The Optimum Addition of Points to Quadrature Formulae*

By T. N. L. Patterson

Abstract. Methods are developed for the addition of points in an optimum manner to the Gauss, Lobatto and general quadrature formulae. A new set of n-point formulae are derived of degree $(3n - 1)/2$.

1. Introduction. In a recent book Kronrod [1] has shown how the n-point Gaussian quadrature formulae may be augmented by a set of $n + 1$ abscissae to yield quadrature formulae of degree $3n + 1$ (n even) or $3n + 2$ (n odd). The importance of these formulae is that the accuracy of a numerical integration can be considerably improved without wasting the integrand evaluations at the Gaussian abscissae. Kronrod has given tables of these extended abscissae including their associated weights for all Gaussian formulae up to 40 points.

Kronrod noted that as n is increased a large number of guarding digits have to be carried to preserve the accuracy of the results and implied that about sixty-five decimal digits were carried to produce the tables correct to sixteen decimal places. It is, unfortunately, the large values of n which are likely to be of the most interest.

In this paper it is shown how the additional abscissae may be derived in a numerically stable fashion by an expansion of the equation for the abscissae in terms of Legendre polynomials. A technique is also discussed to extend the n-point Lobatto quadrature formulae by the addition of $n - 1$ abscissae to yield quadrature formulae of degree $3n - 3$ (n even) or $3n - 2$ (n odd). Finally a method is discussed for the optimum addition of abscissae to general quadrature formulae and a new set of n-point formulae is derived of degree $(3n - 1)/2$.

2. The Extension of Quadrature Formulae. The basic reasoning behind the extension of quadrature formulae is as follows. Let an n-point formula be augmented by the addition of p abscissae and let $G_{n+p}(x)$ be the polynomial whose roots are the $n + p$ abscissae of the new quadrature formula. A general polynomial of degree $n + 2p - 1$ can be expressed as

$$(1) \qquad F_{n+2p-1}(x) = Q_{n+p-1}(x) + G_{n+p}(x) \sum_{k=0}^{p-1} c_k x^k ,$$

where $Q_{n+p-1}(x)$ is a general polynomial of degree $n + p - 1$. This transformation of $F_{n+2p-1}(x)$ is possible since the number of unknown coefficients on the left- and right-hand sides of (1) is equal. $Q_{n+p-1}(x)$ can always be exactly integrated by a $(n + p)$-point formula and if $G_{n+p}(x)$ is such that

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Received December 1, 1967.

* This research was supported by the National Aeronautics and Space Administration under Grant NSG 269.

$$\int_{-1}^{1} G_{n+p}(x)x^k dx = 0\,, \qquad k = 0, 1, \cdots, p-1\,, \tag{2}$$

then all of (1) can be exactly integrated by an $(n + p)$-point quadrature formula. Thus it should, in principle, be possible to derive formulae having $n + p$ abscissae and of degree $n + 2p - 1$.

2.1. *The Extension of the Gauss Formulae.* Kronrod [1] has considered the case $p = n + 1$ for the n-point Gauss formula. This choice of p yields the number of points required to subdivide the intervals spanned by the n original Gauss points. The resulting quadrature formula should have degree $3n + 1$. Since the formulae are symmetrical in the range $[-1, 1]$ odd functions are always integrated exactly with value zero. Hence the effective degree can be increased to $3n + 2$ when n is odd. For this choice of p the polynomial $K_{n+1}(x)$ whose $n + 1$ roots are the additional abscissae must satisfy, corresponding to (2),

$$\int_{-1}^{1} K_{n+1}(x)P_n(x)x^k dx = 0 \quad \text{for } k = 0, 1, \cdots, n\,, \tag{3}$$

where $P_n(x)$ is the Legendre polynomial. Kronrod determines $K_{n+1}(x)$ and hence its zeros by substituting its polynomial expansion into (3) and solving the resulting triangular system of equations to find the polynomial coefficients. It is at this point that the numerical difficulties arise. When n is large, the polynomial coefficients of $K_{n+1}(x)$ differ greatly in magnitude, so the significant inaccuracies due to both rounding and cancellation errors can appear in their calculation and when they are used to evaluate $K_{n+1}(x)$.

These numerical difficulties can be circumvented by expanding $K_{n+1}(x)$ in terms of orthogonal polynomials, in particular the Legendre polynomials. Writing

$$K_{n+1}(x) = \sum_{i=1}^{r} a_i P_{2i-1-q}(x)\,, \tag{4}$$

where $[x]$ denotes the integer part of x, $q = n - 2[n/2]$ and $r = [(n + 3)/2]$, then (3) becomes

$$\sum_{i=1}^{r} a_i \int_{-1}^{1} P_{2i-1-q}(x)P_n(x)x^k dx = 0\,. \tag{5}$$

Since the points should be added symmetrically (that is, if x is an abscissa then so is $-x$) $K_{n+1}(x)$ must be an odd or an even function and can be expressed in the form (4). The notation insures that odd and even values of n are correctly dealt with. Since x^k can be expanded in terms of Legendre polynomials and vice versa, condition (5) can also be expressed with x^k replaced by $P_k(x)$. In addition, since odd functions automatically satisfy (5) because of symmetry, then only odd values of k in (5) need be considered (note that $P_{2i-1-q}(x)P_n(x)$ is an odd function). Thus (5) finally becomes

$$\sum_{i=1}^{r} a_i \int_{-1}^{1} P_{2i-1-q}(x)P_n(x)P_{2k-1}(x)dx = 0\,, \qquad k = 1, 2, \cdots, r-1\,. \tag{6}$$

Writing

$$S_{i,k} = \int_{-1}^{1} P_{2i-1-q}(x)P_n(x)P_{2k-1}(x)dx\,, \tag{7}$$

then (6) can be expressed as

$$\sum_{i=1}^{r} a_i S_{i,k} = 0\,, \qquad k = 1, 2, \cdots, r-1\,. \tag{8}$$

It can be shown [2] that

$$S_{i,k} = 0 \quad \text{if } i + k < r\,. \tag{9}$$

Equation (8) then becomes

$$\sum_{i=r-k}^{r} a_i S_{i,k} = 0\,, \tag{10}$$

or expanding in full

$$\begin{aligned} a_{r-1} &= -a_r \frac{S_{r,1}}{S_{r-1,1}}\,, \\ a_{r-2} &= -a_r \frac{S_{r,2}}{S_{r-2,2}} - a_{r-1}\frac{S_{r-1,2}}{S_{r-2,2}}\,, \\ &\;\;\vdots \\ a_1 &= -a_r \frac{S_{r,r-1}}{S_{1,r-1}} - a_{r-1}\frac{S_{r-1,r-1}}{S_{1,r-1}} \cdots - a_2 \frac{S_{2,r-1}}{S_{1,r-1}}\,, \end{aligned} \tag{11}$$

which can be solved recursively. The coefficient a_r can be arbitrarily set equal to 1 without affecting the calculation of the roots of $K_{n+1}(x)$. It can be shown that

$$\frac{S_{i,k}}{S_{r-k,k}} = \frac{S_{i-1,k}}{S_{r-k,k}} \frac{\{n-q+2(i+k-1)\}\{n+q+2(k-i+1)\}\{n-1-q+2(i-k)\}\{2(k+i-1)-1-q-n\}}{\{n-q+2(i-k)\}\{2(k+i-1)-q-n\}\{n+1+q+2(k-i)\}\{n-1-q+2(i+k)\}} \tag{12}$$

Thus the quantities appearing in (11) can be recursively calculated using (12) with $i = r + 1 - k, \cdots, r$ in steps of one for each of $k = 1, 2, \cdots, r - 1$. Even for high values of n the a_i do not vary excessively in magnitude and in calculating the roots of $K_{n+1}(x)$ very few digits are lost through cancellation and round-off. For example, using sixteen-decimal-digit arithmetic at most two digits were lost for the case of $n = 65$.

The expansion in terms of Legendre polynomials can easily be summed by a simple algorithm which is based on the recurrence properties of the polynomials. To evaluate $S = \sum_{j=0}^{n} a_j P_j(x)$, a series of coefficients b_j is calculated from the recurrence relation

$$b_j = \{(2j+1)xb_{j+1} - (j+1)b_{j+2} - a_j\}/j \tag{13}$$

for $j = n, n-1, \cdots, 1$ with $b_{n+1} = b_{n+2} = 0$. Then $S = a_0 + b_2 - b_1 x$.

2.2. *The Extension of Lobatto Formulae.* The addition of $n - 1$ points ($p = n - 1$) to the n-point Lobatto formula should allow the derivation of a quadrature formula of degree $3n - 3$ (n even) or $3n - 2$ (n odd). This choice of p gives sufficient points to subdivide the intervals spanned by the original n abscissae.

Noting that the n Lobatto abscissae are the roots of the polynomial $(x^2 - 1)P'_{n-1}(x)$ the polynomial $W_{n-1}(x)$ whose roots are the required additional points must satisfy, corresponding to (2),

$$\int_{-1}^{1} W_{n-1}(x)(x^2 - 1)P'_{n-1}(x)x^k dx = 0\,, \qquad k = 0, 1, \cdots, n-2\,. \tag{14}$$

Again, as discussed earlier, x^k may be replaced by $P_k(x)$ and taking account of symmetry and the recurrence relations between the Legendre polynomials and their derivatives (14) may be reduced to,

$$\sum_{i=1}^{r-1} g_i \int_{-1}^{1} P_{2i-1-q}(x)\{P_n(x) - P_{n-2}(x)\}P_{2k-1}(x)dx = 0\,, \qquad k = 1, 2, \cdots, r-2\,, \tag{15}$$

where

$$W_{n-1}(x) = \sum_{i=1}^{r-1} g_i P_{2i-1-q}(x)\,, \tag{16}$$

the expansion again being in terms of Legendre polynomials. As before $q = n - 2[n/2]$ and $r = [(n+3)/2]$. Defining

$$S_{i,k} = \int_{-1}^{1} P_n(x)P_{2i-1-q}(x)P_{2k-1}(x)dx\,, \tag{17}$$

$$D_{i,k} = \int_{-1}^{1} P_{n-2}(x)P_{2i-1-q}(x)P_{2k-1}(x)dx\,, \tag{18}$$

and

$$U_{i,k} = S_{i,k} - D_{i,k}\,, \tag{19}$$

then Eq. (15) reduces to

$$\sum_{i=1}^{r-1} g_i U_{i,k} = 0\,, \qquad k = 1, 2, \cdots, r-2\,. \tag{20}$$

It can be shown that

$$U_{i,k} = 0 \quad \text{if } i + k < r - 1\,. \tag{21}$$

Thus (20) becomes

$$\sum_{i=r-1-k}^{r-1} g_i U_{i,k} = 0\,, \qquad k = 1, 2, \cdots, r-2\,, \tag{22}$$

or

$$
(23)\qquad
\begin{aligned}
g_{r-2} &= -g_{r-1}\frac{U_{r-1,1}}{U_{r-2,1}},\\
g_{r-3} &= -g_{r-1}\frac{U_{r-1,2}}{U_{r-3,2}} - g_{r-2}\frac{U_{r-2,2}}{U_{r-3,2}},\\
&\;\;\vdots\\
g_1 &= -g_{r-1}\frac{U_{r-1,r-2}}{U_{1,r-2}} - \cdots - g_2\frac{U_{2,r-2}}{U_{1,r-2}}.
\end{aligned}
$$

Writing

$$
(24)\qquad \frac{U_{i,k}}{U_{i-1,k}} = \frac{D_{i,k}}{D_{i-1,k}}\frac{\{S_{i,k}/D_{i,k} - 1\}}{\{S_{i-1,k}/D_{i-1,k} - 1\}},
$$

it can be shown that

$$
(25)\qquad \frac{S_{i,k}}{D_{i,k}} = \frac{\{n+1-q+2(i-k-1)\}\{n-1+q+2(k-i)\}\{n-q+2(i+k-1)\}\{2(k+i)-n-q\}}{\{n+1-q+2(i+k-1)\}\{n-q+2(i-k)\}\{n+q+2(k-i)\}\{2(k+i-1)-n+1-q\}}
$$

and

$$
(26)\qquad \frac{D_{i,k}}{D_{i-1,k}} = \frac{\{n-q+2(i+k-2)\}\{n+q+2(k-i)\}\{n-1-q+2(i-k-1)\}\{2(k+i)-n-1-q\}}{\{n-q+2(i-k-1)\}\{2(k+i)-n-q\}\{n+1+q+2(k-i-1)\}\{n-1-q+2(i+k-1)\}}.
$$

The quantities in (23) can then be recursively calculated using the relation

$$
(27)\qquad \frac{U_{i,k}}{U_{r-1-k,k}} = \frac{U_{i-1,k}}{U_{r-1-k,k}}\frac{U_{i,k}}{U_{i-1,k}}
$$

with $i = r - k, \cdots, r - 1$ in steps of one for each of $k = 1, 2, \cdots, r - 2$. The calculations again show that the g_i do not vary greatly in magnitude, and the roots of $W_{n-1}(x)$ can be calculated with little loss of accuracy due to cancellation.

TABLE 1

Davis-Rabinowitz σ_R for $a = 1.05$

Formula	*Number of points used*			
	7	15	31	63
Gauss	.118	1.12×10^{-3}	6.75×10^{-8}	1.31×10^{-21}
Curtiss-Clenshaw	.254	3.01×10^{-3}	9.95×10^{-7}	5.73×10^{-12}
Tables M10–M13	.132	2.07×10^{-3}	3.99×10^{-7}	1.20×10^{-14}

2.3. *The Extension of General Quadrature Formulae.* The methods described in Sections 2.1 and 2.2 for the extension of the Gaussian and Lobatto formulae are specific in that they make use of a knowledge of the properties of the polynomial whose zeros are the abscissae of the quadrature formula. In general no useful properties may be known and it is necessary to resort to an alternative technique.

The basic equation (2) can be written in the equivalent form,

$$\int_{-1}^{1} G_{n+p}(x)P_k(x)dx = 0\,, \qquad k = 0, 1, \cdots, p-1\,, \tag{28}$$

where $P_k(x)$ is the Legendre polynomial. $G_{n+p}(x)$ can however be expanded as

$$G_{n+p}(x) = \sum_{i=0}^{n+p} t_i P_i(x) \tag{29}$$

and substitution in (28) gives

$$\sum_{i=0}^{n+p} t_i \int_{-1}^{1} P_i(x)P_k(x)dx = 0\,, \qquad k = 0, 1, \cdots, p-1\,. \tag{30}$$

It is clear that this implies, due to the orthogonality properties of the Legendre polynomials, that $t_i = 0$ for $i = 0, 1, \cdots, p-1$. Thus (29) becomes

$$G_{n+p}(x) = \sum_{i=p}^{n+p} t_i P_i(x)\,. \tag{31}$$

Taking account of the symmetry of the abscissae, (31) may be rewritten as

$$G_{n+p}(x) = \sum_{i=1}^{[n/2]+1} c_i P_{2i-2+p+q}(x)\,, \tag{32}$$

where again $q = n - 2[n/2]$. Since the original abscissae x_j, $j = 1, \cdots, n$ are roots of $G_{n+p}(x)$ and since $c_{[n/2]+1}$ may be arbitrarily taken to be unity, then

$$\sum_{i=1}^{[n/2]} c_i P_{2i-2+p+q}(x_j) = -P_{n+p}(x_j)\,, \qquad j = 1, 2, \cdots, \left[\frac{n}{2}\right]. \tag{33}$$

The symmetry of the abscissae about the origin has also been taken into account in (33). The coefficients $c_1, c_2, \cdots, c_{[n/2]}$ may be found by solving the $[n/2]$ simultaneous equations which comprise (33) and hence the p additional abscissae determined as the zeros of $G_{n+p}(x)$ as given by (32).

When this analysis is applied to the Gauss and Lobatto formulae, it can be shown that p must be at least $n+1$ in the former case and $n-1$ in the latter case. The reason for this is that if $a_1, a_2, \cdots, a_p$ denote the abscissae added to the n-point Gauss formula, then the weight associated with a_j in the resulting quadrature formula is (cf. [3])

$$\omega_j \propto \int_{-1}^{1} P_n(x)S_j(x)dx\,,$$

where

$$S_j(x) = \prod_{i=1;\, i\neq j}^{p} (x - a_i)\,.$$

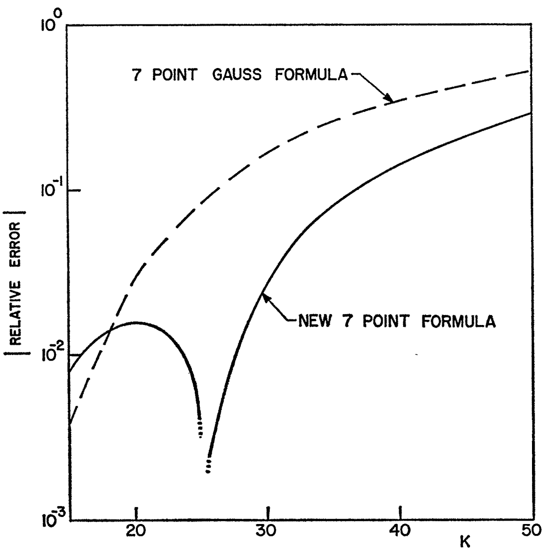

FIGURE 1. Absolute relative error in integrating $\int_{-1}^{1} x^K dx$ using the formula of Table M10. The corresponding result for the Gauss formula is also shown.

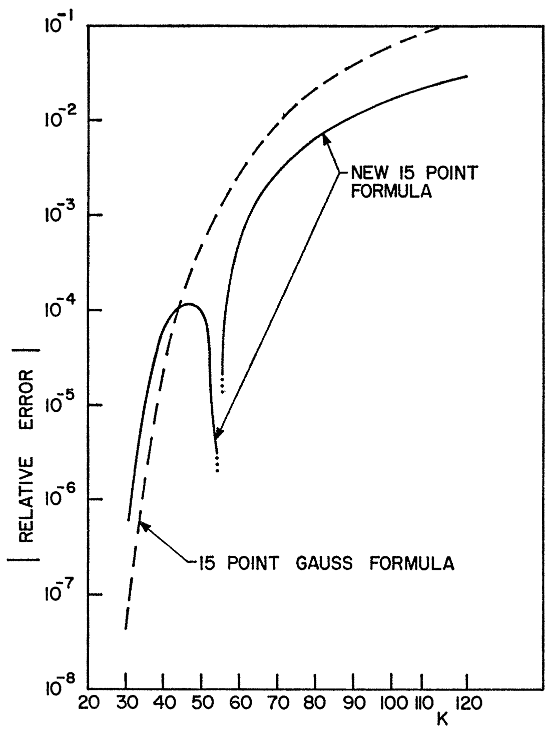

FIGURE 2. Absolute relative error in integrating $\int_{-1}^{1} x^K dx$ using the formula of Table M11. The corresponding result for the Gauss formula is also shown.

Noting that $S_j(x)$ is of degree $p - 1$ and that $P_n(x)$ is orthogonal to all polynomials of degree less than n then p must be greater than n otherwise ω_j would be zero. A similar argument extends to the n-point Lobatto formula whose abscissae are the roots of $[P_n(x) - P_{n-2}(x)]$. In this case p must exceed $n - 2$. The formulae given in Sections 2.1 and 2.2 thus represent the minimum extensions of the Gauss and Lobatto formulae. It may be noted that the extension of the integrating power of the n-point Gauss formula to degree $3n + 1$ by the addition of $n + 1$ points as discussed by Kronrod [1] is not a property restricted to the Gauss formulae. Any n-point formula irrespective of its original integrating degree will have its degree increased to $3n + 1$ by the addition of $n + 1$ points by the method discussed in this section. An example of this will be given later.

3. Some Extended Quadrature Formulae. In this section some examples of the applications of the techniques discussed earlier will be given. It has tacitly been assumed in Section 2 that the roots of the polynomial which defines the additional abscissae for any quadrature formula are all real. It has not in fact been possible to derive general conditions under which this is assured and the procedure has been to apply the techniques assuming that real roots exist but numerically checking for the occurrence of imaginary roots. All calculations have been carried out using not less than thirty decimal digits and any formulae quoted are correct to all digits given. The usual checks of integration of powers were successfully carried out.

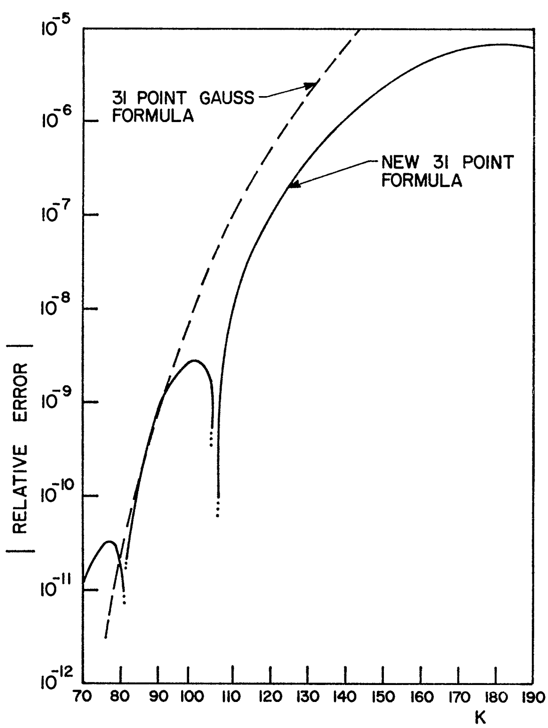

FIGURE 3. Absolute relative error in integrating $\int_{-1}^{1} x^K dx$ using the formula of Table M12. The corresponding result for the Gauss formula is also shown.

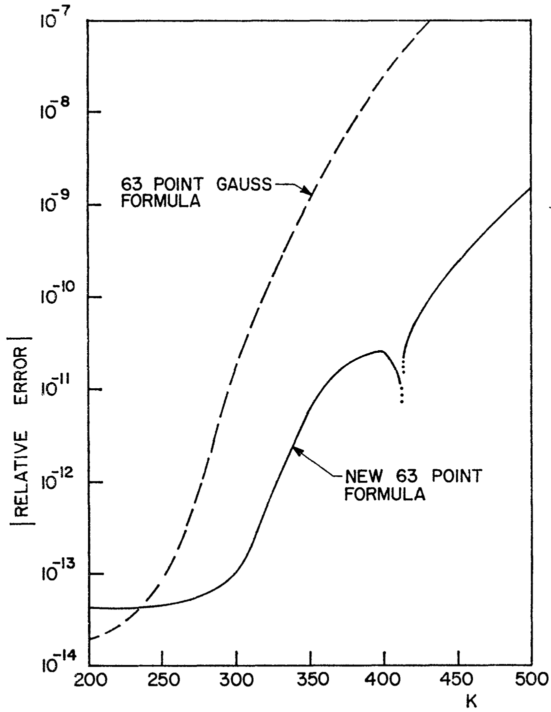

FIGURE 4. Absolute relative error in integrating $\int_{-1}^{1} x^K dx$ using the formula of Table M13. The corresponding result for the Gauss formula is also shown.

3.1. *Gauss and Lobatto Formulae.* In a recent paper [3] a set of economical quadrature formulae have been proposed which are based on the 65-point Gauss and 65-point Lobatto formulae. These formulae can be further extended by the addition of points as described in Sections 2.1 and 2.2. The weights and abscissae of the extended 65-point Gauss and 65-point Lobatto formulae, which are of respective degree 197 and 193 are given in Tables* M1 and M2. As further examples of the extension of Lobatto formulae, Tables M3 to M9 give the extended Lobatto formulae for $n = 3$ to 9.

3.2. *General Formulae.* Using the method of Section 2.3 a group of quadrature formulae were derived in the following sequence. Beginning with the 3-point Gauss formula, 4 abscissae were added to produce a 7-point formula of degree 11. Then 8 abscissae were added to this formula to produce a 15-point formula of degree 23. The process was continued until a 127-point formula of degree 191 was obtained. The effective degree of these n-point formulae is $(3n - 1)/2$. The weights and abscissae for these formulae are given in Tables M10 to M14. It may be noted that the weights associated with all these formulae are positive so that they are likely to converge in a satisfactory manner.

A detailed assessment was made of the integrating power of the first four of the new formulae of degree $(3n - 1)/2$. Figs. 1–4 show the results obtained when they are applied to integrate powers of x which they would not be expected to

* The letter M preceding a table number refers to the microfiche card.

integrate exactly. The absolute relative error (defined as $|(I - I_t)/I_t|$ where I_t is the true value of the integral and I is the value obtained by the formula) is plotted against K, the power of x being integrated. The results for the Gauss formulae using the same number of points are also shown for comparison. The sharp dips that appear on the curves are a result of a sign change in the relative error which give the formulae superior integrating performance to the Gauss formulae for high powers.

The performance of the new formulae has also been assessed by applying them to a large number of badly behaved integrands such as those having near singularities, cusps or singularities in their derivatives. In general it was found that the formulae had the important property of converging *uniformly* towards the true values of the integrals and were more accurate than the Gauss formula using the same number of points.

The quantities, σ_R, introduced by Davis and Rabinowitz [4] have also been calculated. An upper bound to the error E of a quadrature formula can be expressed as

$$|E| \leqq \sigma_R \|f\|$$

where $\|f\|$ is the norm of the integrand over a region R of the complex plane containing the range of integration. Table 1 shows the values of σ_R obtained for the new formulae of Tables M10 to M13 together with the values of σ_R for the Gauss and Curtiss-Clenshaw [5] formulae using the same number of points. The region R has been taken as an ellipse with semimajor axis a and semiminor axis $(a^2 - 1)^{1/2}$ with $a = 1.05$. As a further comparison it may be noted that the Romberg formulae using 5, 9, 17 and 33 points have σ_R respectively equal to 1.24, 0.422, 0.102, and 0.0155.

It should be emphasized again that the abscissae of these new formulae interlace with one another so that no computational labor is lost in going from a particular formula to the one of next higher degree. In practice, the successive application of the formulae would be used to monitor the convergence of the integral to which they are applied so that they are well suited to automatic quadrature. It thus appears that these new formula may form the basis of a very powerful technique for economically carrying out numerical integration.

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TABLE M1. CONTINUED

ABSCISSAE	WEIGHTS
.37434 86151 22066 01201(0)	.22236 82112 60932 01241(-1)
.35200 61722 81730 08657(0)	.22446 00162 44882 51186(-1)
.32946 09198 37486 40765(0)	.22642 26064 47799 23162(-1)
.30672 62021 15123 88900(0)	.22824 91863 75609 59218(-1)
.28381 54539 02248 73062(0)	.22994 46106 40362 40261(-1)
.26074 14822 33917 68599(0)	.23151 34607 45955 92614(-1)
.23751 72033 46416 80657(0)	.23294 90804 87832 60072(-1)
.21415 63336 35879 86339(0)	.23424 51772 98641 87027(-1)
.19067 26556 26142 76977(0)	.23540 66327 53178 38515(-1)
.16707 93211 20095 94785(0)	.23643 81778 54511 60402(-1)
.14338 95546 98975 17113(0)	.23733 36953 23917 10621(-1)
.11961 73280 71724 17312(0)	.23808 73208 13563 43176(-1)
.95776 65320 91975 05652(-1)	.23870 40624 60168 11199(-1)
.71880 89690 73580 96290(-1)	.23918 88850 30427 93629(-1)
.47943 46235 31718 57523(-1)	.23953 61295 62549 52650(-1)
.23978 45654 56066 46133(-1)	.23974 02895 11659 71961(-1)
.00000 00000 00000 00000(0)	.23980 65844 38877 59351(-1)

TABLE M2. EXTENDED 65 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.14875 67001 37033 21556(-3)
.99945 98836 91787 55946(0)	.89717 94413 52264 12811(-3)
.99823 58589 85168 15870(0)	.15331 59291 25636 89622(-2)
.99642 80987 37359 63413(0)	.20710 34513 43963 78925(-2)
.99409 01501 18423 12124(0)	.26192 48187 14872 94448(-2)
.99115 75391 85599 61536(0)	.32549 56332 72898 57984(-2)
.98758 59307 69509 13551(0)	.38761 99921 49446 27670(-2)
.98343 19318 44141 33376(0)	.44227 67750 29190 55801(-2)
.97873 91331 91998 88949(0)	.49725 42973 74324 20981(-2)
.97346 35980 55045 81704(0)	.55858 59307 95072 34980(-2)
.96757 08302 66913 61434(0)	.61900 99429 85505 99570(-2)
.96110 59454 76496 33351(0)	.67316 91358 48302 88834(-2)
.95410 75374 60022 21278(0)	.72726 27960 42309 54170(-2)
.94654 19692 82292 03315(0)	.78646 25022 15294 47519(-2)
.93838 11976 82665 55698(0)	.84485 68867 43230 53357(-2)
.92966 49704 69001 31780(0)	.89765 20867 61906 60146(-2)
.92042 91162 42990 77848(0)	.95013 24659 39016 50868(-2)
.91064 65260 74030 59442(0)	.10069 06621 81197 70227(-1)
.90029 38755 61607 01088(0)	.10628 57951 85375 23813(-1)
.88940 80238 70587 26396(0)	.11136 13297 56674 46988(-1)
.87802 32353 24945 71004(0)	.11638 61527 54537 57055(-1)
.86611 71113 34507 00665(0)	.12178 13400 87558 38093(-1)
.85367 00196 81449 30870(0)	.12708 81429 67153 30303(-1)
.84071 71799 25226 56725(0)	.13190 10427 66871 19974(-1)
.82729 19921 66940 42060(0)	.13664 70582 14242 31247(-1)
.81337 57127 98435 33915(0)	.14171 77918 86257 62629(-1)
.79895 17188 04367 38062(0)	.14669 23629 58410 72262(-1)
.78405 42989 70844 45245(0)	.15118 98137 50331 37667(-1)
.76871 64197 61627 53936(0)	.15560 62445 19087 04982(-1)
.75292 22943 84155 42500(0)	.16031 07621 09431 61498(-1)
.73665 78099 44952 39569(0)	.16491 08511 95226 89389(-1)
.71995 67534 15456 71617(0)	.16904 48545 85644 91029(-1)
.70285 19289 17937 01945(0)	.17308 53092 02300 25469(-1)
.68532 99967 38069 21827(0)	.17738 38451 83188 30929(-1)
.66737 89605 55587 34134(0)	.18156 98258 53511 16755(-1)
.64903 23720 53465 15085(0)	.18529 69222 84873 80044(-1)
.63032 30428 64267 42291(0)	.18891 93638 91523 45903(-1)
.61123 96782 24221 54911(0)	.19277 50960 03525 25773(-1)
.59177 20684 20342 07923(0)	.19651 06619 58357 75538(-1)
.57195 36940 97625 58981(0)	.19979 19494 05068 97351(-1)
.55181 74759 01825 37997(0)	.20295 88266 29681 46906(-1)
.53135 38267 74265 10187(0)	.20633 85405 85430 18524(-1)
.51055 40332 08072 03040(0)	.20959 12434 83918 25854(-1)
.48945 16007 67358 35880(0)	.21239 25161 93490 41272(-1)
.46807 96126 82275 66264(0)	.21507 09550 14254 10543(-1)
.44642 98929 54926 61478(0)	.21794 55489 45386 13791(-1)
.42449 49589 70138 40821(0)	.22068 72305 73369 58660(-1)
.40230 83896 91663 52316(0)	.22297 91590 68734 00110(-1)
.37990 34500 65463 99621(0)	.22514 11671 23776 34058(-1)

TABLE M2. CONTINUED

ABSCISSAE	WEIGHTS
.35727 31045 49699 22750(0)	.22748 60492 07580 88138(-1)
.33441 08521 09538 43246(0)	.22969 31932 54275 73946(-1)
.31135 03598 11223 82476(0)	.23145 15062 45817 61099(-1)
.28812 50685 25947 47573(0)	.23307 41637 59425 12861(-1)
.26472 88295 02593 24883(0)	.23486 95683 35202 70315(-1)
.24115 58840 85896 52051(0)	.23652 35843 89897 49961(-1)
.21743 99780 07708 34883(0)	.23772 92310 80624 35668(-1)
.19361 47045 11110 18185(0)	.23879 48534 47524 70952(-1)
.16967 45590 25373 25589(0)	.24002 60882 99875 18612(-1)
.14561 42922 31964 90496(0)	.24111 35346 11917 82791(-1)
.12146 77021 86137 49557(0)	.24175 28147 07499 66966(-1)
.97268 49892 76217 03334(-1)	.24224 90774 39986 61180(-1)
.73011 58657 31465 23061(-1)	.24290 67089 58171 57297(-1)
.48691 99548 25551 17357(-1)	.24341 94575 69411 69670(-1)
.24343 53857 59553 41108(-1)	.24348 41110 42346 40483(-1)
.00000 00000 00000 00000(0)	.24340 41309 66124 73185(-1)

TABLE M3. EXTENDED 3 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.10000 00000 00000 00000(0)
.65465 36707 07977 14380(0)	.54444 44444 44444 44444(0)
.00000 00000 00000 00000(0)	.71111 11111 11111 11111(0)

TABLE M4. EXTENDED 4 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.52380 95238 09523 80952(-1)
.81649 65809 27726 03273(0)	.29387 75510 20408 16327(0)
.44721 35954 99957 93928(0)	.42517 00680 27210 88435(0)
.00000 00000 00000 00000(0)	.45714 28571 42857 14286(0)

TABLE M5. EXTENDED 5 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.30643 73897 70723 10406(-1)
.89040 55275 12668 78657(0)	.17926 26995 53207 35598(0)
.65465 36707 07977 14380(0)	.28397 87780 48121 11381(0)
.34098 22659 10992 97151(0)	.33423 37398 16417 68358(0)
.00000 00000 00000 00000(0)	.34376 20872 10363 07243(0)

TABLE M6. EXTENDED 6 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.20762 24429 16560 56362(-1)
.92570 36801 44929 57797(0)	.12162 22276 47966 68599(0)
.76505 53239 29464 69285(0)	.19488 50876 62446 84709(0)
.54490 26063 54830 86190(0)	.24204 51378 29853 06725(0)
.28523 15164 80645 09631(0)	.27549 92224 98276 34328(0)
.00000 00000 00000 00000(0)	.29037 21601 39602 00007(0)

TABLE M7. EXTENDED 7 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.14665 88966 58896 65890(-1)
.94713 03475 88262 18140(0)	.87184 82155 76185 43400(-1)
.83022 38962 78566 92987(0)	.14379 01116 29255 26092(0)
.66573 36632 73037 63957(0)	.18238 00341 40545 79620(0)
.46884 87934 70714 21380(0)	.21091 91306 78497 31549(0)
.24442 33913 97794 40864(0)	.23688 52275 48611 97533(0)
.00000 00000 00000 00000(0)	.24834 95695 59162 88554(0)

TABLE M8. EXTENDED 8 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.11089 78697 17030 50426(-1)
.96004 76286 86628 49341(0)	.65848 51614 92001 24816(-1)
.87174 01485 09606 61534(0)	.10876 63209 86566 54815(0)
.74633 66718 39601 53459(0)	.14063 74454 09474 94571(0)
.59170 01814 33142 30214(0)	.16850 77260 19600 97971(0)
.41030 34809 13798 97178(0)	.19304 98391 53946 41368(0)
.20929 92179 02478 86877(0)	.20689 25912 07336 21511(0)
.00000 00000 00000 00000(0)	.21041 55482 04343 44480(0)

TABLE M9. EXTENDED 9 POINT LOBATTO FORMULA.

ABSCISSAE	WEIGHTS
.10000 00000 00000 00000(1)	.85716 93958 88041 96400(-2)
.96900 62363 96496 10536(0)	.51267 30533 78337 06561(-1)
.89975 79954 11460 15731(0)	.85832 98210 83995 35644(-1)
.80029 59699 78342 09738(0)	.11184 47962 75368 17706(0)
.67718 62795 10737 75345(0)	.13451 99355 65912 64943(0)
.53078 87048 68355 21544(0)	.15805 00535 81448 05318(0)
.36311 74638 26178 15871(0)	.17563 45320 57268 04001(0)
.18317 60550 68777 29799(0)	.18260 59906 19940 54825(0)
.00000 00000 00000 00000(0)	.18334 54209 89897 74045(0)

TABLE M10. GENERAL EXTENDED FORMULA OF DEGREE 11.

ABSCISSAE	WEIGHTS
.96049 12687 08020 28342(0)	.10465 62260 26467 26519(0)
.77459 66692 41483 37704(0)	.26848 80898 68333 44073(0)
.43424 37493 46802 55800(0)	.40139 74147 75962 22291(0)
.00000 00000 00000 00000(0)	.45091 65386 58474 14235(0)

TABLE M11. GENERAL EXTENDED FORMULA OF DEGREE 23.

ABSCISSAE	WEIGHTS
.99383 19632 12755 02221(0)	.17001 71962 99402 60339(-1)
.96049 12687 08020 28342(0)	.51603 28299 70797 39697(-1)
.88845 92328 72256 99889(0)	.92927 19531 51245 37686(-1)
.77459 66692 41483 37704(0)	.13441 52552 43784 22036(0)
.62110 29467 37226 40294(0)	.17151 19091 36391 38079(0)
.43424 37493 46802 55800(0)	.20062 85293 76989 02103(0)
.22338 66864 28966 88163(0)	.21915 68584 01587 49640(0)
.00000 00000 00000 00000(0)	.22551 04997 98206 68739(0)

TABLE M12. GENERAL EXTENDED FORMULA OF DEGREE 47.

ABSCISSAE	WEIGHTS
.99909 81249 67667 59766(0)	.25447 80791 56187 44154(-2)
.99383 19632 12755 02221(0)	.84345 65739 32110 62463(-2)
.98153 11495 53740 10687(0)	.16446 04985 43878 10934(-1)
.96049 12687 08020 28342(0)	.25807 59809 61766 53565(-1)
.92965 48574 29740 05667(0)	.35957 10330 71293 22097(-1)
.88845 92328 72256 99889(0)	.46462 89326 17579 86541(-1)
.83672 59381 68868 73550(0)	.56979 50949 41233 57412(-1)
.77459 66692 41483 37704(0)	.67207 75429 59907 03540(-1)
.70249 62064 91527 07861(0)	.76879 62049 90035 31043(-1)
.62110 29467 37226 40294(0)	.85755 92004 99903 51154(-1)
.53131 97436 44375 62397(0)	.93627 10998 12644 73617(-1)
.43424 37493 46802 55800(0)	.10031 42786 11795 57877(0)
.33113 53932 57976 83309(0)	.10566 98935 80234 80974(0)
.22338 66864 28966 88163(0)	.10957 84210 55924 63824(0)
.11248 89431 33186 62575(0)	.11195 68730 20953 45688(0)
.00000 00000 00000 00000(0)	.11275 52567 20768 69161(0)

TABLE M13. GENERAL EXTENDED FORMULA OF DEGREE 95.

ABSCISSAE	WEIGHTS
.99987 28881 20357 61194(0)	.36322 14818 45530 65969(-3)
.99909 81249 67667 59766(0)	.12651 56556 23006 80114(-2)
.99720 62593 72221 95908(0)	.25790 49794 68568 82724(-2)
.99383 19632 12755 02221(0)	.42176 30441 55885 48391(-2)
.98868 47575 47429 47994(0)	.61155 06822 11724 63397(-2)
.98153 11495 53740 10687(0)	.82230 07957 23592 96693(-2)
.97218 28747 48581 79658(0)	.10498 24690 96213 21898(-1)
.96049 12687 08020 28342(0)	.12903 80010 03512 65626(-1)
.94634 28583 73402 90515(0)	.15406 75046 65594 97802(-1)
.92965 48574 29740 05667(0)	.17978 55156 81282 70333(-1)
.91037 11569 57004 29250(0)	.20594 23391 59127 11149(-1)
.88845 92328 72256 99889(0)	.23231 44663 99102 69443(-1)
.86390 79381 93690 47715(0)	.25869 67932 72147 46911(-1)
.83672 59381 68868 73550(0)	.28489 75474 58335 48613(-1)
.80694 05319 50217 61186(0)	.31073 55111 16879 64880(-1)
.77459 66692 41483 37704(0)	.33603 87714 82077 30542(-1)
.73975 60443 52694 75868(0)	.36064 43278 07825 72640(-1)
.70249 62064 91527 07861(0)	.38439 81024 94555 32039(-1)
.66290 96600 24780 59546(0)	.40715 51011 69443 18934(-1)
.62110 29467 37226 40294(0)	.42877 96002 50077 34493(-1)
.57719 57100 52045 81484(0)	.44914 53165 36321 97414(-1)
.53131 97436 44375 62397(0)	.46813 55499 06280 12403(-1)
.48361 80269 45841 02756(0)	.48564 33040 66731 98716(-1)
.43424 37493 46802 55800(0)	.50157 13930 58995 37414(-1)
.38335 93241 98730 34692(0)	.51583 25395 20484 58777(-1)
.33113 53932 57976 83309(0)	.52834 94679 01165 19862(-1)
.27774 98220 21824 31507(0)	.53905 49933 52660 63927(-1)
.22338 66864 28966 88163(0)	.54789 21052 79628 65032(-1)
.16823 52515 52207 46498(0)	.55481 40435 65593 63988(-1)
.11248 89431 33186 62575(0)	.55978 43651 04763 19408(-1)
.56344 31304 65927 89972(-1)	.56277 69983 12543 01273(-1)
.00000 00000 00000 00000(0)	.56377 62836 03847 17388(-1)

TABLE M14. GENERAL EXTENDED FORMULA OF DEGREE 191.

ABSCISSAE	WEIGHTS
.99998 24303 54910 25124(0)	.50536 09519 01452 06798(-4)
.99987 28881 20357 61194(0)	.18073 95644 73257 95500(-3)
.99959 87996 71903 79848(0)	.37774 66463 19067 34571(-3)
.99909 81249 67667 59766(0)	.63260 73193 59580 60423(-3)
.99831 66353 18407 86870(0)	.93836 98485 43070 95542(-3)
.99720 62593 72221 95908(0)	.12895 24082 61058 90073(-2)
.99572 41046 98407 16409(0)	.16811 42865 42143 32069(-2)
.99383 19632 12755 02221(0)	.21088 15245 72662 43327(-2)
.99149 57211 78106 13362(0)	.25687 64943 79402 21893(-2)
.98868 47575 47429 47994(0)	.30577 53410 17553 15775(-2)
.98537 14995 98520 37105(0)	.35728 92783 51729 95547(-2)
.98153 11495 53740 10687(0)	.41115 03978 65469 30221(-2)
.97714 15146 39705 71416(0)	.46710 50372 11432 17529(-2)
.97218 28747 48581 79658(0)	.52491 23454 80885 91267(-2)
.96663 78515 58416 56709(0)	.58434 49875 83563 95072(-2)
.96049 12687 08020 28342(0)	.64519 00050 17573 69227(-2)
.95373 00064 25761 13641(0)	.70724 89995 43355 54681(-2)
.94634 28583 73402 90515(0)	.77033 75233 27974 18482(-2)
.93832 03977 79592 88365(0)	.83428 38753 96815 77056(-2)
.92965 48574 29740 05667(0)	.89892 75784 06413 57233(-2)
.92034 00254 70012 42073(0)	.96411 77729 70253 66953(-2)
.91037 11569 57004 29250(0)	.10297 11695 79563 55524(-1)
.89974 48997 76940 03664(0)	.10955 73338 78379 01648(-1)
.88845 92328 72256 99889(0)	.11615 72331 99551 34727(-1)
.87651 34144 84705 26974(0)	.12275 83056 00827 70087(-1)
.86390 79381 93690 47715(0)	.12934 83966 36073 73455(-1)
.85064 44947 68350 27976(0)	.13591 57100 97655 46790(-1)
.83672 59381 68868 73550(0)	.14244 87737 29167 74306(-1)
.82215 62543 64980 40737(0)	.14893 64166 48151 82035(-1)
.80694 05319 50217 61186(0)	.15536 77555 58439 82440(-1)
.79108 49337 99848 36143(0)	.16173 21872 95777 19942(-1)
.77459 66692 41483 37704(0)	.16801 93857 41038 65271(-1)
.75748 39663 80513 63793(0)	.17421 93015 94641 73747(-1)
.73975 60443 52694 75868(0)	.18032 21639 03912 86320(-1)
.72142 30853 70098 91548(0)	.18631 84825 61387 90186(-1)
.70249 62064 91527 07861(0)	.19219 90512 47277 66019(-1)
.68298 74310 91079 22809(0)	.19795 49504 80974 99488(-1)
.66290 96600 24780 59546(0)	.20357 75505 84721 59467(-1)
.64227 66425 09759 51377(0)	.20905 85144 58120 23852(-1)
.62110 29467 37226 40294(0)	.21438 98001 25038 67246(-1)
.59940 39302 42242 89297(0)	.21956 36630 53178 26939(-1)
.57719 57100 52045 81484(0)	.22457 26582 68160 98707(-1)
.55449 51326 31932 54887(0)	.22940 96422 93877 4876[illegible](-1)
.53131 97436 44375 62397(0)	.23406 77749 53140 06201(-1)
.50768 77575 33716 60215(0)	.23854 05210 60385 40080(-1)
.48361 80269 45841 02756(0)	.24282 16520 33365 99358(-1)
.45913 00119 89832 33287(0)	.24690 52474 44876 76909(-1)
.43424 37493 46802 55800(0)	.25078 56965 29497 68707(-1)
.40897 98212 29888 67241(0)	.25445 76996 54647 65813(-1)

continued

TABLE M14. GENERAL EXTENDED FORMULA OF DEGREE 191
Continued -

ABSCISSAE	WEIGHTS
.38335 93241 98730 34692(0)	.25791 62697 60242 29388(-1)
.35740 38378 31532 15238(0)	.26115 67337 67060 97680(-1)
.33113 53932 57976 83309(0)	.26417 47339 50582 59931(-1)
.30457 64415 56714 04334(0)	.26696 62292 74503 59906(-1)
.27774 98220 21824 31507(0)	.26952 74966 76330 31963(-1)
.25067 87303 03483 17661(0)	.27185 51322 96247 91819(-1)
.22338 66864 28966 88163(0)	.27394 60526 39814 32516(-1)
.19589 75027 11100 15392(0)	.27579 74956 64818 73035(-1)
.16823 52515 52207 46498(0)	.27740 70217 82796 81994(-1)
.14042 42331 52560 17459(0)	.27877 25147 66137 01609(-1)
.11248 89431 33186 62575(0)	.27989 21825 52381 59704(-1)
.84454 04008 37108 83710(-1)	.28076 45579 38172 46607(-1)
.56344 31304 65927 89972(-1)	.28138 84991 56271 50636(-1)
.28184 64894 97456 94339(-1)	.28176 31903 30166 02131(-1)
.00000 00000 00000 00000(0)	.28188 81418 01923 58694(-1)

ON SOME GAUSS AND LOBATTO BASED INTEGRATION FORMULAE

T. N. L. PATTERSON

See article in this issue for explanation of symbols in table.

Table 1. Lobatto based formulae.

Points $\pm x_i$	33 point weights	17 point weights	9 point weights	5 point weights
1.0	.14744 39062 23614 8234(-2)*	.49234 91033 20187 2454(-2)	.17783 47054 02406 5912(-1)	.69828 28769 65013 4788(-1)
.99409 01501 18423 1212	.10507 33497 23075 8424(-1)			
.97873 91331 91998 8895	.20086 78872 89725 5026(-1)	.39243 42612 08195 0222(-1)		
.95410 75374 60022 2128	.29165 31798 05123 9577(-1)			
.92042 91162 42990 7785	.38125 74379 87640 8717(-1)	.76530 96914 73047 2610(-1)	.14862 75777 29162 8906	
.87802 32353 24945 7100	.46625 58462 59996 0899(-1)			
.82729 19921 66940 4206	.54752 71035 04295 2529(-1)	.10929 80423 28228 0073		
.76871 64197 61627 5394	.62308 29987 95408 6359(-1)			
.70285 19289 17937 0194	.69314 79561 00290 2891(-1)	.13873 41487 88251 0442	.27887 52850 37865 5454	.53341 01230 39479 2226
.63032 30428 64267 4229	.75629 43786 68315 4870(-1)			
.55181 74759 01825 3800	.81256 86216 92831 6828(-1)	.16240 81994 65222 5713		
.46807 96126 82275 6826	.86087 22333 18844 9298(-1)			
.37990 34500 65463 9962	.90125 17887 28598 7215(-1)	.18031 61235 10764 7891	.35938 95132 26807 1560	
.28812 50685 25947 4757	.93206 72515 67080 2941(-1)			
.19361 47045 11110 1818	.95584 29447 52008 6836(-1)	.19108 97507 22245 3075		
.97268 49892 76217 033(-1)	.96955 83498 74074 2776(-1)			
.0	.97427 28215 20666 0017(-1)	.19491 16977 67924 3597	.39064 83069 31847 4977	.79352 31785 28038 8600

* (-r) indicates that the number should be multiplied by 10^{-r}.