Efficient Calculation of Clausen's Integral

By Van E. Wood

In evaluating certain singular integral equations, it is often necessary to compute values of the dilogarithm of complex argument [1], [2], which in turn requires the calculation of Clausen's integral [1], [3], [4],

$$\mathrm{Cl}\,(x) = -\int_0^x \ln\,(2\sin\,(t/2))dt\,,$$

where it is sufficient to consider the range $0 \leqq x \leqq \pi$. This function is somewhat difficult to compute from its series expansion or by numerical integration, but a rapidly convergent Chebyshev expansion is readily found. Integrating by parts, we see that

$$\mathrm{Cl}\,(x) = -x\ln\,(2\sin\,(x/2)) + \tfrac{1}{2}\pi^2\int_0^{x/\pi} z\cot\,(\pi z/2)dz\,.$$

Making use of the Chebyshev expansion [5] for $z\cot\,(\pi z/2)$ and of the properties of the Chebyshev polynomials, we obtain the approximation

$$\mathrm{Cl}\,(x) = -x\ln\,(2\sin\,(x/2)) + \tfrac{1}{2}\pi^2 N(x)\,, \tag{1}$$

where

$$N(\pi y) = y\sum_{r=0}{}' a_{2r}T_{2r}(y)\,, \qquad 0 \leqq y \leqq 1\,, \tag{2}$$

and the coefficients a_{2r} are given to 8 decimals in Table I. The prime on the summation indicates that the first term is to be divided by 2. For purposes of checking or extension of the table, we note that $\mathrm{Cl}\,(\pi) = 0$, hence $\sum' a_{2r} = (\ln 4)/\pi$. Also $\mathrm{Cl}\,(\pi/2) = G$ (Catalan's constant) $= 0.91596559\ldots$ The maximum of the function occurs at $x = \pi/3$; from (1) we obtain $\mathrm{Cl}\,(\pi/3) = 1.0149416\ldots$, correcting in the seventh decimal the value given by Lewin [1].

Table I

Numerical values of expansion coefficients occurring in Eq. 2

$2r$	a_{2r}
0	1.08360758
2	−.09753546
4	−. 285064
6	−. 13838
8	−. 763
10	−. 45
12	−. 3

Received February 26, 1968.

The coefficients a_{2r} may alternatively be obtained from the series

$$a_{2r} = \sum_{n=r}^{\infty} \frac{(-1)^n B_{2n} \pi^{2n-1}}{(2n+1)4^{n-1}(n-r)!(n+r)!},$$

where the B's are Bernoulli numbers. The series are easily shown to converge, and as a practical matter the convergence is fairly rapid; yet for most purposes it is easier to obtain the coefficients from the numerical values of those in the cotangent series.

The real part of the dilogarithm of complex argument is a function of two variables, and there is apparently no better way to calculate it than by numerical integration or, when the modulus is small, from the series expansion. Expressions for Chebyshev coefficients as series involving cosines of odd multiples of the argument generally converge slowly.

In the course of this work we noticed the relation (doubtless already known to many)

$$G = \frac{\pi}{4}\left(\ln 2 + \sum_{k=1}^{\infty} \frac{(4^k - 1)\zeta(2k)}{4^{2k-1}(2k+1)}\right).$$

This relation, which is similar to some given by Glasser [6], provides a fairly rapid method for calculating G to any desired accuracy, given sufficiently accurate values of the zeta functions involved.

Battelle Memorial Institute
Columbus Laboratories
Columbus, Ohio 43201

1. L. Lewin, *Dilogarithms and Associated Functions*, Macdonald, London, 1958. MR **21** #4264.

2. B. I. Lundqvist, "Single-particle spectrum of the degenerate electron gas. I," *Physik der kondensierten Materie*, v. 6, 1967, pp. 193–205.

3. A. A. Ashour & A. Sabri, "Tabulation of the function $\Psi(\theta) = \sum (\sin n\theta)/n^2$," *Math. Comp.*, v. 10, 1956, pp. 57–65. MR **18**, 339.

4. M. Abramowitz & I. A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Appl. Math. Series, No. 55, U. S. Government Printing Office, Washington, D. C., 1964, pp. 1005–1006. MR **29** #4914.

5. C. W. Clenshaw, "Chebyshev series for mathematical functions," in *National Physical Laboratory Mathematical Tables*, Vol. 5, HMSO, London, 1962. MR **26** #362.

6. M. L. Glasser, "Some integrals of the arctangent function," *Math. Comp.*, v. 22, 1968, pp. 445–447.