High Accuracy Gamma Function Values for Some Rational Arguments

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During the course of determining accurate Gaussian quadrature formulas for certain nonclassical weight functions, it was necessary to compute $\Gamma(p/q)$ to 60D to obtain moments to high accuracy. In particular, we needed $\Gamma(p/q)$ for $p = 1, 2, \cdots, q - 1,\ 2p \neq q$, and $q = 3, 4, 5, 8, 10$. The quantities $\Gamma(p/q)$, which for special cases also arise in the computation of Bessel functions of fractional order or of the Airy functions, were calculated using an interpretive routine that treated floating-point numbers of 70 significant figures.

TABLE 1

p	q	*Values of* $\ln \Gamma(p/q)$
1	3	0.98542 06469 27767 06918 71740 36977 96139 17355 56496 38588 58542 34757
2	3	0.30315 02751 47523 56867 58628 17372 01103 56634 93171 97830 62455 32199
1	4	1.28802 25246 98077 45737 06104 40219 71729 59253 77565 11286 05504 99987
3	4	0.20328 09514 31295 37148 14329 71862 42969 97596 67314 98257 86480 73977
1	5	1.52406 38224 30784 52488 10564 93926 30219 25659 33737 40640 34751 04287
2	5	0.79667 78177 01783 76654 47359 62391 62647 40394 48412 45829 74362 09729
3	5	0.39823 38580 69234 89961 68542 20400 87768 42343 54029 05730 96991 15903
4	5	0.15205 96783 99837 58877 82926 02290 57038 88430 53038 49486 41798 82363
1	8	2.01941 83575 53796 34532 02905 21167 09958 99482 80952 13444 96051 31965
3	8	0.86307 39822 70647 46240 50890 94134 01549 53324 70629 34842 71350 12142
5	8	0.36082 94954 88940 18118 49576 85822 77948 78573 69120 20625 81717 15344
7	8	0.08585 87072 25334 32350 23655 83769 48770 22697 19125 68187 11123 48816
1	10	2.25271 26517 34205 95986 97016 46368 49511 86156 27222 29495 37650 41740
3	10	1.09579 79948 18075 52167 71681 42370 10727 84451 48450 76420 34066 38624
7	10	0.26086 72465 31666 51438 57324 17016 75957 81424 62162 12570 28993 34661
9	10	0.06637 62397 34742 97118 87167 39867 10858 42423 52059 36627 35802 53581

As in Sherry and Fulda [1], use is made of the asymptotic expansion

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(1) $\ln \Gamma(p/q) \sim L(p/q, n) + \sum_{r=1}^{\infty} C_r/(n + p/q)^{2r-1}$ (n a positive integer) ,

where $C_r = B_{2r}/2r(2r - 1)$, B_{2r} are Bernoulli numbers, and

$$L(p/q, n) = (n - 1/2 + p/q) \ln (nq + p) - (n + p/q) + 1/2 \ln (2\pi) + (1/2 - p/q) \ln q - \sum_{j=0}^{n-1} \ln (p + jq) . \quad (2)$$

The function $\Gamma(x)$ is then calculated from

$$\Gamma(x) = \exp (0.1m) \exp [\ln \Gamma(x) - 0.1m] , \quad (3)$$

where m is taken to be the greatest integer in $10 \ln \Gamma(x)$. Mansell's tables [2] provided the values of $(1/2) \ln (2\pi)$ and logarithms of integers; the numbers B_k are found in Davis [3].

With the above procedure, we obtained Tables 1 and 2 giving values of $\ln \Gamma(p/q)$ and $\Gamma(p/q)$ to 60 decimal places.

TABLE 2

p	q	*Values of* $\Gamma(p/q)$
1	3	2.67893 85347 07747 63365 56929 40974 67764 41286 89377 95730 11009 50428
2	3	1.35411 79394 26400 41694 52880 28154 51378 55193 27266 05679 36983 94022
1	4	3.62560 99082 21908 31193 06851 55867 67200 29951 67682 88006 54674 33378
3	4	1.22541 67024 65177 64512 90983 03362 89052 68512 39248 10807 06112 30119
1	5	4.59084 37119 98803 05320 47582 75929 15200 34341 09998 29340 30177 88853
2	5	2.21815 95437 57688 22305 90540 21907 67945 07705 66501 77146 95822 41978
3	5	1.48919 22488 12817 10239 43333 88321 34228 13205 99038 75992 47353 38680
4	5	1.16422 97137 25303 37363 63209 38268 45869 31419 61768 89118 77529 84894
1	8	7.53394 15987 97611 90469 92298 41215 13362 46104 19588 14907 59409 83128
3	8	2.37043 61844 16600 90864 64735 04176 65250 98874 00803 35892 49877 75127
5	8	1.43451 88480 90556 77563 60197 39456 42313 66322 07772 20666 73307 70680
7	8	1.08965 23574 22896 95125 23767 55102 89297 11478 70067 76756 51205 13704
1	10	9.51350 76986 68731 83629 24871 77265 40219 25505 78626 08837 73430 50001
3	10	2.99156 89876 87590 62831 25165 15904 91779 11128 06024 92171 51127 44120
7	10	1.29805 53326 47557 78568 11711 79152 81161 77841 41170 55394 62479 21645
9	10	1.06862 87021 19319 35489 73053 35694 48077 81698 38785 06097 31790 49371

The calculated values were checked by means of the identity

$$\Gamma(x)\Gamma(1 - x) = \pi/\sin (\pi x) . \quad (4)$$

In all cases, the two sides agreed to 64 decimal places. Also, the values of $\Gamma(1/3)$, $\Gamma(2/3)$, $\ln\,\Gamma(1/3)$, and $\ln\,\Gamma(2/3)$ agree with the 35D calculations of Sherry and Fulda [1].

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1. M. E. Sherry & S. Fulda, "Calculation of gamma functions to high accuracy," *Math. Comp.*, v. 13, 1959, pp. 314–315. MR **21** #7603.
2. W. E. Mansell, *Tables of Natural and Common Logarithms to* 110 *Decimals*, Roy. Soc. Math. Tables, v. 8, Cambridge Univ. Press, New York, 1964. MR **29** #3675.
3. H. T. Davis, *Tables of the Higher Mathematical Functions*. Vol. II, Principia Press of Trinity University, San Antonio, Texas, 1935; rev. ed., 1963.