Error Bounds for the Gauss-Chebyshev Quadrature Formula of the Closed Type

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1. Introduction. We are concerned with the Gauss-Chebyshev quadrature formula of the closed type,

$$(1)\qquad \int_{-1}^{1} (1 - t^2)^{-1/2} f(t)dt = \sum_{k=0}^{n} A_k f(t_k) + E_n(f) \qquad (n \geqq 2)$$

with the abscissas

$$t_k = \cos(k\pi/n), \qquad k = 0, \cdots, n,$$

and the Christoffel numbers

$$A_0 = A_n = \pi/2n, \qquad A_k = \pi/n, \qquad k = 1, \cdots, n-1.$$

The quadrature formula (1) is exact for all polynomials of degree $\leqq 2n - 1$. For a general discussion of the Gauss formulas of the closed type, see Krylov [1, Chapter 9].

The usual real-variable theory estimate for the error $E_n(f)$ is given (Krylov [1, p. 171]) in terms of derivatives of f:

$$(2)\qquad E_n(f) = -\frac{\pi}{2^{2n-1}} \frac{f^{(2n)}(\eta)}{(2n)!}$$

for some $\eta \in [-1, 1]$. The error expression (2) is valid for the class of functions which are $2n$-times differentiable. In most cases, the exact value of η will be unknown, and the estimate $\max_{-1 \leq t \leq 1} |f^{(2n)}(t)|$ is used. But in many cases it will be far from convenient to obtain $f^{(2n)}$ or the bounds on it.

In the following, we use the complex-variable method to obtain a contour integral representation for $E_n(f)$, applied to analytic functions, and give bounds for the error in terms of the size of the integrand in the complex plane.

2. Error Bounds. Let $f(t)$ be analytic on $[-1, 1]$, then it can be continued analytically so as to be single-valued and analytic in a domain D of the z-plane containing the interval $[-1, 1]$ in its interior.

Let C be a closed contour in D enclosing the interval $[-1, 1]$ in its interior and let $U_{n-1} = 2^{n-1} \prod_{k=1}^{n-1} (t - t_k)$ be the Chebyshev polynomial of the second kind. On applying the residue theorem to the contour integral

$$(3)\qquad \frac{1}{2\pi i} \int_C \frac{f(z)dz}{w(z)}, \qquad w(t) = (t^2 - 1)U_{n-1}(t),$$

we get

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$$f(t) = \sum_{k=0}^{n} \frac{w(t)}{(t - t_k)w'(t_k)} f(t_k) + \frac{1}{2\pi i} \int_C \frac{w(t)f(z)}{w(z)} dz . \tag{4}$$

Multiplying (4) by the weight $(1 - t^2)^{-1/2}$ and integrating on $[-1, 1]$, there results the quadrature formula (1), with the error

$$E_n(f) = \frac{i}{\pi} \int_C \frac{Q_{n-1}^*(z) f(z) dz}{U_{n-1}(z)(z^2 - 1)} \tag{5}$$

where we have put

$$Q_n^*(z) = \frac{1}{2} \int_{-1}^{1} (1 - t^2)^{1/2} \frac{U_n(t)}{z - t} dt . \tag{6}$$

In a recent paper (Chawla [2]), the following result was proved. For sufficiently large $|z|$,

$$\left| \frac{Q_n^*(z)}{U_n(z)} \right| \leqq \frac{\pi}{2^{2n+2}} |z|^{-2n-1} . \tag{7}$$

Taking C: $|z| = R$ with sufficiently large R, from (5) and (7), we find

$$|E_n(f)| \leqq \frac{\pi}{2^{2n-1}} \frac{R^2 M(R)}{(R^2 - 1)R^{2n}} , \tag{8}$$

where $M(R) = \max_{|z|=R} |f(z)|$.

These error bounds are simple to obtain and they will not be unduly pessimistic, but are valid for the class of functions which are continuable analytically in a sufficiently large domain of the z-plane containing the range of integration $[-1, 1]$.

We obtain next estimates for $E_n(f)$ for all functions analytic on $[-1, 1]$. For this purpose, we introduce the ellipse $\mathcal{E}_\rho$ $(\rho > 1)$ defined by

$$z = \tfrac{1}{2}(\xi + \xi^{-1}) , \qquad \xi = \rho e^{i\theta} \quad (0 \leqq \theta \leqq 2\pi) \tag{9}$$

with foci at $z = \pm 1$ and semiaxes $\frac{1}{2}(\rho + \rho^{-1})$ and $\frac{1}{2}(\rho - \rho^{-1})$.

Let $f(t)$ be analytic on $[-1, 1]$. Then, for some $\rho > 1$, f can be continued analytically into the closure of an ellipse $\mathcal{E}_\rho$. It has been proved (Chawla [2]) that for z on $\mathcal{E}_\rho$,

$$Q_n^*(z) = (\pi/2)\xi^{-n-1} . \tag{10}$$

Since on $\mathcal{E}_\rho$,

$$U_n(z) = (\xi^{n+1} - \xi^{-n-1})/(\xi - \xi^{-1}) \tag{11}$$

and by virtue of (10), (5) becomes

$$E_n(f) = i \int_{|\xi|=\rho} \frac{f[\frac{1}{2}(\xi + \xi^{-1})] d\xi}{\xi(\xi^{2n} - 1)} \tag{12}$$

or, equivalently,

$$E_n(f) = i \int_{\mathcal{E}_\rho} \frac{f(z) dz}{(z^2 - 1)^{1/2}[(z \pm (z^2 - 1)^{1/2})^{2n} - 1]} , \tag{13}$$

where the sign in the integrand is chosen so that $|z \pm (z^2 - 1)^{1/2}| > 1$. From (12) follows the following estimate for the error:

$$|E_n(f)| \leqq 2\pi M(\rho)/(\rho^{2n} - 1) \tag{14}$$

where $M(\rho) = \max |f|$ on $|\xi| = \rho$.

By experimenting with various "admissible" ρ, a conservative upper bound can be established. A similar remark applies to the estimate (8).

3. Example. Consider the estimation of error in the evaluation of the integral

$$J = \int_{-1}^{1} (1 - t^2)^{-1/2} \frac{at}{4 + t} = \frac{\pi}{(15)^{1/2}} \doteq 0.811155735192$$

by the quadrature formula (1). For $n = 4$, the approximate value found by the quadrature formula is $\doteq 0.811155845096$. Thus, the true error $E_4 \doteq -0.0000001099$.

In this case, the real-variable estimate (2) gives $|E_4| \leqq 0.0000012$.

Taking $R = 3.5$, the estimate (8) gives $|E_4| \leqq 0.0000023$.

However, evaluating the contour integral in (13), we find

$$E_4 = -\frac{2\pi}{(15)^{1/2}[(4 + (15)^{1/2})^8 - 1]} \doteq -0.0000001099$$

which is the exact error.

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1. V. I. Krylov, *Approximate Calculation of Integrals*, Macmillan, New York, 1962. MR 26 #2008.

2. M. M. Chawla, "On the Chebyshev polynomials of the second kind," *SIAM Rev.*, v. 9, 1967, p. 729.