in modulus is 1.0, obtaining

$$\left|v_k - \frac{(XU)_k}{(XU)_{\max}}\right| \leqq \frac{\sum_{j=1}^n |X_{kj}|s_j^{(2)}}{|(XU)_{\max}|} , \qquad k = 1, \cdots, n .$$

2. Basic Roundoff Errors. To bound the errors in machine calculation, we let $\eta_1 = 2\cdot(1.06)\cdot\beta^{1-t}$ as in [3, p. 19] where β is the floating-point number base of the machine and t, the number of base β digits carried in each single-precision floating-point number. Thus η_1 is an upper bound for the relative rounding error committed in each basic real single-precision floating-point operation. That is,

$$\begin{aligned} |fl(x + y) - (x + y)| &\leqq \eta_1(|x| + |y|) , \\ |fl(x\cdot y) - (x\cdot y)| &\leqq \eta_1(|x\cdot y|) , \\ |fl(x/y) - (x/y)| &\leqq \eta_1(|x/y|) , \qquad y \neq 0 . \end{aligned}$$

We also assume the square root routine on the machine gives answers of comparable accuracy, i.e.,

$$|fl(\sqrt{x}) - \sqrt{x}| \leqq \eta_1 \sqrt{x} .$$

The factor 1.06 makes for easier accumulation of errors. η_1 could probably be taken smaller on most machines by a factor between 2 and 4, and the user may wish to adjust its value in the program. For further information, see Wilkinson [3].

To bound the errors in complex operations, let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then the following results are easily obtained.

1. $|fl(z_1 + z_2) - (z_1 + z_2)| \leqq \eta_1(|z_1| + |z_2|)$.
2. $|fl(z_1\cdot z_2) - (z_1\cdot z_2)| \leqq (2\sqrt{2})\eta_1|z_1\cdot z_2|$.
3.(a) If

$$fl(z_1/z_2) \equiv fl\left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\cdot fl\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right) ,$$

then $|fl(z_1/z_2) - (z_1/z_2)| \leqq (5\sqrt{2})\eta_1|z_1/z_2|$.

(b) Suppose $|x_2| \leqq |y_2|$ and set $r = x_2/y_2$, $d = y_2 + rx_2$.
If

$$fl(z_1/z_2) \equiv fl\left(\frac{x_1 r + y_1}{d}\right) + i \cdot fl\left(\frac{y_1 r - x_1}{d}\right),$$

then $|fl(z_1/z_2) - (z_1/z_2)| \leqq (6\sqrt{2})\eta_1 |z_1/z_2|$.

4.(a) If $fl(|z|) \equiv fl((x^2 + y^2)^{1/2})$, then $|fl(|z|) - |z|| \leqq 2\eta_1|z|$.

(b) If $|x| \leqq |y|$ and $fl(|z|) \equiv fl(|y| \cdot (1 + |x/y|^2)^{1/2})$, then $|fl(|z|) - |z|| \leqq 4\eta_1|z|$.

5. In a real single-precision floating-point inner product,

$$\left| fl\left(\sum_1^n x_i y_i\right) - \left(\sum_1^n x_i y_i\right) \right| \leqq n\eta_1 \left(\sum_1^n |x_i y_i|\right) \equiv \epsilon_1 \left(\sum_1^n |x_i y_i|\right).$$

6. In a complex floating-point inner product of single-precision factors accumulated in double precision and rounded to single precision,

$$\left| fl_2\left(\sum_1^n z_i w_i\right) - \left(\sum_1^n z_i w_i\right) \right| \leqq \eta_1 \left| \sum_1^n z_i w_i \right| + \epsilon_2 \left(\sum_1^n |z_i w_i|\right)$$

where $\epsilon_2 = (n + 1/2)\eta_2$ and $\eta_2 = 2 \cdot (1.06)\beta^{1-2t}$ bounds the basic double-precision rounding error.

3. Machine Bounds for the Error Matrix Q. To bound the elements of Q, we need bounds for the quantities used to bound Q in Section 1. We use barred symbols to denote the machine bounds.

1. $\alpha_i \leqq \bar{\alpha}_i = fl(\sum_{j=1}^n |Y_{ij}|) \cdot (1 + \epsilon_1 + 3\eta_1)$, $i = 1, \cdots, n$.
2. $\beta_i \leqq \bar{\beta}_i = fl(\sum_{j=1}^n |X_{ji}|) \cdot (1 + \epsilon_1 + 3\eta_1)$, $i = 1, \cdots, n$.
3. $\gamma_i \leqq \bar{\gamma}_i = \max_{1 \leqq j \leqq n} (fl(|F_{ji}|)) \cdot (1 + 5\eta_1)$, $i = 1, \cdots, n$.
4. $A_{\max} \leqq \bar{a} = \max_{1 \leqq i,j \leqq n} (fl(|A_{ij}|)) \cdot (1 + 5\eta_1)$.
5. To bound $\|E\|_\infty$, we have for $j \neq i$,

$$fl(E_{ij}) = -fl_2\left(\sum_{k=1}^n X_{ik} Y_{kj}\right)$$
$$= E_{ij}(1 + \theta\eta_1) + \theta\epsilon_2\left(\sum_{k=1}^n |X_{ik}||Y_{kj}|\right)$$

and

$$fl(E_{ii}) = fl_2\left(1 - \sum_{k=1}^n X_{ik} Y_{ki}\right)$$
$$= E_{ii}(1 + \theta\eta_1) + \theta(\epsilon_2 + \eta_2)\left(\sum_{k=1}^n |X_{ik}||Y_{ki}|\right) + \theta\eta_2 .$$

Here θ denotes a generic multiplier less than or equal to 1 in modulus. Thus if we set

$$\bar{\alpha} = fl\left(\sum_{i=1}^n \bar{\alpha}_i\right)(1 + \epsilon_1) ,$$

we have, since $|X_{ik}| \leqq 1$ for all i and k,

$$\|E\|_\infty \leqq \bar{e} = \left(\max_{1 \leqq i \leqq n} \left[fl\left(\sum_{j=1}^n |E_{ij}|\right)\right](1 + \epsilon_1 + 4\eta_1) + (\epsilon_2 + \eta_2)\bar{\alpha} + \eta_2\right)(1 + 5\eta_1).$$

6. We also need a bound for $\tau_i = \sum_{j=1}^n |(YE)_{ij}|$. Now

$$\begin{aligned} fl(YE)_{ij} &= fl_2\left(\sum_{k=1}^n Y_{ik}\cdot fl(E_{kj})\right) \\ &= \left(\sum_{k=1}^n Y_{ik}E_{kj}\right)(1+\theta\eta_1) + \sum_{k=1}^n Y_{ik}(E_{kj} - fl(E_{kj})) \\ &\quad + \theta\epsilon_2\cdot \sum_{k=1}^n (|Y_{ik}|\cdot|fl(E_{kj})|)\,. \end{aligned}$$

Thus

$$\begin{aligned} \tau_i \leqq \bar{\tau}_i = \Bigg[\left(\sum_{j=1}^n |fl_2(YE)_{ij}|\right)&(1+\epsilon_1+4\eta_1) + \bar{\alpha}_i(\epsilon_2+\eta_2)\bar{\alpha} \\ &+ (\eta_1+\epsilon_2)\bar{e} + \eta_2\Bigg](1+6\eta_1)\,. \end{aligned}$$

7. To bound σ_j, recall

$$\begin{aligned} fl(F_{ij}) &= fl_2\left(\left(\sum_{k=1}^n A_{ik}X_{kj}\right) - X_{ij}\Lambda_{jj}\right) \\ &= (AX - X\Lambda)_{ij} + \theta\eta_1|F_{ij}| + \theta(\epsilon_2+\eta_2)\left(\sum_{k=1}^n |A_{ik}||X_{kj}|\right) \\ &\quad + \theta((2\sqrt{2})\eta_2)|X_{ij}||\Lambda_{jj}| \\ &= (AX - X\Lambda)_{ij} + (Q_1)_{ij}\,, \end{aligned}$$

and thus

$$\sigma_j \leqq \bar{\sigma}_j = (\epsilon_2+\eta_2)\bar{\alpha}\bar{\beta}_j + 3\eta_2|\Lambda_{jj}| + \eta_1\bar{\gamma}_j\,.$$

8. Finally to bound $|(Q_2)_{ij}|$, we have

$$\begin{aligned} fl(P_{ij}) &= fl_2\left(\sum_{k=1}^n Y_{ik}F_{kj}\right) \\ &= (YF)_{ij} + \theta\eta_1|P_{ij}| + \theta\epsilon_2\left(\sum_{k=1}^n |Y_{ik}||F_{kj}|\right), \end{aligned}$$

so that

$$|(Q_2)_{ij}| \leqq \eta_1|P_{ij}| + \epsilon_2\bar{\alpha}_i\bar{\gamma}_j\,.$$

To give a rigorous machine bound for Q, we have to account for the errors made in computing the above bounds as well. Thus, for example,

$$|P_{ij}| \leqq fl(|P_{ij}|)\cdot(1+4\eta_1)\,.$$

Finally, we obtain

$$|Q_{ij}| \leqq \bar{Q}_{ij} = fl\left\{\eta_1|P_{ij}| + \epsilon_2\bar{\alpha}_i\bar{\gamma}_j + \frac{\bar{\alpha}_i(\delta\bar{\alpha}\bar{\beta}_j+\bar{\sigma}_j)+\bar{\tau}_i\bar{\gamma}_j}{1-\bar{e}-\eta_1(1+\bar{e})}\right\}(1+12\eta_1)\,.$$

For this to exist, we must have the denominator positive. Thus corresponding to the theoretical condition $\|E\|_\infty < 1$ of Section 1, we have the machine condition $\bar{e} < (1-\eta_1)/(1+\eta_1)$.

EIGENSYSTEM BOUNDS: FOURTH ITERATION

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ABS(LAMBDA[10]-( 8.12276,59240,40516,95387,418@-02)) ≤ 7,0877366625@-13

EIGENVECTOR BOUNDS ARE:
ABS(X[ 1,10] - (-9.80976,06019,70599,72072,042@-10)) ≤ 1,1068600432@-14
ABS(X[ 2,10] - ( 1.10958,96145,28815,97919,513@-08)) ≤ 1,0313422383@-14
ABS(X[ 3,10] - ( 7.33905,41289,82543,43573,171@-09)) ≤ 5,2100951406@-15
ABS(X[ 4,10] - (-1.44903,69720,68494,60904,979@-06)) ≤ 6,8429513045@-15
ABS(X[ 5,10] - ( 1.55770,04376,90727,26666,813@-05)) ≤ 6,9846249985@-15
ABS(X[ 6,10] - (-3.34790,51622,02285,25975,705@-05)) ≤ 2,7379035124@-14
ABS(X[ 7,10] - (-9.63703,80170,00350,37346,562@-04)) ≤ 1,9301862161@-13
ABS(X[ 8,10] - ( 1.33735,07467,49867,89746,583@-02)) ≤ 1,2611504385@-12
ABS(X[ 9,10] - (-9.19476,17614,76852,50508,680@-02)) ≤ 6,4441614122@-12
ABS(X[10,10] - ( 3.81457,47745,22297,68340,198@-01)) ≤ 2,3154536600@-11
ABS(X[11,10] - (-9.18772,34075,95947,12999,605@-01)) ≤ 5,1921389637@-11
ABS(X[12,10] - ( 1.00000,00000,00000,00000,000@+00)) ≤ 5,4842010918@-11


ABS(LAMBDA[11]-( 1.43646,51976,92204,86973,645@-01)) ≤ 1,5416429665@-13

EIGENVECTOR BOUNDS ARE:
ABS(X[ 1,11] - ( 6.20702,69189,10568,26641,505@-09)) ≤ 1,7434299658@-14
ABS(X[ 2,11] - (-3.70033,96340,09112,79938,779@-08)) ≤ 1,6308351611@-14
ABS(X[ 3,11] - (-2.54717,68428,92117,74995,888@-07)) ≤ 6,7782059504@-15
ABS(X[ 4,11] - ( 4.09451,15813,68153,62233,487@-06)) ≤ 9,3479387732@-15
ABS(X[ 5,11] - (-8.45053,59816,38507,96475,652@-06)) ≤ 8,2431766586@-15
ABS(X[ 6,11] - (-1.77654,47289,81438,28545,117@-04)) ≤ 1,3763628955@-14
ABS(X[ 7,11] - ( 1.47089,38187,69934,90432,824@-03)) ≤ 7,4496003126@-14
ABS(X[ 8,11] - (-1.34829,72206,76638,38936,038@-03)) ≤ 5,5882308860@-13
ABS(X[ 9,11] - (-4.31604,61433,37340,37369,166@-02)) ≤ 3,0773949604@-12
ABS(X[10,11] - ( 2.94847,38166,70738,33609,435@-01)) ≤ 1,1497666843@-11
ABS(X[11,11] - (-8.56353,48023,07795,40335,046@-01)) ≤ 2,6381681437@-11
ABS(X[12,11] - ( 1.00000,00000,00000,00000,000@+00)) ≤ 2,8151569027@-11


ABS(LAMBDA[12]-( 2.84749,72055,84781,88282,617@-01)) ≤ 1,3873439522@-14

EIGENVECTOR BOUNDS ARE:
ABS(X[ 1,12] - (-1.19061,05278,84764,66934,149@-07)) ≤ 6,0302198146@-15
ABS(X[ 2,12] - ( 2.99064,21369,10219,92917,680@-07)) ≤ 5,6408659548@-15
ABS(X[ 3,12] - ( 3.84817,17071,07064,87942,594@-06)) ≤ 2,3714995476@-15
ABS(X[ 4,12] - (-2.01687,57372,24147,30515,215@-05)) ≤ 3,2382383130@-15
ABS(X[ 5,12] - (-7.09670,09125,93027,69031,150@-05)) ≤ 2,8686450523@-15
ABS(X[ 6,12] - ( 7.44897,06845,14587,61341,644@-04)) ≤ 5,0253794403@-15
ABS(X[ 7,12] - (-1.26492,74040,69843,68850,891@-04)) ≤ 2,6846351293@-14
ABS(X[ 8,12] - (-1.53780,95663,08615,36267,732@-02)) ≤ 1,7839195852@-13
ABS(X[ 9,12] - ( 4.08486,82483,41952,45413,714@-02)) ≤ 1,0032388667@-12
ABS(X[10,12] - ( 1.13416,62084,13488,33735,374@-01)) ≤ 3,8657327999@-12
ABS(X[11,12] - (-7.15250,27944,15221,20985,641@-01)) ≤ 9,0072377961@-12
ABS(X[12,12] - ( 1.00000,00000,00000,00000,000@+00)) ≤ 9,6801259988@-12
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4. Machine Bounds for the Eigenvalues and Eigenvectors. We first form

$$\bar{\lambda}_i = fll(\Lambda_{ii} + P_{ii}) \,, \qquad i = 1, \cdots, n$$

as improved estimates for the eigenvalues, where *fll* denotes double-precision add and store. We include roundoff here by adding $\eta_2(|\Lambda_{ii}| + |P_{ii}|)$ to $\bar{Q}_{ii}$. We also form

$$t_i = fl\left(\sum_{k \neq i} (|P_{ik}| + \bar{Q}_{ik})\right) \cdot (1 + \epsilon_1 + 5\eta_1) \,, \qquad i = 1, \cdots, n \,,$$

bounding the off-diagonal row sums of $(P + Q)$.

To bound the error in $\bar{\lambda}_i$, we use the Gerschgorin bounds obtained in Section 1. We form for $j \neq i$,

$$\bar{\rho}_j = fl_2(\bar{\lambda}_i - \bar{\lambda}_j)\,,$$
$$\mu_j = fl(|\bar{\rho}_j|)\,,$$

and

$$\nu_j = fl((\eta_1 + \eta_2)(|\bar{\lambda}_i| + |\bar{\lambda}_j|))\cdot(1 + 7\eta_1)\,,$$

so that $|\bar{\rho}_j - (\bar{\lambda}_i - \bar{\lambda}_j)| \leqq \nu_j$. Then we can say rigorously that

$$|\lambda_i - \bar{\lambda}_i| \leqq \bar{r}_i = fl(Q_{ii} + \beta^{-m}t_i)\cdot(1 + \eta_1)\,,$$

where m is the largest nonnegative integer such that the ith Gerschgorin disk is isolated, i.e. so that for all $j \neq i$,

$$\mu_j > fl(4\eta_1\mu_j + \nu_j + \bar{r}_i + Q_{jj} + t_j + \beta^m(|P_{ji}| + \bar{Q}_{ji}))\cdot(1 + 7\eta_1)\,.$$

To choose m initially, note that the largest term in the above expression is usually the last, so we pick the largest m such that

$$\beta^m(|P_{ji}| + \bar{Q}_{ji}) < \mu_j\,, \qquad j \neq i\,.$$

Then we test each of the above more stringent requirements, decreasing m until they all hold or until $m < 0$. If the latter is true, we conclude that the ith eigenvalue cannot be isolated. Otherwise, we proceed to bound the corresponding eigenvector.

We first bound the eigenvector u of $B = X^{-1}A°X$. Corresponding to identity (1) of Section 1, we have in terms of machine quantities, for $k \neq i$,

$$\bar{\rho}_k u_k = P_{ki} + \theta\bar{Q}_{ki} + \theta(\bar{Q}_{kk} + \bar{r}_i + \nu_k)u_k + \sum_{j\neq k,i}(P_{kj} + \theta Q_{kj})u_j\,. \tag{2}$$

Again we first obtain a crude bound for $|u_k|$ by taking moduli and replacing $|u_j|$, $j = 1, \cdots, n$ on the right-hand side by the upper bound 1.0, obtaining for $k \neq i$,

$$|u_k| \leqq \bar{s}_k^{(1)} = fl\left(\frac{\bar{Q}_{kk} + \bar{r}_i + \nu_k + t_k}{\mu_k}\right)\cdot(1 + 8\eta_1)\,.$$

Now we use this bound for $|u_j|$ $(j \neq i)$ in the right-hand side of (2), obtaining the estimate $\bar{u}_k = fl(P_{ki}/\bar{\rho}_k)$ for u_k and the bound, for $k \neq i$,

$$\begin{aligned}|u_k - \bar{\mu}_k| &\leqq \bar{s}_k^{(2)}\\ &= fl\left\{\frac{\bar{Q}_{ki} + \bar{s}_k^{(1)}(\bar{Q}_{kk} + \bar{r}_i + \nu_k) + \sum_{j\neq k,i}(|P_{kj}| + Q_{kj})\bar{s}_j^{(1)}}{\mu_k} + 9\eta_1|\bar{u}_k|\right\}\\ &\qquad\cdot(1 + \epsilon_1 + 11\eta_1)\,.\end{aligned}$$

To bound the corresponding eigenvector $v = Xu$ of $A°$, we multiply through by X. Thus our estimate for v is $\bar{v} = X\bar{u}$, which we form and store in double precision. So

$$\bar{v}_j = fll\left(\sum_{k=1}^{n} X_{jk}\bar{u}_k\right)$$

and $|\bar{v}_j - (X\bar{u})_j| \leqq \epsilon_2\xi$, where

$$\xi = fl\left(\sum_{k=1}^{n} |\bar{u}_k|\right) \cdot (1 + \epsilon_1 + 3\eta_1) .$$

Also, from the above bound for $|u_k - \bar{u}_k|$, we have

$$|v_j - (X\bar{u})_j| \leqq \sum_{k=1}^{n} |X_{jk}|\bar{s}_k^{(2)} ,$$

where we know $u_i = \bar{u}_i = 1.0$ and $\bar{s}_i^{(2)} = 0$. This gives for $j = 1, \cdots, n$,

$$|v_j - \bar{v}_j| \leqq \bar{s}_j^{(3)} = fl\left(\sum_{k=1}^{n} |X_{jk}|\bar{s}_k^{(2)} + \epsilon_2\xi\right) \cdot (1 + \epsilon_1 + 5\eta_1) .$$

Finally we normalize $\bar{v}$ so its largest component in modulus is 1.0 by dividing by the largest component $\bar{v}_{j\max}$ in double precision, obtaining

$$\left| v_j - fll\left(\frac{\bar{v}_j}{\bar{v}_{j\max}}\right)\right| \leqq fl\left(\frac{\bar{s}_j^{(3)} + 9\eta_2|\bar{v}_j|}{|\bar{v}_{j\max}|}\right) \cdot (1 + 8\eta_1) .$$

5. Use of the Program. The program, M-10, given in the microfiche section herein, is coded in standard Algol 60 except for the addition of *complex* and *long* (double-precision) declarations. Arithmetic operations between two long variables is assumed to be done in double precision, and we assume the *abs* function is defined for a complex argument and gives the modulus. It is important to note that the technique can be applied repeatedly, using the output improved eigensystem as the input for the next iteration.

The program, translated into Burroughs Algol for real matrices, has been tested on scores of matrices using the Burroughs B5500 at Stanford University. The one example given here is the 12×12 Frank matrix for which the approximate eigensystem is given in [2]. Because the approximations to the smallest eigenvalues were so poor, on the first iteration we could not isolate the smallest three eigenvalues. But the improved eigensystem was more accurate, and on the fourth iteration, we obtained estimates for the eigenvalues which agreed with those published by Wilkinson [3, p. 152] to all 15 decimal places he gives, and which were guaranteed to at least 12 decimal places, as were the eigenvectors. We list the fourth iteration results for the smallest three eigenvalues.

Acknowledgment. The author would like to thank Professor G. E. Forsythe for his supervision and guidance, and Dr. J. H. Wilkinson for his many helpful suggestions.

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2. J. M. Varah, "The calculation of the eigenvectors of a general complex matrix by inverse iteration," *Math. Comp.*, v. 22, 1968, pp. 785–791.

3. J. H. Wilkinson, *Rounding Errors in Algebraic Processes*, Notes on Applied Science No. 32, HMSO, London; Prentice-Hall, Englewood Cliffs, N. J., 1963. MR **28** #4661.

4. J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965. MR **32** #1894.

```
COMMENT NOW BEGIN MAIN PROCEDURE EIGENSYSTEMBOUNDS;
REAL S1,S2,S3,S4,S5,AMAX,YSUM,EMAXROWSUM,ETA1,ETA2,EPS1,EPS2,
   FACTOR,RADIUS,RADIISUM,USUM;
LONG REAL MAXCOMP; COMPLEX SC1; LONG COMPLEX SCD1;
INTEGER I,J,K,M,JMAX;
REAL ARRAY Q,PQ[1:N,1:N],YROWSUM,XCOLSUM,FMAXCOL,PQROWSUM,ROOTDISTANCE,
   ABSBOUND,DISTERROR,UBND,VBND,YEROWSUM[1:N];
COMPLEX ARRAY F,P[1:N,1:N],ROOTDIFFERENCE,U[1:N];
LONG COMPLEX ARRAY V[1:N];

COMMENT FIRST CALCULATE BASIC ROUNDING ERRORS;
ETA1:=1.06×2×BETA↑(1-T); COMMENT THIS IS A BOUND FOR THE ROUNDING ERROR
   MADE IN EACH OF THE FOLLOWING REAL SINGLE PRECISION OPERATIONS:
   ADD,SUBTRACT,MULTIPLY,DIVIDE,SQUARE ROOT, IT IS MULTIPLIED BY 1.06
   TO PERMIT EASIER CALCULATION OF ACCUMULATED ERROR. THIS ROUNDING
   ERROR BOUND IS PESSIMISTIC, AND COULD PROBABLY BE DECREASED FOR
   THE ACTUAL MACHINE USED;
ETA2:=1.06×2×BETA↑(1-2×T); COMMENT A BOUND FOR THE CORRESPONDING REAL
   DOUBLE PRECISION OPERATIONS;
EPS1:=N×ETA1; COMMENT USED IN THE BOUND FOR A REAL SINGLE PRECISION
   INNER PRODUCT OF N TERMS;
EPS2:=(N+0.5)×ETA2; COMMENT USED IN THE BOUND FOR A COMPLEX DOUBLE
   PRECISION INNER PRODUCT OF N TERMS;
POORINVERSE:=FALSE;
COMMENT NOW FORM APPROXIMATE X-INVERSE;
INVERSE(N,X,Y,F,SINGULAR);

COMMENT NOW FIND ROW SUMS OF Y AND Y×E,COLUMN SUMS OF X, MAXIMUM
   ELEMENT OF A, AND MAXIMUM ROW SUM OF E, WHERE E=I-XY IS STORED IN F;
AMAX:=YSUM:=EMAXROWSUM:=0;
FOR I:=1 STEP 1 UNTIL N DO
BEGIN S1:=S2:=S3:=S4:=0;
   FOR J:=1 STEP 1 UNTIL N DO
   BEGIN
      S1:=S1+ABS(Y[I,J]);
      S2:=S2+ABS(X[J,I]);
      S3:=S3+ABS(F[I,J]);
      SCD1:=0;
      FOR K:=1 STEP 1 UNTIL N DO
         SCD1:=SCD1+Y[I,K]×F[K,J]; COMMENT COMPLEX DOUBLE PRECISION
            OPERATION;
      S4:=S4+ABS(ROUND(SCD1));
      AMAX:=MAX(AMAX,ABS(A[I,J]))
   END J;
   YROWSUM[I]:=S1×(1+EPS1+3×ETA1);
   YSUM:=YSUM+YROWSUM[I];
   YEROWSUM[I]:=S4;
   XCOLSUM[I]:=S2×(1+EPS1+3×ETA1);
   IF S3 > EMAXROWSUM THEN EMAXROWSUM := S3
END I;
AMAX:=AMAX×(1+5×ETA1);
YSUM:=YSUM×(1+EPS1);
EMAXROWSUM:=(EMAXROWSUM×(1+EPS1+4×ETA1) + (EPS2+ETA2)×YSUM + ETA2)
            ×(1+5×ETA1);
FOR I:=1 STEP 1 UNTIL N DO
```

```
   YERONSUM[I]:=(YERONSUM[I]×(1+EPS1+6×ETA1) + YROWSUM[I]
                 ×((EPS2+ETA2)×YSUM+(ETA1+EPS2)×EMAXROWSUM+ETA2))
                ×(1+6×ETA1);
IF EMAXROWSUM ≥ 1 THEN
BEGIN COMMENT THE CALCULATED INVERSE Y IS SUCH A POOR INVERSE FOR X
      THAT OUR BOUND FOR THE L-INFINITY NORM OF THE RESIDUAL MATRIX
      IS LARGER THAN 1.0;
   POORINVERSE:=TRUE;
'  GO TO FIN
END;

COMMENT NOW FORM RESIDUAL F=A×X-X×(DIAG(LAMBDA));
FOR I:=1 STEP 1 UNTIL N DO
BEGIN S1:=0; SC1:=LAMBDA[I];
   FOR J:=1 STEP 1 UNTIL N DO
   BEGIN SCD1:=0;
      FOR K:=1 STEP 1 UNTIL N DO
         SCD1:=SCD1+A[J,K]×X[K,I]; COMMENT COMPLEX DOUBLE PRECISION
            OPERATION;
      SCD1:=SCD1-X[J,I]×SC1; COMMENT COMPLEX DOUBLE PRECISION
         OPERATION;
      F[J,I]:=ROUND(SCD1);
      S1:=MAX(S1,ABS(F[J,I]))
   END J;
   FMAXCOL[I]:=S1×(1+4×ETA1)
END I;

COMMENT NOW FORM P=Y×F;
FOR I:=1 STEP 1 UNTIL N DO
BEGIN
   FOR J:=1 STEP 1 UNTIL N DO
   BEGIN SCD1:=0;
      FOR K:=1 STEP 1 UNTIL N DO
         SCD1:=SCD1+Y[I,K]×F[K,J]; COMMENT COMPLEX DOUBLE PRECISION
            OPERATION;
      P[I,J]:=ROUND(SCD1)
   END J;
   LAMBDAIMP[I]:=LAMBDA[I]+P[I,I]; COMMENT COMPLEX DOUBLE PRECISION
      OPERATION;
END I;

COMMENT NOW CALCULATE Q, THE MATRIX BOUNDING THE ERROR IN P;
S1:=1-EMAXROWSUM-(1+EMAXROWSUM)×ETA1;
S2:=AMAX×(ETA2+EPS2+DELTA);
S3:=3×ETA2;
FOR I:=1 STEP 1 UNTIL N DO
BEGIN S4:=EPS2×FMAXCOL[I]+(ETA1×FMAXCOL[I]+S2×XCOLSUM[I]
                          +S3×ABS(LAMBDA[I])×(1+4×ETA1))/S1;
   S5:=FMAXCOL[I]/S1;
   FOR J:=1 STEP 1 UNTIL N DO
      Q[J,I]:=(ETA1×ABS(P[J,I])×(1+4×ETA1)+YROWSUM[J]×S4
              +YERONSUM[J]×S5) × (1+10×ETA1);
   Q[I,I]:=Q[I,I]+ETA2×(ABS(LAMBDA[I])+ABS(P[I,I]))
END I;
```

```
COMMENT NOW FIND MATRIX PQ = ABS(P)+Q AND ITS OFF-DIAGONAL ROW SUMS;
FOR I:=1 STEP 1 UNTIL N DO
BEGIN S1:=0;
   FOR J:=1 STEP 1 UNTIL I-1,I+1 STEP 1 UNTIL N DO
   BEGIN PQ[I,J]:=(ABS(P[I,J])+Q[I,J])×(1+5×ETA1);
      S1:=S1+PQ[I,J]
   END;
   PQROWSUM[I]:=S1×(1+EPS1)
END;

COMMENT NOW BOUND EIGENVALUES AND EIGENVECTORS - I IS THE INDEX OF THE
   ROOT CONSIDERED;
FOR I:=1 STEP 1 UNTIL N DO
BEGIN COMMENT FIRST FIND DISTANCES BETWEEN ROOT I AND REST;
   FOR J:=1 STEP 1 UNTIL I-1,I+1 STEP 1 UNTIL N DO
   BEGIN ROOTDIFFERENCE[J]:=ROUND(LAMBDAIMP[I]-LAMBDAIMP[J]);
         COMMENT COMPLEX DOUBLE PRECISION OPERATION;
      DISTERROR[J]:=(ETA1+ETA2)×(ABS(ROUND(LAMBDAIMP[I]))
         +ABS(ROUND(LAMBDAIMP[J])))×(1+8×ETA1);
      ROOTDISTANCE[J]:=ABS(ROOTDIFFERENCE[J])
   END;
   COMMENT NOW BOUND THE I-TH EIGENVALUE USING THE GERSCHGORIN THEOREM.
      THE SMALLEST ISOLATED DISC IS FOUND USING A DIAGONAL SIMILARITY
      TRANSFORMATION WITH MULTIPLIER (BETA↑M). FIRST FIND THE
      APPROXIMATE EXPONENT M;
   IF I=1 THEN K:=2 ELSE K:=1;
   S1:=IF ROOTDISTANCE[K]=0 THEN 0 ELSE ROOTDISTANCE[K]/PQ[K,I];
   FOR J:=K+1 STEP 1 UNTIL I-1,I+K STEP 1 UNTIL N DO
      S1:=MIN(S1,IF ROOTDISTANCE[J]=0 THEN 0
            ELSE ROOTDISTANCE[J]/PQ[J,I]);
   IF S1<1 THEN M:=0 ELSE M:=ENTIER(LN(S1)/LN(BETA));
   FACTOR:=BETA↑M;

   COMMENT NOW MAKE SURE I-TH DISC IS ISOLATED;
NEWDISC:
   RADIUS:=(PQROWSUM[I]/FACTOR+Q[I,I])×(1+ETA1);
   FOR J:=1 STEP 1 UNTIL I-1,I+1 STEP 1 UNTIL N DO
   BEGIN
      RADIISUM:=(4×ETA1×ROOTDISTANCE[J]+DISTERROR[J]+Q[J,J]+PQROWSUM[J]
         +RADIUS+FACTOR×PQ[J,I])×(1+7×ETA1);
      IF ROOTDISTANCE[J]≤RADIISUM THEN
      BEGIN COMMENT DISC IS NOT ISOLATED - REDUCE M IF POSSIBLE;
         IF M>0 THEN
         BEGIN M:=M-1;
            FACTOR:=FACTOR/BETA;
            GO TO NEWDISC
         END ELSE
         BEGIN EIGENVALUERADIUS[I]:=-1;
            COMMENT DISCS I AND J CANNOT BE SEPARATED-HERE ONE COULD
               OUTPUT THIS FACT AND GIVE THE OVERLAPPING DISCS:
               DISCI=ABS(LAMBDA-LAMBDA[I])<RADIUS,
               DISCJ=ABS(LAMBDA-LAMBDA[J])<(RADIISUM-RADIUS);
            FOR K:=1 STEP 1 UNTIL N DO XIMP[K,I]:=X[K,I];
            GO TO ENDVECTOR
         END
```

```
    END
END J;
EIGENVALUERADIUS[I]:=RADIUS;

COMMENT NOW BOUND I-TH EIGENVECTOR OF A BY USING THE EQUATION
   ((X-INVERSE)×A×X)×U = LAMBDA[I]×U AND GERSCHGORIN BOUND FOR
   LAMBDA. FIRST FIND CRUDE UPPER BOUND FOR ABS(U[J]) BY ASSUMING
   LARGEST COMPONENT OF U IS U[I], AND LETTING THIS BE 1.0, SO
   ABS(U[J])<1,J≠I;
FOR J:=1 STEP 1 UNTIL I-1,I+1 STEP 1 UNTIL N DO
   ABSBOUND[J]:=(PQRUWSUM[J]+DISTERROR[J]+Q[J,J]+RADIUS)
      /ROOTDISTANCE[J]×(1+8×ETA1);

COMMENT NOW USE CRUDE BOUNDS IN SAME EQUATIONS TO GIVE MORE PRECISE
   BOUNDS. FIRST FIND ERROR BOUNDS FOR EACH COMPONENT OF U;
USUM:=0;
FOR J:=1 STEP 1 UNTIL I-1,I+1 STEP 1 UNTIL N DO
BEGIN S1:=Q[J,I];
   FOR K:=1 STEP 1 UNTIL MIN(I,J)-1,MIN(I,J)+1 STEP 1 UNTIL
      MAX(I,J)-1,MAX(I,J)+1 STEP 1 UNTIL N DO
         S1:=S1+PQ[J,K]×ABSBOUND[K];
   U[J]:=P[J,I]/ROOTDIFFERENCE[J];
   S2:=ABS(U[J]); USUM:=USUM+S2;
   UBND[J]:=((S1+ABSBOUND[J]×(Q[J,J]+RADIUS+DISTERROR[J]))
      /ROOTDISTANCE[J]+9×ETA1×S2)×(1+EPS1+11×ETA1)
END J;
USUM:=(USUM+1)×(1+EPS1+3×ETA1);

COMMENT NOW TRANSFORM TO A-BASIS BY MULTIPLICATION BY X;
MAXCOMP:=0;
FOR J:=1 STEP 1 UNTIL N DO
BEGIN SCD1:=X[J,I]; S1:=0;
   FOR K:=1 STEP 1 UNTIL I-1,I+1 STEP 1 UNTIL N DO
   BEGIN SCD1:=SCD1+X[J,K]×U[K]; COMMENT COMPLEX DOUBLE PRECISION
         OPERATION;
      S1:=S1+ABS(X[J,K])×UBND[K]
   END;
   V[J]:=SCD1; COMMENT COMPLEX DOUBLE PRECISION OPERATION;
   VBND[J]:=(S1+EPS2×USUM)×(1+EPS1+5×ETA1);
   IF ABS(SCD1)>MAXCOMP THEN
   BEGIN JMAX:=J;
      MAXCOMP:=ABS(SCD1); COMMENT COMPLEX DOUBLE PRECISION
         OPERATION;
   END
END J;

COMMENT NOW NORMALIZE VECTOR IN A-BASIS SO THAT ITS LARGEST
   COMPONENT IN MODULUS IS 1.0;
SCD1:=V[JMAX]; COMMENT COMPLEX DOUBLE PRECISION OPERATION;
S1:=ROUND(MAXCOMP);
FOR J:=1 STEP 1 UNTIL JMAX-1,JMAX+1 STEP 1 UNTIL N DO
BEGIN XIMP[J,I]:=V[J]/SCD1; COMMENT COMPLEX DOUBLE PRECISION
      OPERATION;
   EIGENVECTORRADIUS[J,I]:=(VBND[J]/S1+9×ETA2×ABS(ROUND(XIMP[J,I])))
      ×(1+8×ETA1)
```

```
      END;
      XIMP[JMAX,I]:=1.0; COMMENT COMPLEX DOUBLE PRECISION OPERATION;
      EIGENVECTORRADIUS[JMAX,I]:=VBNO[JMAX]/S1×(1+2×ETA1)1
   ENDVECTOR:
   END I;
   GO TO FIN;
   SINGULAR:
   POORINVERSE:=TRUE; COMMENT THE MATRIX X COULD NOT BE INVERTED;
   FIN:
   END EIGENSYSTEMBOUNDS;
```

THE OPTIMUM ADDITION OF POINTS TO QUADRATURE FORMULAE

T. N. L. PATTERSON

See article in this issue for explanation of symbols in table.

TABLE M1. EXTENDED 65 POINT GAUSS FORMULA.

ABSCISSAE	WEIGHTS
.99988 81764 53396 71498(0)*	.30125 85132 31673 70699(-3)
.99932 60970 75412 87727(0)	.84426 66418 02694 82767(-3)
.99818 32778 99995 05756(0)	.14409 74396 85300 28047(-2)
.99645 09480 61849 16306(0)	.20195 44741 84109 25814(-2)
.99414 97909 64034 94977(0)	.25819 35728 28895 91273(-2)
.99128 52761 76801 66872(0)	.31489 33385 29716 53928(-2)
.98784 91483 00927 63193(0)	.37236 23338 50821 03764(-2)
.98383 98121 87034 94138(0)	.42933 86024 84055 43766(-2)
.97926 54811 15988 37707(0)	.48543 12435 51340 42934(-2)
.97413 15398 33551 16907(0)	.54139 51002 81884 65190(-2)
.96843 69354 87597 49514(0)	.59751 42661 20618 34295(-2)
.96218 27547 18055 23771(0)	.65319 80449 96366 85264(-2)
.95537 55216 23831 03017(0)	.70814 75843 77401 14549(-2)
.94802 09281 68407 50637(0)	.76274 99793 84036 25927(-2)
.94012 09431 86899 50834(0)	.81720 14635 78137 01977(-2)
.93167 86282 28749 33797(0)	.87114 10032 06977 89083(-2)
.92270 06530 10921 93985(0)	.92434 20840 07523 21865(-2)
.91319 34405 42846 26174(0)	.97704 20249 91696 17917(-2)
.90316 09568 42872 39210(0)	.10293 81123 81518 72365(-1)
.89260 78805 04738 93142(0)	.10811 05280 05173 41970(-1)
.88154 15522 70033 47585(0)	.11320 34350 81627 65169(-1)
.86996 92949 26407 03619(0)	.11823 30098 99908 66315(-1)
.85789 66653 52267 05451(0)	.12320 97919 56166 62245(-1)
.84532 97528 99930 28394(0)	.12811 43161 06823 08620(-1)
.83227 67481 01741 59374(0)	.13293 16580 73810 40171(-1)
.81874 59259 22651 45343(0)	.13767 36769 94471 00941(-1)
.80474 42111 45924 39263(0)	.14234 86650 53417 32000(-1)
.79027 89574 92121 84305(0)	.14694 08698 21101 41522(-1)
.77535 92543 28223 24389(0)	.15143 75431 59682 28436(-1)
.75999 43224 41999 78687(0)	.15584 78852 34809 74670(-1)
.74419 22979 56146 71150(0)	.16017 87071 59279 99315(-1)
.72796 16763 29424 67901(0)	.16441 67767 69837 96055(-1)
.71131 24334 09271 95767(0)	.16855 09574 88589 45597(-1)
.69425 46952 13991 63355(0)	.17258 87401 53060 28112(-1)
.67679 76784 95260 92905(0)	.17653 59543 19965 17863(-1)
.65895 09061 93625 13304(0)	.18038 11905 95004 77916(-1)
.64072 51936 71929 56062(0)	.18411 45579 11501 90913(-1)
.62213 15090 85400 24158(0)	.18774 24289 77393 72243(-1)
.60318 00285 61739 61277(0)	.19126 99948 27276 24330(-1)
.58388 11896 60487 31333(0)	.19468 72317 37824 28613(-1)
.56424 65780 75511 99218(0)	.19798 52457 44320 03595(-1)
.54428 79248 62227 13855(0)	.20116 96842 85209 81585(-1)
.52401 62464 88573 96841(0)	.20424 53465 55819 49284(-1)
.50344 27804 55006 88234(0)	.20720 33047 03184 58503(-1)
.48257 97979 90000 38780(0)	.21003 54856 40188 00892(-1)
.46143 97015 69145 05770(0)	.21274 70868 14018 64581(-1)
.44003 42279 16276 66208(0)	.21534 27003 95674 04187(-1)
.41837 52966 23409 00926(0)	.21781 42923 66157 09489(-1)
.39647 57680 67884 94155(0)	.22015 44795 99525 62797(-1)

*The integer in brackets denotes the power of ten by which the number should be multiplied.