Uniform Asymptotic Solution of Second Order Linear Differential Equations without Turning Varieties

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Abstract. The purpose of this paper is to initiate the study of a new kind of asymptotic series expansion for solutions of differential equations containing a parameter. We obtain uniform asymptotic solutions for certain equations of the form

$$\epsilon^{2n}y'' = a(t, \epsilon)y , \qquad (\ \ )' = d/dt ,$$

where n is a positive integer, t and ϵ are real variables ranging over $|t| \leqq t_0$, $0 < \epsilon \leqq \epsilon_0$, and a is a function infinitely differentiable on the closure of this domain. We require that $a(t, \epsilon)$ satisfy conditions which can be regarded as generalized non-turning-point conditions. These conditions imply the absence of secondary turning points, and reduce in the simplest case to the condition $a(t, 0) \neq 0$, but also include cases (the interesting ones) in which $a(0, 0) = 0$. ∎

1. Introduction. The purpose of this paper is to initiate the study of a new kind of asymptotic series expansion for solutions of differential equations containing a parameter. We shall obtain uniform asymptotic solutions for certain equations of the form

$$\epsilon^{2n}y'' = a(t, \epsilon)y , \qquad (\ \ )' = d/dt , \tag{1-1}$$

where n is a positive integer, t and ϵ are real variables ranging over $|t| \leqq t_0$, $0 < \epsilon \leqq \epsilon_0$, and a is a function infinitely differentiable on the closure of this domain. We shall thus present a purely real variable theory.

The asymptotic theory of (1-1) (as ϵ approaches 0) has been divided into two cases, depending on whether or not $a(t, 0)$ has isolated zeros. If $a(t, 0)$ does not vanish then (1-1) falls within the scope of a systematic theory ([1, Chapter 6]; [2]). However, if $a(t, 0)$ has isolated zeros, individual representatives of (1-1) become highly idiosyncratic, and there exists a profound literature devoted to what can be fairly called the investigation of special cases (see the bibliography of [3]). Such problems are called "turning point" or "transition point" problems and the zeros of $a(t, 0)$ are called "turning" or "transition" points.

We shall solve a class of problems for which the above classification is inadequate. We shall require that (1-1) satisfy conditions (given in Section 2) which can be regarded as generalized non-turning-point conditions. These conditions imply the absence of what have come to be known informally as secondary turning points, and reduce in the simplest case to the condition $a(t, 0) \neq 0$, but also include cases (the interesting ones) in which $a(0, 0) = 0$. Our conditions have their basis in a formal analysis which has been carried out for general linear equations by Iwano and Sibuya [4]. They determine a finite set of differential equations which arise

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from (1-1) by transformations of the independent variable depending on the parameter ϵ. Roughly put, each transformed equation describes (1-1) in a different asymptotic scale in ϵ. The values of t corresponding to turning points of these problems, with some possible exceptions, are the secondary turning points. Since these values of t in general depend on ϵ, the secondary turning points typically lie on a curve in (t, ϵ) space, that is they lie on a subvariety of (t, ϵ) space (which in our case is usually an algebraic subvariety). We do not wish to perpetuate the distinction between turning point and secondary turning point since the point of [6] and of this paper is that the secondary turning points, properly defined, are in fact the "true" turning points of (1-1). Ideally the best terminology would simply use the term "turning point" in an extended sense. However, it does not seem possible to do this without confusion (e.g. change "varieties" to "points" in our title and we refer to a problem solved by Liouville). We have therefore selected the term *turning variety*. We are also reinforced in this choice by the fact that in applying the ideas of this paper to problems containing several parameters, sets arise which in standard mathematical usage are described by no term other than "variety."

We can now describe our main result, Theorem 2 below, as *uniform asymptotic solution of* (1-1) *in the absence of turning varieties.*

The following formal scheme can be regarded as a point of departure for our analysis.

The transformation $\epsilon^n y^{-1} y' = r$ leads to the Ricatti equation

$$\epsilon^n r' + r^2 - a = 0\,. \tag{1-2}$$

The equation $\sigma r' + r^2 - a(t, \epsilon) = 0$ has formal σ-power series solutions

$$\sum_{k=0}^{\infty} \sigma^k R_k$$

where R_0 is a root, call it λ, of

$$R_0^2 - a = 0 \tag{$(1\text{-}2)_0$}$$

and

$$R_{k+1} = -\frac{1}{2\lambda}\left\{\frac{dR_k}{dt} + \sum_{i+j=k+1;\ i,j>0} R_i R_j\right\}. \tag{$(1\text{-}2)_k$}$$

Replacing σ by ϵ^n we obtain the formal expressions

$$r = \sum_{k=0}^{\infty} \epsilon^{nk} R_k(t, \epsilon), \tag{1-3}$$

$$y = \exp\left\{\epsilon^{-n} \sum_{k=0}^{\infty} \epsilon^{nk} \int_{\tau}^{t} R_k(s, \epsilon)\,ds\right\}. \tag{1-4}$$

There is a considerable arbitrariness in this procedure and the terms of the resulting formal series have no simple distinguishing algebraic features purely as functions of (t, ϵ). In fact there are many slight variants of this scheme which would be equally satisfactory for our purposes. We mention the following example solely for purposes of illustration. Let $a(t, \epsilon) = \tilde{a}(t, \epsilon) + \epsilon^{2n} b(t, \epsilon)$ where $\tilde{a}$ is, say, the $(2n-1)$th partial sum of the ϵ-Taylor series of a. Then the formal σ-power series solution of

$$\sigma r' + r^2 - \tilde{a} - \sigma^2 b = 0 \tag{1-5}$$

with σ replaced by ϵ^n could be used instead of (1-3).

We will expend our main effort in making a detailed formal analysis of formulas (1-3) and (1-4), accounting in a general way for the nonuniqueness illustrated above (Section 6), establishing precise formal properties (Theorem 1, Section 9) and elucidating their asymptotic character (Theorem 2, Section 10). To accomplish this we require new definitions of formal and asymptotic series (Section 6) using notions from the local theory of functions of several variables which are standard but which do not seem to have been applied before to asymptotics.

Finally, to emphasize that we are dealing with a natural and effective means for solving certain kinds of problems, we have attempted to include an informal sketch of our analysis in a continued section consisting of specific examples which also illustrate our hypotheses and results. In particular we analyse the scattering-like problem

$$\epsilon^2 y'' + (\phi(t) + \epsilon)y = 0 \tag{1-6}$$

where ϕ is a nonnegative, infinitely differentiable function which has a single zero of even order greater than two (say 6) at $t = 0$ and is constant for $|t| \geqq 1$.

Some of our results appear in Stengle [5]. We also note that Evgrafov and Fedoryuk [6] study the leading term of our expansion in a function-theoretic context. The author is indebted to Professor N. Kazarinoff for calling his attention to problem (1-6).

2. Location of Turning Varieties. Basic Hypotheses. We first state some preliminary restrictions which we shall either take for granted without further mention or which happen to be consequences of hypotheses stated more formally below. A necessary condition which we must impose is that $a(t, \epsilon)$ does not vanish for $\epsilon \neq 0$. This requirement has been used as a condition precluding turning points by Evgrafov and Fedoryuk [6] in a context where it is viable. However, this condition is not sufficiently discerning for our purposes and must be regarded as a necessary, but by no means sufficient, condition. We are of course interested in the case where $a(t, 0)$ does have zeros and we suppose that this function has only one zero at $t = 0$. We thus have assumed that $a(t, 0)$ and $a(0, \epsilon)$ both have an isolated zero at $(0, 0)$. We also suppose that these zeros are of finite order. These restrictions already clearly delimit the scope of our results. For example the famous problem $\epsilon^2 y'' = a(t)y$, where $a(0) = 0$, is immediately excluded.

We now analyse $a(t, \epsilon)$ in terms of its formal power series at $t = \epsilon = 0$.

Notation. Given an infinitely differentiable function $\phi(t, \epsilon)$, let $\hat{\phi}$ denote the formal power series of ϕ at $t = \epsilon = 0$. We also use circumflexed quantities to denote formal power series not necessarily associated with a given function or the product of such a formal power series and an infinitely differentiable function.

Since $a(t, 0)$ has a zero of finite order, say m_0, at $t = 0$, it follows from the Weierstrass preparation theorem for formal power series [7] that $\hat{a}$ has the unique factorization

$$\hat{a} = \left(t^{m_0} + \sum_{m=0}^{m_0-1} t^m \epsilon^{k(m)} \hat{p}_m(\epsilon)\right)\hat{U}_0 = \hat{P}_0 \hat{U}_0 \tag{2-1}$$

where $\hat{p}_m(\epsilon)$ is identically zero or else a unit in the ring of formal power series, $k(m)$ is a positive integer defined only if $\hat{p}_m(\epsilon)$ is a unit, and $\hat{U}_0$ is a unit.

We now apply the characteristic polygon of Iwano and Sibuya. We obtain the characteristic polygon $\mathcal{C}$ of (1-1) in the (k, m) plane by forming the convex hull of the vertical ray, $S_0 = \{(0, m) | m \geqq m_0\}$, the points $(k(m), m)$ from (2-1) and the point $(2n, -2)$. The boundary of this hull is the characteristic polygon, and evidently the points $(0, m_0)$ and $(2n, -2)$ are vertices. Since $a(0, \epsilon)$ has a zero of finite order, $\hat{p}_0(\epsilon) \neq 0$ and the point $(k(0), 0)$ appears in the hull. As our first crucial technical hypothesis we assume: *$(k(0), 0)$ is a vertex of $\mathcal{C}$.* We denote the sides of $\mathcal{C}$ between $(0, m_0)$ and $(k(0), 0)$ from left to right by $S_1, S_2, \cdots, S_p$. Let S_j be described by the equation

$$(2\text{-}2)_j \qquad k + \delta_j m = \gamma_j$$

where δ_j and γ_j are positive rationals with least common denominator n_j, and let the lower right-hand vertex of S_j be (k_j, m_j). It can then be seen that the change of variables

$$(2\text{-}3)_j \qquad \begin{aligned} t &= \epsilon^{\delta_j} s\,, \\ y(t, \epsilon) &= w(s, \epsilon)\,, \end{aligned} \qquad j = 1, 2, \cdots, p\,,$$

transforms (1-1) into

$$(2\text{-}4)_j \qquad \epsilon^{2n-2\delta_j-\gamma_j} \frac{d^2 w}{ds^2} = [s^{m_j}\alpha_j(s) + \epsilon^{1/n_j}\beta_j(s, \epsilon^{1/n_j})]w$$

where α_j is the polynomial

$$(2\text{-}5)_j \qquad \alpha_j(s) = \hat{U}_0(0, 0) \sum_{(k(m), m) \in S_j} \hat{p}_k(0) s^{m-m_j}$$

and $\beta_j(s, u)$ is an infinitely differentiable function. The original equation (1-1) can be designated as $(2\text{-}4)_0$ if we let $\delta_0 = 0$ and append

$$(2\text{-}5)_0 \qquad \alpha_0(s) = s^{-m_0} a(s, 0)\,.$$

Briefly summarized, the sides S_j determine the essential ways in which a leading part, $s^{m_j}\alpha_j(s)$, can be brought out of the transformed coefficient.

We can now state the following

Conditions Precluding a Turning Variety.

1. $(k(0), 0)$ is a vertex of $\mathcal{C}$.

2. $\alpha_0(s) \neq 0$, $-t_0 \leqq s \leqq t_0$.

3_j. $\alpha_j(s) \neq 0$, $-\infty < s < \infty$, $j = 1, 2, \cdots, p$.

Condition 1 has the following significance. Since $\mathcal{C}$ is concave upwards, the vertex $(2n, -2)$ must lie strictly above the line $k + \delta_p m - \gamma_p = 0$ which contains S_p. Hence $2n - 2\delta_p - \gamma_p > 0$. This implies that $(2\text{-}4)_p$, and hence each problem $(2\text{-}4)_j$, $j = 1, 2, \cdots, p$, depends on ϵ in a singular way. In other words we must require that each of the related problems has the singular kind of dependence on the parameter which disposes toward an asymptotic theory. Condition 2 merely restates that $a(t, 0)$ has no zeros other than $t = 0$. Conditions 3_j, $j = 1, \cdots p - 1$ assert that $(2\text{-}4)_j$ has no turning points other than $s = 0$. Condition 3_p refers to $(2\text{-}4)_p$ which has the special form

$$\epsilon^{2n-2\delta_p-\gamma_p} d^2 w/ds^2 = [\alpha_p + \epsilon^{1/n_p}\beta_p]w$$

since $m_p = 0$. Since $0 = \delta_0 < \delta_1 \cdots < \delta_p$, the transformation $t = s\epsilon^{\delta_p}$ describes (1-1) in the smallest or "innermost" asymptotic scale. Thus condition 3_p asserts that this innermost problem has no turning points.

The absence of turning varieties will insure the formal validity of our results in the precise sense of Theorem 1 below. In addition we require a more analytical condition of a familiar kind which will imply that we can determine solutions of (1-1) which do not undergo a transition from violent growth to violent decay. In the absence of such a condition we would be forced to subdivide $[-t_0, t_0]$ into contiguous closed subintervals. Since $a(t, \epsilon)$ does not vanish and the underlying (t, ϵ) domain is contractable, a continuous choice of arg $a(t, \epsilon)$ is uniquely determined by its value at a single point. We refer to such a choice in the following.

Nontransition Condition. There is a continuous determination of arg $a(t, \epsilon)$ satisfying

$$|\arg a(t, \epsilon)| \leqq \pi .$$

We could use other inequivalent conditions instead: for example, $|\arg \tilde{a}(t, \epsilon)| \leqq \pi$ where $\tilde{a}$ is the "leading part" of a used in Eq. (1-5). Also in some cases a shorter section of the ϵ-power series of a could be used.

3. Examples.

3.1. We consider (1-6) assuming that $\phi(t)$ has a zero of order 6 at $t = 0$. The vertices of the characteristic polygon are (0, 6), (1, 0) and (2, −2) and there is only one relevant change of scale, $t = \epsilon^{1/6} s$, determined by the line joining (0, 6) and (1, 0). This leads to the transformed problem

$$\epsilon^{2/3} d^2w/ds^2 + (\alpha s^6 + 1 + \epsilon^{1/6}\beta(s, \epsilon^{1/6}))w = 0 \tag{3-1}$$

where $\alpha = \phi^{(6)}(0)/6! > 0$. This problem has no turning points. Accordingly the original problem, which has a turning point at $t = 0$, has no turning varieties. Clearly the nontransition condition is also fulfilled.

3.2. The problem

$$\epsilon^5 y'' = (it^3 + \epsilon t + i\epsilon^2)y$$

determines two changes of scale, $t = \epsilon^{1/2}s$, $t = \epsilon s$, and corresponding equations

$$\epsilon^{5/2} d^2w/ds^2 = (s[1 + is^2] + i\epsilon^{1/2})w$$

$$\epsilon d^2w/ds^2 = (i + s + i\epsilon s^3)w .$$

Here $\alpha_0 = i$, $\alpha_1(s) = 1 + is^2$, $\alpha_2(s) = i + s$, and the problem has no turning varieties. However, the nontransition condition fails since the range of a is a curve in the complex plane crossing the negative real axis at $a = -\epsilon^{5/3}$ when $t = -\epsilon^{2/3}$. This is not serious since we can obtain separate results on two contiguous domains of the form $-t_0 \leqq t \leqq -\epsilon^{2/3}$, $0 < \epsilon \leqq \epsilon_1$ and $-\epsilon^{2/3} \leqq t \leqq t_0$, $0 < \epsilon \leqq \epsilon_1$.

(In considering the extension of this problem to complex values of t and ϵ, it is clear that the complex algebraic curves determined by $it^2 + \epsilon = 0$ and $t + i\epsilon = 0$ are the turning varieties. However, extensions of the nontransition condition must account for geometric phenomena of great complexity.)

3.3. For each of the problems $\epsilon^2 y'' + (t^2 + \epsilon^2)y = 0$ and $\epsilon^2 y'' + t(t - \epsilon^2)y = 0$, we obtain a polygon with only one numbered side S_0, to which corresponds the leading part $t^{m_0}\alpha_0(t) = t^2$. Thus $t = 0$ is a turning variety of each problem. In these examples the varieties $t + i\epsilon = 0$, $t - i\epsilon = 0$, $t = 0$, and $t - \epsilon^2 = 0$, which account for the vanishing of the coefficient functions, are too close in the relevant scales to justify distinguishing among them. It is also natural to describe $t = 0$ as a *double* turning variety for each problem.

3.4. The problem $\epsilon^2 y'' = (t^3\psi(t) + \epsilon)y$, $\psi(0) = 1$ has a turning variety $t = -\epsilon^{1/3}$. Thus our results taken in total do not apply. However, it is not hard to extract partial results which elude the classical methods. For example our formulas give uniform asymptotic solutions on $[0, t_0]$ (more generally on $[-\epsilon^{1/3} + C\epsilon^{4/9-\delta}, t_0]$ where $\delta > 0$, although we shall not prove these results). This is significant because this one-sided failure of our hypotheses (a turning variety intersecting only the negative real axis) produces a two-sided failure of the classical asymptotic expansions. Finally it appears that a single uniform theory of solutions on $[-t_0, t_0]$ can be obtained by combining our methods with the comparison methods of R. E. Langer.

4. The Connection Problem for the Classical Expansions. Our hypotheses bring problem $(2\text{-}4)_j$ within the scope of the standard theory if s is restricted to a finite closed interval not containing $s = 0$ for $j = 0, \cdots, p - 1$ or to any finite closed interval if $j = p$. It is then the case that on each such interval, Eq. $(2\text{-}4)_j$ has a fundamental pair of solutions $w_j^{(i)}(s, \epsilon)$, $i = 1, 2$, which possess simple asymptotic representations $\hat{w}_j^{i}(s, \epsilon)$. However, these results do not reveal to what extent the formal expressions $\hat{w}_j^{(i)}(t\epsilon^{-\delta_j}, \epsilon)$ will describe the limiting behavior of the solutions, $w_j^{(i)}(t\epsilon^{-\delta_j}, \epsilon)$, of (1-1) obtained by reversing the transformation $(2\text{-}3)_j$. Moreover the solution pairs $w_j^{(i)}(t\epsilon^{-\delta_j}, \epsilon)$ are, in general, different solution pairs of (1-1). Among these pairs must persist linear relations depending on ϵ. It is the case that the formal expressions $\hat{w}_j^{(i)}$ are of limited usefulness unless we have an asymptotic description of these relations.

We will determine the full range of validity of the classical expansions and the linear relations or connection formulas relating them by the simple procedure of breaking down our uniform asymptotic solution into classical forms on suitable subdomains.

5. Examples (Continued).

5.1. Formal methods given in [1] lead to formal solutions of (1-6) having the form

$$\varphi^{-1/4}(t) \exp\left[\pm \frac{i}{\epsilon}\int_\tau^t \phi^{1/2}(s)ds \pm i\int_\tau^t \phi^{-1/2}(s)ds\right]\{1 + \cdots\} \tag{5-1}$$

where the brackets enclose a formal power series in ϵ with coefficients which are infinitely differentiable functions of t. Evidently these formulas are meaningless at $t = 0$, but they are known to represent solutions of (1-6) on any closed interval not containing $t = 0$.

Similarly the transformed problem (3-1) has formal solutions

$$(\alpha s^6 + 1)^{-1/4} \exp\left[\pm \frac{i}{\epsilon^{1/3}}\int_0^s (\alpha\sigma^6 + 1)^{1/2}d\sigma \pm \frac{i}{\epsilon^{1/6}}\int_0^s \frac{\beta(\sigma, 0)}{(\alpha\sigma^6 + 1)^{1/2}}d\sigma + \cdots\right] \tag{5-2}$$

which represent solutions of (3-1) on any closed s-interval. It is not hard to see that the formulas (5-2) cannot be meaningful if we reverse the transformation $t = \epsilon^{1/6}s$ since, for example, if $\phi = t^6$ near $t = 0$, then $\hat{\beta}(s, \epsilon) = 0$, and (5-2) does not even depend on the function ϕ.

We are thus concerned with three solution pairs: a pair represented by (5-1) on some negative t interval (pick $\tau = -t_0$ in (5-1)), a pair represented by (5-2) on a finite s interval, and a pair represented by (5-1) on some positive t-interval ($\tau = t_0$). But for small ϵ the three corresponding t-domains do not overlap. For this reason the results we have described are not sufficient to determine asymptotically the linear relations connecting the three solution pairs. A simpler problem of a similar nature is the scattering problem: to find the relations expressing the solution pair of (1-6) which has complex exponential form for large positive t in terms of the pair which has the same form for large negative t.

5.2 *An algebraic connection problem.* We consider the problem of finding asymptotic formulas for the roots of $\lambda^2 + \phi(t) + \epsilon = 0$ where ϕ, t, ϵ are as in problem (1-6). Let $\phi(t) = t^6\psi^2(t)$ where $\psi > 0$. Then the global continuous roots of the previous equation have asymptotic representations

$$\begin{aligned} \lambda_\pm &\sim \pm it^3\psi \pm (i\epsilon/2)t^{-3}\psi^{-1} + \cdots \qquad 0 < t_1 \leqq t \leqq t_0\,, \\ &\sim \pm i\epsilon^{1/2}(\alpha s^6 + 1)^{1/2} + \cdots \qquad t = \epsilon^{1/6}s,\ |s| \leqq s_0\,, \\ &\sim \mp it^3\psi \mp (i\epsilon/2)t^{-3}\psi^{-1} + \cdots \qquad -t_0 \leqq t \leqq -t_1 < 0\,. \end{aligned}$$

It is perhaps appropriate to regard the choice of sign here as a little connection problem since the series can be computed by formal procedures which provide no clue as to the correct choice. Also the series have formal irregularities resembling those in series (5-1) and (5-2). Finally the roots undergo a *Stokes' phenomenon* or change in the analytic form of their asymptotic representations. As we will see, these irregularities will partly account for more recondite irregularities in the asymptotic solutions of (1-6).

6. General Formal and Asymptotic Series. In this section our object is to give convenient definitions of formal and asymptotic series depending on a single small parameter, ϵ. We first describe a bare context in which we can place the notion of uniform asymptotic expansion. Let X be a quite general set and let Ω be a subset of $X \times (0, \epsilon_0]$ on which the function $1/\epsilon$ is unbounded. Let $B'(\Omega)$ denote the ring of complex valued bounded functions on Ω. Then $\epsilon B'(\Omega)$ is a proper ideal in $B'(\Omega)$ and can be used in the following way to induce a notion of formal convergence in the ring of complex valued functions on Ω.

Definition 1.′ A sequence $\{f_k\}$, $k = 1, 2, \cdots$, of complex-valued functions on Ω is formally convergent to zero if for any positive integer N there is a $k(N)$ such that $f_k \in \epsilon^N B'(\Omega)$ for all $k \geqq k(N)$.

We note that if f_k converges formally to zero then all but a finite number of the f_k belong to $B'(\Omega)$. In the next definition we put a restriction on these possibly unbounded terms.

Definition 2′. A series of complex-valued functions on Ω, $\sum_{k=0}^\infty f_k$, is formally convergent if for some N and all k, $f_k \in \epsilon^{-N}B'(\Omega)$, and if the sequence f_k converges formally to zero.

We define the sum and product of formally convergent series by $\sum_{k=0}^\infty f_k + \sum_{k=0}^\infty g_k = \sum_{k=0}^\infty (f_k + g_k)$ and $(\sum_{k=0}^\infty f_k)(\sum_{k=0}^\infty g_k) = \sum_{k=0}^\infty(\sum_{j=0}^k f_j g_{k-j})$. Since these operations lead to series which are again formally convergent, the collection of formally convergent series is a ring $\mathcal{B}'(\Omega)$. However, this ring is too large to be of much use (e.g. the series $f + 0 + 0 + \cdots$ and $0 + f + 0 + \cdots$ are different elements of $\mathcal{B}'(\Omega)$). It is not hard to verify that the subcollection $\mathcal{B}_0'(\Omega)$ of series formally convergent to zero, that is, series for which the sequence of partial sums converges formally to zero, forms an ideal in $\mathcal{B}'(\Omega)$. We can thus form the quotient ring $\mathcal{B}'/\mathcal{B}_0'$ or equivalently introduce the following terminology.

Definition 3′. The symbol $\doteq$ or the term "formal equality" denotes equality in $\mathcal{B}'(\Omega)$ modulo $\mathcal{B}_0'(\Omega)$.

The preceding definitions are partly analytic in nature and do not have the purely algebraic character, for example, of the theory of formal power series. They will be applied to such series as the right-hand side of (1-3), the terms of which are not distinguished by any simple algebraic properties, and which itself has no significant uniqueness properties.

We next give a related definition of uniform (more precisely, uniform in x) asymptotic expansion.

Definition 4′. The series $\sum_{k=1}^{\infty} f_k$ is a uniform asymptotic expansion of f if for some N, $f_k \in \epsilon^{-N}B'(\Omega)$ and the sequence $f - \sum_{j=1}^{k} f_j$ converges formally to zero. We indicate this relationship by writing

$$f \sim \sum_{k=1}^{\infty} f_k .$$

Remarks. Evidently a uniform asymptotic expansion is a formally convergent series. If we identify f with the trivial formal series $f + 0 + \cdots$ then $f \sim \sum f_k$ is equivalent to $f \doteq \sum f_k$. Also if $f \sim \sum f_k$ and $\sum f_k \doteq \sum g_k$ then $f \sim \sum g_k$ so that asymptotic expansions are well defined modulo $\mathcal{B}_0'$. They are also unique modulo $\mathcal{B}_0'$, that is, as elements of $\mathcal{B}'/\mathcal{B}_0'$, but since elements of $\mathcal{B}_0'$ abound, asymptotic expansions are not unique in any more refined sense.

The above notion of asymptotic expansion is determined (in Definition 1′) by the nested sequence of ideals, $\epsilon^k B'(\Omega)$, $k = 1, 2, \cdots$. It is possible to define more general kinds of asymptotic character by introducing more general sequences of ideals but the preceding notions seem to include very many cases of interest. Indeed our most delicate results involve other sequences of ideals but since these depend on the individual features of (1-1) it is most natural to let these ideals appear more explicitly in our results.

Using the formal convergence described above we can consider formal solutions in $\mathcal{B}'(\Omega)$ of an equation $\pi(x, \epsilon, y) \doteq 0$ if π is a polynomial in y with coefficients in $B'(\Omega)$ or $\mathcal{B}'(\Omega)$. We now refine and elaborate the above scheme in order to be able to consider analogous solutions of differential and partial differential equations. We restrict our previous definitions to the case where X is Euclidean q-space R^q with coordinates $x_1, \cdots, x_q$. Thus Ω is a subset of $R^q \times (0, \epsilon_0]$. Let $K(\Omega)$ be the ring of complex valued functions f on Ω such that the restriction of f to each ϵ-cross section of Ω is infinitely differentiable. (We recall that a function on a subset Y of R^q is defined to be differentiable if it is the restriction to Y of a function differentiable on some open set containing Y.) We want to specify a subring of $K(\Omega)$ in which the function $f(x, \epsilon) = \epsilon$ generates a proper ideal and in which this ideal induces a formal convergence which is preserved under x-differentiation. The ring $K(\Omega)$ is itself much too large ($\epsilon K(\Omega) = K(\Omega)$) and likewise the ring $K(\Omega) \cap B'(\Omega)$ contains sequences such as $\epsilon^k \sin (e^{k/\epsilon} x_1)$, formal convergence of which is destroyed by differentiation. At the other extreme the ring of functions with bounded x-derivatives of all orders could be used, but this is too special for our purposes since we are forced to consider functions which respond to differentiation in an unbounded manner (solutions of (1-1) for example). A suitable intermediate choice is the following. Given a fixed nonnegative integer M let $B(\Omega, M)$ be the subring of $K(\Omega) \cap B'$ consisting of functions such that derivatives of all orders with respect to the differentiations $\epsilon^M \partial/\partial x_j$, $j = 1, \cdots, q$ belong to $B'(\Omega)$. We then proceed as above. We shall retain the same terminology: "formal convergence," "uniform asymptotic expansion" etc., with the understanding that the preceding preliminary uses are being superseded.

Definition 1. A sequence f_k, $k = 1, 2, \cdots$, of elements of $K(\Omega)$ is formally convergent to zero if for any positive integer N there is a $k(N)$ such that $f_k \in \epsilon^N B(\Omega, M)$ for all $k \geqq k(N)$.

Definition 2. A series of elements of $K(\Omega)$, $\sum_{k=1}^{\infty} f_k$, is formally convergent if for some N and all k, $f_k \in \epsilon^{-N} B(\Omega, M)$ and if the sequence f_k converges formally to zero.

Again the collection of formally convergent series is a ring $\mathcal{B}(\Omega, M)$, actually a differential ring with derivations $\partial/\partial x_k$, $k = 1, 2, \cdots, q$. Similarly it is easy to see that the collection $\mathcal{B}_0(\Omega, M)$ of series formally convergent to zero is a differential ideal, that is, $\mathcal{B}_0$ is an ideal closed under differentiation. This means that the $\partial/\partial x_i$ act naturally on the quotient ring $\mathcal{B}/\mathcal{B}_0$.

Definition 3. The symbol $\doteq$ or the term "formal equality" denotes equality in $\mathcal{B}$ modulo $\mathcal{B}_0$.

We can now consider formal solutions in $\mathcal{B}$ of partial differential equations $\pi(y) \doteq 0$ where π is a polynomial in y and its x-partial derivatives with coefficients in B or $\mathcal{B}$. Such a problem corresponds in a well-defined way to the corresponding problem $\pi(y) = 0$ in $\mathcal{B}/\mathcal{B}_0$ with coefficients in the same quotient ring, or less elegantly expressed: we are sure that if y_1 is a formal solution of $\pi(y) \doteq 0$ and $y_2 \doteq y_1$ then y_2 is also a formal solution.

Finally we have the corresponding notion of uniform asymptotic expansion.

Definition 4. The series $\sum_{k=1}^{\infty} f_k$ is a uniform asymptotic expansion of an element f of $K(\Omega)$ if for some N, $f_k \in \epsilon^{-N} B(\Omega, M)$ and the sequence $f - \sum_{j=1}^{k} f_j$ converges formally to zero. We indicate this relationship by writing

$$f \sim \sum_{k=1}^{\infty} f_k .$$

It happens that a uniform asymptotic expansion in this sense can be termwise differentiated with respect to x. For if $f \sim \sum_{k=0}^{\infty} f_k$, then for some N the sequence $f - \sum_{j=0}^{k} f_k$, $k = 1, 2, \cdots$, is a sequence of elements of $\epsilon^{-N} B$ convergent to zero. This implies that the sequence $(\partial f/\partial x_i) - \sum_{j=0}^{k} \partial f_j/\partial x_i$, $k = 1, 2, \cdots$ is a sequence of elements of $\epsilon^{-N-M} B$ convergent to zero, which means that $(\partial f/\partial x_i) \sim \sum_{k=1}^{\infty} \partial f_k/\partial x_i$.

Similar properties are enjoyed by the formal ϵ-power series expansion of an infinitely differentiable function of (x, ϵ) of which our kind of expansion is a generalization. It is known that any formal power series is the asymptotic expansion of some infinitely differentiable function. We require a similar result for our kind of formal series. For purposes of exposition we suppose that x is one-dimensional although the conclusion and proof are easily modified to include the case where $x \in R^q$.

LEMMA. *Any formally convergent series is the uniform asymptotic expansion of some function.*

Proof. If $\sum_{k=1}^{\infty} f_k(x, \epsilon)$ is formally convergent then there exist integers M, N such that

$$\left(\frac{d}{dx}\right)^m f_k(x, \epsilon) \in \epsilon^{-N-mM+\psi(k)} B(\Omega, M)$$

where $\psi(k)$ is a nondecreasing function of k approaching $+\infty$ as $k \to \infty$. Otherwise expressed, there exist constants b_{mk} such that

$$|(d/dx)^m f_k| < b_{mk}\epsilon^{-N-mM+\psi(k)}$$

where we can suppose that $b_{mk} \geqq 1$. Let $\theta(u)$ be an infinitely differentiable function identically 1 near zero, identically 0 for $u \geqq 1$ and satisfying $0 \leqq \theta \leqq 1$ for all u. For any $\delta > 0$ let F_k be the function

$$F_k = \theta\left(\epsilon^{\delta} k! \sum_{m=0}^{k} b_{mk}\right) f_k .$$

The sum $f = \sum_{=1}^{\infty} F_k$ is finite for each ϵ since F_k is zero unless $\epsilon^{\delta} k! \sum_{m=0}^{k} b_{mk} < 1$. This implies that for $k \geqq m$

$$\left|\left(\frac{d}{dx}\right)^m F_k\right| < \frac{\epsilon^{-\delta-N-mM+\psi(k)}}{k!} .$$

This ensures that for large k

$$\left(\frac{d}{dx}\right)^m \left(f - \sum_{j=1}^{k-1} F_k\right) \in \epsilon^{-\delta-N-mM+\psi(k)} B'(\Omega) ,$$

which implies $f \sim \sum_{k=1}^{\infty} F_k$. And since $\sum_{k=1}^{\infty} f_k \doteq \sum_{k=1}^{\infty} F_k$ we conclude that

$$f \sim \sum_{k=1}^{\infty} f_k .$$

7. Examples (Continued).

7.1. If the functions $a_k(t)$ are each infinitely differentiable and the $\pi_k(t, u)$ are polynomials in u with infinitely differentiable coefficients for $|t| \leqq t_0$, then the series $\sum_{k=-N}^{\infty} a_k(t)\epsilon^k$, $\sum_{k=-N}^{\infty} a_k(t)\epsilon^{k/h}$, where $h > 0$ and $\sum_{k=2}^{\infty} \pi_k(t, \log \epsilon)\epsilon^{\log \log k}$ are rather conventional examples of formally convergent series. Here $\Omega = [-t_0, t_0] \times (0, \epsilon_0]$ and the underlying ring can be taken to be $\mathcal{B}(\Omega, 0)$. However there seems to be no advantage in strengthening our definitions to exclude bizarre dependence on ϵ. For example if $b_k(\epsilon)$ is any sequence of individually bounded *unmeasurable* functions $\sum_{k=-N}^{\infty} a_k(t)b_k(\epsilon)\epsilon^k$ is also formally convergent in the same sense.

7.2. If $f(\epsilon)$ is infinitely differentiable at $\epsilon = 0$, then its formal Taylor series $\sum_{k=0}^{\infty} c_k\epsilon^k$ is an asymptotic expansion in our sense. Moreover, on the domain $\Omega = \{(t, \epsilon) : |t - \epsilon| > \epsilon^{1-\delta}, \delta > 0\}$, we have

$$f\left(\frac{\epsilon}{t - \epsilon}\right) \sim \sum_{k=0}^{\infty} c_k \left(\frac{\epsilon}{t - \epsilon}\right)^k$$

where the underlying ring can be taken to be $\mathcal{B}(\Omega, 1)$. However on the domain $\Omega' = \{(t, \epsilon): |t - \epsilon| > \epsilon \log 1/\epsilon\}$ this series has no asymptotic character in our sense. Roughly speaking, on Ω the series behaves at worst like an asymptotic power series in ϵ^δ, while on Ω' its behavior degenerates to that of an asymptotic power series in $(\log \epsilon^{-1})^{-1}$. A weaker kind of convergence induced by an ideal generated by $(\log \epsilon^{-1})^{-1}$ would however be relevant in the latter case.

8. An Ad Hoc Preparation Lemma. Thus far our analysis of (1-1) has depended chiefly on the formal factorization (2-1) of $\hat{a}$ and the derived characteristic polygon. We next obtain a related factorization of the function a itself. We recall that the leading parts of the transformed problems $(2\text{-}4)_j$ are determined by the exponent pairs (k, m) which lie on the characteristic polygon. In the following definition we identify the simplest polynomial which determines essentially the same leading parts.

Definition 8.1. Let $P(t, \epsilon)$ be the polynomial in t and ϵ derived from the formal factorization (2-1) and from the characteristic polygon according to

$$P = t^{m0} + \sum_{(k,m) \in S_1 \cup S_2 \dots \cup S_p} t^m \epsilon^k \hat{P}_m(0) .$$

We now consider the factorization $a = P(P^{-1}a)$. Following the point of view of Section 6 we will characterize the quotient $U = P^{-1}a$ by its membership in a differential ring.

Definition 8.2. Let $\Omega(t_0, \epsilon_1)$ be the domain $|t| \leqq t_0$, $0 < \epsilon \leqq \epsilon_1 \leqq \epsilon_0$. Let $z(t, \epsilon) = (t^2 + \epsilon^{2\delta_p})^{1/2}$. Let $A = A(t_0, \epsilon_1)$ be the subset of $K(\Omega(t_0, \epsilon_1))$ consisting of functions f such that

$$\left(z \frac{d}{dt}\right)^k f \in B'(\Omega(t_0, \epsilon_1)), \quad k = 0, 1, \cdots .$$

Remarks. The collection A is a differential ring with derivation zd/dt. Since we have assumed $2n - 2\delta_p - \gamma_p > 0$ it follows that $\delta_p < n$. This readily implies that A is a differential subring of $B(\Omega(t_0, \epsilon_1), n)$ which we shall henceforth suppose to be the ring underlying our use of $\doteq$ and $\sim$.

We can now state the following

PREPARATION LEMMA. *For ϵ_1 sufficiently small the function a can be represented in the form*

$$a = PU \tag{8-1}$$

where U is a unit in $A(t_0, \epsilon_1)$ and P is defined by (8-1).

Proof. We first observe that given any $t_1 > 0$ there is an ϵ_1 sufficiently small such that for $t_1 \leqq |t| \leqq t_0$ and $0 \leqq \epsilon \leqq \epsilon_1$ both P and a are units in the ring of infinitely differentiable functions, so that on this domain the factorization is trivial. It thus suffices to prove the lemma for $|t| \leqq t_1$, $0 < \epsilon \leqq \epsilon_1$ where both t_1 and ϵ_1 are sufficiently small. But for t_1 and ϵ_1 sufficiently small the Malgrange preparation theorem [8] implies that corresponding to the formal factorization (2-1) there is a factorization of a of the form $a = \pi u$ where π is a monic polynomial in t of degree m_0 with coefficients which are infinitely differentiable functions of ϵ, u is a unit in the ring of infinitely differentiable functions of t and ϵ, and $\hat{\pi} = \hat{P}_0$, $\hat{u} = \hat{U}_0$. We consider the equation $\hat{P}_0(\tau, \epsilon) = 0$ as an algebraic equation for τ with coefficients in the ring of formal ϵ-power series. This equation has m_0 solutions in the ring of formal power series in some fractional power of ϵ. We choose m_0 infinitely differentiable functions of this fractional power of ϵ having the formal roots as asymptotic power series. Let P_1 be the monic polynomial in t having these functions as roots. Then the coefficients of P_1 are easily seen to be infinitely differentiable functions of ϵ. Also P_1 satisfies $\hat{P}_1 = \hat{\pi} = \hat{P}_0$. Hence each coefficient of the polynomial in t, $\pi - P_1$, has a zero of infinite order at $\epsilon = 0$. Thus for any integer M the Malgrange factorization can be written in the form

$$a = (P_1 + \epsilon^M Q)u \tag{8-2}$$

where $Q = \epsilon^{-M}(\pi - P_1)$ is again a polynomial in t with infinitely differentiable coefficients.

We now examine the linear factors of $\hat{P}_0$, P_1 and P. The portion of the characteristic polygon of (1-1) between $(0, m_0)$ and $(k_p, 0)$ is the Newton polygon of $\hat{P}_0$. It follows (Semple and Kneebone [9] or Bliss [10]) that the roots of $\hat{P}_0$ have the following form. Corresponding to each zero ζ of one of the leading parts $\alpha_k(s)$, there is a root of $\hat{P}_0$ having the form

$$\tau_0 = \epsilon^{\delta_k}\zeta + \cdots .$$

By construction this formal series is the asymptotic series for a root τ_1 of P_1. Since the polynomial P agrees with P_1 in terms which lie on the Newton polygon, there is a root τ of P possessing a convergent series with the same leading term. Under the hypothesis that there are no turning varieties none of the possible values for ζ are real. It follows that for ϵ_1 sufficiently small, $|t - \tau_1| > c\epsilon^{\delta_k}$ and $|t - \tau| > c\epsilon^{\delta_k}$, where $c > 0$. In particular, the linear factors of P_1 and P are never zero on $\Omega(t_0, \epsilon_1)$. It then can easily be seen that the functions

$$1, \quad \frac{t-\tau_1}{t-\tau}, \quad \frac{t-\tau}{t-\tau_1}, \quad \frac{t}{z}, \quad \frac{z}{t-\tau}, \quad \frac{z}{t-\tau_1} \text{ and } \frac{\epsilon^n}{t-\tau}$$

are bounded, and therefore generate an algebra of bounded functions. Since the application of zd/dt to each generator again yields an element of this algebra, we conclude that this algebra is a subset of A. In particular $(t - \tau)^{-1}(t - \tau_1)$ is a unit in A.

We now choose $M = m_0 n + 1$ in Eq. (8-2) obtaining

$$a = P\left\{\frac{P_1}{P} + \epsilon\frac{\epsilon^{m_0 n}}{P} Q\right\}u .$$

Since P_1/P is a product of factors of the form $(t - \tau)^{-1}(t - \tau_1)$ it is a unit in A. Similarly $P^{-1}\epsilon^{m_0 n}$ is an element of A. For ϵ_1 sufficiently small $P_1/P + \epsilon(\epsilon^{m_0 n}/P)Q$ has a bounded inverse which in turn readily implies that it is also a unit in A. Finally $U = P^{-1}a = \{P^{-1}P_1 + \epsilon^{m_0 n+1}P^{-1}Q\}u$ is also a unit in A which concludes the proof.

We remark that P is uniquely determined by (1-1) but the functions π and P_1 appearing in the proof of the preparation lemma are not.

9. Formal Solutions. The following theorem establishes that series (1-3) is formally convergent and that it behaves roughly like a power series in the quantity $I = \epsilon^n z^{-1} P^{-1/2}$.

THEOREM 1. *If Eq.* (1-1) *has no turning varieties, then for* ϵ_1 *sufficiently small the sequence* R_k *defined by* (1-2) *satisfies*

(9-1) $$\epsilon^{nk}R_k \in P^{1/2}I^kA \quad \textit{where } I = \epsilon^n z^{-1}P^{-1/2}.$$

Moreover the series

$$r \doteq \sum_{k=0}^{\infty} \epsilon^{nk}R_k$$

is a formal solution of $\epsilon^n r' + r^2 - a \doteq 0$.

Proof. 1. We first show that the sets $P^{1/2}I^kA$ are differential ideals in A, that is, ideals in A closed under the action of zd/dt. Evidently $P^{1/2}$ is bounded. Also $zP^{-1}dP/dt$ is a sum of elements of the form $z(t - \tau)^{-1}$, where τ is a root of P. These have been established as elements of A in the proof of the preparation lemma. Hence $zP^{-1}dP/dt \in A$. The identity

(9-2) $$\begin{aligned}\left(z\frac{d}{dt}\right)^{N+1}P^{1/2} &= \frac{1}{2}\left(z\frac{d}{dt}\right)^{N}P^{1/2}\left(zP^{-1}\frac{dP}{dt}\right)\\ &= \frac{1}{2}\sum\binom{N}{i}\left\{\left(z\frac{d}{dt}\right)^{i}P^{1/2}\right\}\left\{\left(z\frac{d}{dt}\right)^{N-i}zP^{-1}\frac{dP}{dt}\right\}\end{aligned}$$

and induction shows that $(zd/dt)^N P^{1/2}$ is bounded for all N, that is, $P^{1/2} \in A$.

We next verify that I is bounded. In the proof of the preparation lemma it was established that for small ϵ_1 the linear factors of P corresponding to the kth side of the characteristic polygon satisfy $|t - \tau| > c\epsilon^{\delta_k}$. Hence, since $|z^{-1}| \leqq \epsilon^{-\delta_p}$,

$$|I| < C\epsilon^{n-(\Sigma\delta_k-\delta_p)/2}$$

where the indicated sum is over all linear factors of P. But the sum, $\sum\delta_k$, is just the order of the zero of $a(0, \epsilon)$ at $\epsilon = 0$ which is k_p or γ_p (see Eq. (2-2)). Hence

$$|I| < C\epsilon^{n-\delta_p-\gamma_p/2}.$$

Since the inequality $2n - 2\delta_p - \gamma_p > 0$ is a consequence of our hypotheses precluding turning varieties, we conclude that I is bounded, and even more, that the powers I^k form a sequence formally convergent to zero. Now, carrying over the above argument for P, since $zI^{-1}dI/dt = -(z/2)P^{-1}dP/dt - t/z$ belongs to A, identities similar to (9-2) show that $I \in A$.

We now apply zd/dt to $P^{1/2}I^kA$ obtaining

$$\begin{aligned} z\frac{d}{dt}P^{1/2}I^kA &\subset \left(z\frac{d}{dt}P^{1/2}\right)I^kA + P^{1/2}\left(z\frac{d}{dt}I^k\right)A + P^{1/2}I^k\left(z\frac{d}{dt}A\right)\\ &\subset P^{1/2}\left(zP^{-1}\frac{dP}{dt}\right)I^kA + P^{1/2}I^k\left(zI^{-1}\frac{dI}{dt}\right)A + P^{1/2}I^kA\\ &\subset P^{1/2}I^kA.\end{aligned}$$

Thus $P^{1/2}I^kA$ is a differential ideal in A.

2. We now prove the inclusions (9-1) by induction. By the preparation lemma

$$R_0 = \pm a^{1/2} = \pm P^{1/2}U^{1/2} \in P^{1/2}A$$

since the square root of the unit U is easily seen to belong to A. Thus (9.1) is true for $k = 0$. Suppose it is true for $k \leqq i$. Then by $(1\text{-}2)_k$

$$\begin{aligned}\epsilon^{n(i+1)}R_{i+1} &= -\frac{\epsilon^n}{2R_0 z}\left(z\frac{d}{dt}\right)\epsilon^{ni}R_i - \frac{1}{2R_0}\sum_{k+j=i+1;\,k+j>0}\epsilon^{kn}R_k\epsilon^{jn}R_j\\ &\in I\left(z\frac{d}{dt}\right)P^{1/2}I^iA + P^{-1/2}\sum_{j+k=i+1}P^{1/2}I^jP^{1/2}I^kA\\ &\in P^{1/2}I^{i+1}A\,.\end{aligned}$$

3. It follows from estimates of I made above that

$$I \in \epsilon^{n-\delta_p-\gamma p/2}A\,.$$

Hence (9-1) implies that the series $\sum_{k=0}^{\infty}\epsilon^{nk}R_k$ is formally convergent. Finally, substitution into $\epsilon^n dr/dt + r^2 - a$ leads to a series which is formally equivalent to zero.

10. Formal Solutions are Asymptotic Solutions. We now show that the Riccati equation has solutions which are represented asymptotically by the above formal solutions.

THEOREM 2. *If $\epsilon^{2n}y'' = ay$ has no turning varieties and $|\arg a(t,\epsilon)| \leqq \pi$ then for ϵ_1 sufficiently small and $|t| \leqq t_0$, $0 < \epsilon \leqq \epsilon_1$, solutions exist such that*

$$\epsilon^n y'/y \sim \sum_{k=0}^{\infty}\epsilon^{nk}R_k(t,\epsilon)$$

where

$$R_0^{\,2} - a = 0\,, \qquad R_{k+1} = -\frac{1}{2}R_0^{-1}\left\{R_k' + \sum_{i+j=k+1;\,i,j>0}R_iR_j\right\}.$$

Proof. As a preliminary step we select a function S such that $S \sim \sum_{k=2}^{\infty}\epsilon^{nk}R_k$. Imitating the construction in the Lemma of Section 6, for any $\delta > 0$ it easily follows that we can construct an S also satisfying

$$S \in (\epsilon^{2n-\delta}/P^{1/2}z^2)A\,.$$

Let $R = R_0 + \epsilon^n R_1 + S$. Then R is an element of $B(\Omega, n)$ (actually of $A(\Omega)$) such that $R \sim \sum_{k=0}^{\infty}\epsilon^{nk}R_k$. We will establish the existence of suitable solutions of the Riccati equation by regarding R as a very excellent initial guess for Newtonian successive approximations.

We transform the initial value problem

$$\epsilon^n r' + r^2 - a = 0\,, \qquad r(-t_0) = R(-t_0)$$

by the change of variable $r = R + \epsilon^n\rho$ into the equivalent integral equation

$$\rho(t) - \int_{-t_0}^{t}\exp\left[-2\epsilon^{-n}\int_s^t R\,d\sigma\right]\{\alpha + \rho^2\}ds = 0$$

where $\alpha = \epsilon^{-n}(a - R^2 - \epsilon^n R')$ is a function asymptotic to zero. We consider the left side of the integral equation to be a function, $F(\rho)$, on the Banach space of continuous functions on $[-t_0, t_0]$, vanishing at $-t_0$, endowed with the maximum norm. To establish the theorem it suffices to show that $F(\rho) = 0$ has a solution satisfying $\rho \sim 0$.

We use a basic result about Newtonian approximations due to Kantorovich

[11, p. 708]. Kantorovich's theorem requires estimates for F' and F'', the first and second Fréchet derivatives of F, and implies that if $\|\{F'(0)\}^{-1}F(0)\| < \eta$, $\|\{F'(0)\}^{-1}F''(0)\| < L$ and $\eta L < 1/2$, then $F(\rho) = 0$ has a solution satisfying $\|\rho\| < (1 - (1 - 2\eta L)^{1/2})/L$. In our case $F'(0)$ is the identity mapping and $F''(r)$ is independent of r. Hence we require estimates for $\|F(0)\|$ and $\|F''\|$.

In order to establish these estimates we must use the nontransition condition which implies that $a(t, \epsilon)$ has a root $R_0 = a^{1/2}$ satisfying $\operatorname{Re} a^{1/2} \geqq 0$. Then since for any positive δ

$$R = a^{1/2} - \frac{\epsilon^n}{4}\frac{d}{dt}\log a + \frac{\epsilon^{2n-\delta}}{P^{1/2}z^2}b$$

where b is bounded, we conclude that the kernel

$$h(t, s) = \exp - \frac{2}{\epsilon}\int_s^t R d\sigma$$

can be estimated by

$$|h(t, s)| < \left|\frac{a(t, \epsilon)}{a(s, \epsilon)}\right|^{1/2} \exp\left\{C_1\epsilon^{n-\gamma_p-\delta}\int_s^t \frac{d\sigma}{\sigma^2 + \epsilon^{2\delta_p}}\right\}$$

$$< C_1\epsilon^{-\gamma_p/2}\exp\left\{C_2\epsilon^{n-\gamma_p/2-\delta_p-\delta}\int_{-\infty}^{\infty}\frac{d\tau}{1+\tau^2}\right\}.$$

Choosing δ to be smaller than the positive quantity $n - \gamma_p/2 - \delta_p$ we obtain

$$|h(t, s)| < C_3\epsilon^{-\gamma_p/2}.$$

Since $F(0) = \int_{-t_0}^{t} h(t, s)\alpha(s)ds$ we conclude that

$$|F(0)| \leqq C_4\epsilon^{-\gamma_p/2}\|\alpha\| = \eta(\epsilon).$$

Also $F''(r)$ is the bilinear mapping given by $F''(r)(u, v) = 2\int_{-t_0}^{t} h(t, s)u(s)v(s)ds$. Hence

$$\|F''(r)\| < C_5\epsilon^{-\gamma_p/2} = L(\epsilon).$$

Thus the quantity $\eta(\epsilon)L(\epsilon)$ decisive for Kantorovich's result is $C_6\epsilon^{-\gamma_p}\|\alpha\|$. Since $\alpha \sim 0$, $\|\alpha\| \in \epsilon^N B'$ for each N. This implies that for small ϵ, $\eta(\epsilon)L(\epsilon) < 1/2$. Hence there exists a solution ρ of $F = 0$ such that

$$\|\rho\| < (1 - (1 - \eta(\epsilon)L(\epsilon))^{1/2})/L(\epsilon)$$

which implies that $\rho \in \epsilon^N B'$ for each N. This in itself does not insure that $\rho \sim 0$. However, this relation follows from the fact that the functional equation $F(\rho) = 0$ implies that any solution in $B'(\Omega)$ which is continuous in t for each ϵ actually belongs to $B(\Omega, n)$. Finally we observe that for the other choice of square root, $R_0 = -a^{1/2}$, the proof carries over if we replace the boundary condition $\rho(-t_0) = 0$ by $\rho(t_0) = 0$.

COROLLARY. *Under the hypotheses of Theorem 2 there exist solutions $y_\pm$ such that the functions*

$$\exp \mp \left\{\epsilon^{-n}\int_0^t a^{1/2}(s, \epsilon)ds\right\} y_\pm(t, \epsilon)$$

possess uniform asymptotic expansions.

Proof. Theorem 2 implies that $y_\pm$ exist such that

$$\log y_\pm - \epsilon^{-n}\int_0^t a^{1/2} + \frac{1}{4}\log a \sim \sum_{k=2}^{\infty} \epsilon^{nk}\int_0^t R_k(s, \epsilon)ds .$$

Since $\epsilon^{nk}\int_0^t R_k ds \in \epsilon^{n-\delta_p-(1/2)\gamma_p}B$, the expansion for the function on the left leads easily to an asymptotic expansion for its exponential. Termwise multiplication of this expansion by $a^{-1/4}(t, \epsilon)$ gives the required expansion.

11. Examples (Continued). Theorem 2 ensures that (1-6) has solutions $y_\pm$ satisfying

$$\begin{aligned}\epsilon \log y_\pm &\sim \pm i\int_\tau^t (\phi + \epsilon)^{1/2} ds - \frac{\epsilon}{4}\log(\phi + \epsilon) \\ &\mp i\epsilon^2 \int_\tau^t \left\{\frac{1}{8}\phi''(\phi+\epsilon)^{-3/2} - \frac{5}{16}\phi'^2(\phi+\epsilon)^{-5/2}\right\}ds + \cdots .\end{aligned} \tag{11-1}$$

These formulas cannot be further simplified without sacrificing the uniformity of the expansion.

12. Solution of the Connection Problem. Since we possess uniform asymptotic solutions on $[-t_0, t_0]$ we already have, in principle, connection formulas. We need only break down our formulas into the simpler classical forms and read off the connection constants. In carrying out this reduction we will obtain precise results about the range of validity of the classical expansions. However, our main interest in Theorem 3 below is that it contains an effective procedure for computing the connection constants.

To study the relation of our uniform asymptotic solutions to the classical asymptotic solutions we must introduce a covering of $[-t_0, t_0]$ by $2p - 1$ contiguous closed intervals $I_{-p+1}, I_{-p+2}, \cdots, I_0, I_1 \cdots I_{p-1}$. We define these by $I_k = (\operatorname{sgn} k)J_{p-|k|}$ for $k \neq 0$ and $I_0 = J_p \cup -J_p$, where

$$\begin{aligned} J_1 &= \{t \mid s_1\epsilon^{\delta_1'} \leqq t \leqq t_0\} , \\ J_k &= \{t \mid s_k\epsilon^{\delta_k'} \leqq t \leqq s_{k-1}\epsilon^{\delta_{k-1}'}\} \qquad k = 2, \cdots, p-1 , \end{aligned}$$

and $\delta_{k-1} < \delta_k' < \delta_k$ where $s_k > 0$, and

$$J_p = \{t \mid 0 \leqq t \leqq s_{p-1}\epsilon^{\delta_{p-1}'}\} .$$

We remark that the upper end point of J_k, $k > 1$, is large in the asymptotic scale $s = t\epsilon^{-\delta_k}$, but small in the adjacent scale $s = t\epsilon^{-\delta_{k-1}}$.

We will use the technical device of the *Neutrix* or additive class of negligible functions, a notion which has been developed by J. G. van der Corput [12]. For the sake of this device it will be convenient to suppose that the quantities s_k appearing in the above partition of $[0, t_0]$ are each variables ranging over some small interval J containing 1. We can now attack the problem of computing asymptotic expansions for integrals of the form $\int_0^t f(s, \epsilon)ds$ in the following way. If $f(t, \epsilon)$ possesses simple asymptotic expansions on each J_k, then for $t \in J_k$, we can obtain expansions for the integral by decomposing it into $\int_0^t = \int_{J_p} + \int_{J_{p-1}} + \int_{J_{k-1}} + \cdots + \int_{s_k\epsilon^{\delta'_k}}^t$, and inserting asymptotic expansions for the integrand. However, it is self-evident that while the intermediate computations in this procedure depend heavily on the variable end points of the intervals J_i, the resulting asymptotic expansion does not. It is therefore reasonable to hope that since all functions of s_k which occur in the intermediate steps must cancel in the end, these functions can be systematically eliminated in early stages of the computation. We will show that such reasoning is possible. We remark that this technique simplifies the computations in a fundamental way and it is hardly possible to analyze even simple examples without it. However, the relation

$$\epsilon \sim \epsilon/2 + \log se^{\epsilon/2} - \log s$$

shows that some caution is required. Here, although the function on the left is independent of s, we cannot individually neglect the functions of s which appear in the trivial asymptotic expansion on the right. It is thus essential to identify a class of terms which can be individually neglected. The following special class will be sufficient for our purposes.

Definition. We consider formal series in the variables $t, s_1, s_2, \cdots, s_{p-1}, \epsilon$, on the domain $[-t_0, t_0] \times J^{p-1} \times (0, \epsilon_1]$, where J is a small interval containing 1. A series is said to be *negligible* if it is a finite sum of the following: formal Laurent series in $s_j\epsilon^h$ or $s_j^{-1}\epsilon^h$, $h > 0$, with no constant term, a finite number of negative exponents, and coefficients which are formal power series in a fractional power of ϵ; multiples of $\log s_k$ with similar coefficients. If a function f has a uniform asymptotic series of the form

$$f(t, s_1, \cdots, s_p, \epsilon) \sim \hat{F}(t, \epsilon) + \hat{N}(s_1, \cdots, s_{p-1}, \epsilon)$$

where $\hat{F}$ is a formally convergent series on $[-t_0, t_0] \times (0, \epsilon_1]$ and $\hat{N}$ is negligible, we say that $\hat{F}$ is a *neutralized uniform asymptotic expansion* of f.

This definition is useful because the following lemma justifies neglecting negligible functions in our computations.

LEMMA. *If 0 is a neutralized uniform asymptotic expansion of a function $f(t, \epsilon)$ of t and ϵ alone, then $f \sim 0$.*

Proof. By hypothesis $f(t, \epsilon) \sim \hat{N}(s_1, \cdots, s_k, \epsilon)$ where $\hat{N}$ is negligible. It follows that f has a uniform asymptotic expansion of the form

$$f \sim \sum_{k=0}^{\infty} \epsilon^{h_k} \varphi_k(s_1, \cdots, s_p)$$

where h_k is a strictly increasing sequence of real numbers tending to ∞ and φ_k is a finite linear combination of positive powers, negative powers, and logarithms of $s_1, \cdots, s_{p-1}$. By the definition of asymptotic expansion there is an N_1 such that

$$\left| f - \sum_{k=0}^{N_1} \epsilon^{h_k} \varphi_k \right| < M\epsilon^{h_2}$$

which implies that

$$\epsilon^{-h_1} f(t, \epsilon) = \varphi_1(s_1, \cdots, s_{p-1}) + \epsilon^{h_2 - h_1} \phi_1$$

where ϕ_1 is bounded. This implies that

$$|\varphi_1(s_1, \cdots, s_{p-1}) - \varphi_1(s_1', \cdots, s_{p-1}')| < \epsilon^{h_2 - h_1} M$$

for $s_k, s_k' \in J$, which, together with the special form of φ_1, implies $\varphi_1 = 0$. Similarly $\varphi_2 = \varphi_3 = \cdots = 0$. Thus $f \sim \sum \epsilon^{h_k} \varphi_k \sim 0$.

A complete solution of the connection problem of Section 4 can now be expressed in the following result.

THEOREM 3. *On each domain* $t \in I_k$, $0 < \epsilon \leqq \epsilon_1$, $k = -p+1, \cdots, p-1$, *the solution pair* $y_\pm$ *of Theorem* 2 *has an asymptotic representation of the form*

$$y_\pm(t, \epsilon) \sim \hat{w}_k{}^\pm(s, \epsilon)\hat{c}_k{}^\pm(\epsilon)$$

where $\hat{w}_k{}^\pm$ *is a classical formal solution pair of* $(2\text{-}4)_k$. *The* $\hat{c}_k{}^\pm(\epsilon)$ *are determined by neutralized formal expansions of the formal solutions* (1-4) *and satisfy*

$$\epsilon^n \log \hat{c}_k{}^\pm(\epsilon) = \hat{f}_k{}^\pm(\epsilon) + \hat{g}_k{}^\pm(\epsilon) \log \epsilon$$

where $\hat{f}_k{}^\pm$ *and* $\hat{g}_k{}^\pm$ *are formal power series in a fractional power of* ϵ.

We omit the proof of Theorem 3 since it is well represented by our concluding analysis of example (1-6) below.

13. Examples (Concluded). We complete our analysis of

$$\epsilon^2 y'' + (\phi + \epsilon)y = 0 \tag{1-6}$$

by computing connection formulas relating the classical asymptotic forms (see Section 5) valid for negative t to the similar forms valid for positive t. We accomplish this by reducing the uniform formulas given by (11-1) to two different classical forms, depending on whether t is positive or negative. Let $Y_L{}^\mp$ and $Y_R{}^\pm$ denote the solution pairs obtained from (11-2) by choosing $\tau = -1$ and $\tau = +1$, respectively, as the limit of integration in the exponential. Then these solutions are related by

$$Y_R{}^\pm = Y_L{}^\mp C^\pm \tag{13-1}$$

where

$$\begin{aligned}\epsilon \log C^\pm \sim{}& \pm i \int_{-1}^{1} (\phi + \epsilon)^{1/2} dt \\ &\mp i\epsilon^2 \int_{-1}^{1} \left\{ \frac{1}{8}\phi''(\phi+\epsilon)^{-3/2} - \frac{5}{16}\phi'^2(\phi+\epsilon)^{-5/2} \right\} dt + \cdots .\end{aligned} \tag{13-2}$$

Since we will verify that $Y_L{}^\pm$ and $Y_R{}^\pm$ are represented by classical asymptotic formulas for negative and positive t respectively, the relations (13-1) and (13-2) constitute, in principle, a complete solution of the stated connection problem. The main object of our remaining analysis is to obtain more explicit asymptotic formulas for the *connection constants* $C^\pm(\epsilon)$ by the method described in Section 12. We now suppose, for purposes of illustration, that ϕ has the specific form described in Section 1. We assume that

$$\begin{aligned}&\text{(i)} \quad \phi(t) = t^6\psi^2(t), \quad \text{where } \psi > 0, \\ &\text{(ii)} \quad \hat{\psi}^2(t) = \alpha + \sum_{k=1}^{\infty} \beta_k t^k, \\ &\text{(iii)} \quad \phi + \epsilon = 1, \quad \text{for } |t| \geqq 1.\end{aligned} \tag{13-3}$$

We recall Section 3, Example 1, in which we found that the relevant asymptotic scales for this problem are $t = O(1)$ and $t = O(\epsilon^{1/6})$. In this case the subdivision of $[-1, 1]$ into contiguous closed subintervals given in Section 12 has the form

(13-4) $$[-1, 1] = [-1, -s_1\epsilon^\delta] \cup [-s_1\epsilon^\delta, s_1\epsilon^\delta] \cup [s_1\epsilon^\delta, 1]$$

where $0 < \delta < 1/6$ and, say, $s_1 \in [1/2, 3/2]$. The significance of this subdivision is that on each subinterval the function $\phi + \epsilon$ possesses a single, simple leading part in the following manner. On each of the outer intervals $\phi + \epsilon = t^6\psi^2[1 + \epsilon t^{-6}\psi^{-2}] = t^6\psi^2[1 + O(\epsilon^{1-6\delta})]$ while on the inner interval (letting $t = s\epsilon^{1/6}$)

$$\phi + \epsilon \sim \epsilon(\alpha s^6 + 1)\left[1 + (\alpha s^6 + 1)^{-1} \sum_{k=1}^{\infty} \beta_k \epsilon^{k/6} s^{k+6}\right]$$
$$\sim \epsilon(\alpha s^6 + 1)[1 + O(\epsilon^\delta)] .$$

It follows that on these subintervals we can simplify the fractional powers of $\phi + \epsilon$ in (11-1) by inserting the asymptotic expansions

(13-5) $$(\phi + \epsilon)^{k/2} \sim (t^3\psi \operatorname{sgn} t)^k[1 + (k/2)\epsilon t^{-6}\psi^{-2} + \cdots]$$

and

(13-6) $$(\phi + \epsilon)^{k/2} \sim \epsilon^{k/2}(\alpha s^6 + 1)^{k/2}\left[1 + \frac{k}{2}\epsilon^{1/6}\frac{\beta_1 s^7}{\alpha s^6 + 1} + \cdots\right].$$

If we use (13-5) to simplify $Y_L^\pm$ and $Y_R^\pm$ we find that

$(13\text{-}7)_L$ $$Y_L^\pm \sim (-t)^{-3/2}\psi^{-1/2} \exp\left\{\pm \frac{i}{\epsilon}\int_{-1}^{t} s^3\psi ds + \cdots\right\} \text{for } t \in (-\infty, -s_1\epsilon^\delta],$$

and

$(13\text{-}7)_R$ $$Y_R^\pm \sim t^{-3/2}\psi^{-1/2} \exp\left\{\pm \frac{i}{\epsilon}\int_{1}^{t} s^3\psi ds + \cdots\right\} \text{for } t \in [s_1\epsilon^\delta, \infty) .$$

(It can be seen in these formulas that we have willfully used "$\pm$" to label the solutions according to the formal properties of their classical asymptotic formulas.) Thus the constants $C^\pm(\epsilon)$ are actually the constants relating the classical asymptotic forms. The asymptotic behavior of $C^\pm(\epsilon)$ is given by the following result.

Asymptotic Connection Formulas. The asymptotic solutions of (1-6) given by $(13\text{-}7)_L$ for negative t, and by $(13\text{-}7)_R$ for positive t, are related by $Y_R^\pm = Y_L^\mp C^\pm$ where

(13-8)
$$\begin{aligned}
\log C^\pm = \pm i\Bigg\{&\epsilon^{-1}\int_{-1}^{1}|s|^3\psi ds + \epsilon^{-1/3}\int_{-\infty}^{\infty}[(\alpha s^6 + 1)^{1/2} - \alpha^{1/2}|s|^3]ds \\
&+ \log\epsilon\left(\frac{1}{12}\beta_2\alpha^{-3/2} - \frac{1}{16}\beta_1^2\alpha^{-5/2}\right) \\
&+ \Bigg(\frac{1}{2}\int_{-1}^{1}\Bigg[s^{-3}\psi^{-1} - \alpha^{-1/2}s^{-3} + \frac{1}{2}\alpha^{-3/2}\beta_1 s^{-2} \\
&\qquad + \left(\frac{1}{2}\beta_2\alpha^{-3/2} - \frac{3}{8}\beta_1^2\alpha^{-5/2}\right)s^{-1}\Bigg]\operatorname{sgn} s ds - \frac{1}{2}\alpha^{-1/2} \\
&+ \int_{-\infty}^{-1}\left[h(s) - \left(\frac{1}{4}\beta_2\alpha^{-3/2} - \frac{3}{16}\beta_1^2\alpha^{-5/2}\right)s^{-1}\right]ds \\
&+ \int_{-1}^{1}h(s)ds + \int_{1}^{\infty}\left[h(s) + \left(\frac{1}{4}\beta_2\alpha^{-3/2} - \frac{3}{16}\beta_1^2\alpha^{-5/2}\right)s^{-1}\right]ds\Bigg)\Bigg\} \\
&+ O(\epsilon^{1/3}) ,
\end{aligned}$$

and

$$h(s) = (1/2)(\alpha s^6 + 1)^{-1/2}\beta_2 s^8 - (1/8)(\alpha s^6 + 1)^{-3/2}\beta_1^2 s^{14} - \left(\frac{1}{2}\beta_2\alpha^{-1/2} - \frac{1}{8}\beta_1^2\alpha^{-3/2}\right)|s|^5 .$$

Proof. We introduce the subdivision (13-4) into the integration in (13-2), obtaining

$$\begin{aligned}\epsilon \log C^{\pm}(\epsilon) \sim \pm i \int_{-1}^{-s_1\epsilon^\delta} (\phi + \epsilon)^{1/2} dt \pm i \int_{-s_1\epsilon^\delta}^{s_1\epsilon^\delta} (\phi + \epsilon)^{1/2} dt \\ \pm i \int_{s_1\epsilon^\delta}^{1} (\phi + \epsilon)^{1/2} dt \\ \mp i\epsilon^2 \int_{-1}^{-s_1\epsilon^\delta} \left\{\frac{1}{8}\phi''(\phi + \epsilon)^{-3/2} - \frac{5}{16}\phi'^2(\phi + \epsilon)^{-5/2}\right\} dt \\ + \cdots .\end{aligned}$$

Changing the dummy variable from t to $s = t\epsilon^{-1/3}$ in integrals over the inner interval and inserting the asymptotic expansions (13-4) and (13-5) for $(\phi + \epsilon)^{k/2}$, we obtain

$$\begin{aligned}\epsilon \log C^{\pm}(\epsilon) = \mp i \int_{-1}^{-s_1\epsilon^\delta} \{t^3\psi + \tfrac{1}{2}\epsilon t^{-3}\psi^{-1}\} dt \\ \pm \epsilon^{2/3} i \int_{-s_1\epsilon^{\delta-1/6}}^{s_1\epsilon^{\delta-1/6}} \left\{(\alpha s^6 + 1)^{1/2} + \frac{1}{2}\epsilon^{1/6}\frac{\beta_1 s^7}{(\alpha s^6 + 1)^{1/2}} \right. \\ \left. + \frac{1}{2}\epsilon^{1/3}\frac{\beta_2 s^6}{(\alpha s^6 + 1)^{1/2}} - \epsilon^{1/3}\frac{1}{8}\frac{\beta_1^2 s^{14}}{(\alpha s^6 + 1)^{3/2}}\right\} ds \\ \pm i \int_{s_1\epsilon^\delta}^{1} \{t^3\psi + \tfrac{1}{2}\epsilon t^{-3}\psi^{-1}\} dt + \cdots .\end{aligned} \tag{13-9}$$

We now introduce a class of negligible formal series according to the definition of Section 12 in order to compute neutralized asymptotic expansions for the integrals in the last formula. Specifically, we will neglect any formal Laurent series in $s_1\epsilon^\delta$ or $s_1^{-1}\epsilon^{1/6-\delta}$ with a finite number of negative exponents and no constant term, and we also neglect multiples of $\log s_1$. Since $\epsilon \log C^{\pm}$ is independent of s_1, the lemma of Section 12 justifies simplifying the intermediate computations in this manner. We illustrate this procedure with some specimen calculations.

1. $\int_{s_1\epsilon^\delta}^1 t^3\psi(t)dt = \int_0^1 t^3\psi(t)dt - \int_0^{s_1\epsilon^\delta} t^3\psi(t)dt$. But $\int_0^{s_1\epsilon^\delta} t^3\psi(t)dt$ is represented asymptotically by the term by term integration of $\int_0^{s_1\epsilon^\delta} t^3\hat{\psi}(t)dt$ (where $\hat{\psi}$ is the formal power series expansion of ψ), which is a negligible series. Hence in the sense of neutralized computations

$$\int_{s_1\epsilon^\delta}^1 t^3\psi(t)dt = \int_0^1 t^3\psi(t)dt .$$

2. The following computation is more complicated because it involves the extraction of a suitable leading part from the integrand before the previous calculation can be imitated. First

$$\int_{-s_1\epsilon^{\delta-1/6}}^{s_1\epsilon^{\delta-1/6}} (\alpha s^6 + 1)^{1/2} ds = \int_{-s_1\epsilon^{\delta-1/6}}^{s_1\epsilon^{\delta-1/6}} [(\alpha s^6 + 1)^{1/2} - \alpha^{1/2}|s|^3] ds$$

because

$$\int_{-s_1\epsilon^{\delta-1/6}}^{s_1\epsilon^{\delta-1/6}} \alpha^{1/2}|s|^3 ds = \frac{\alpha}{2}\,(s_1\epsilon^{\delta-1/6})^4$$

which is negligible. We now observe that

$$\int_{\pm s_1\epsilon^{\delta-1/6}}^{\pm\infty} [(\alpha s^6+1)^{1/2} - \alpha^{1/2}|s|^3]ds \sim \int_{\pm s_1\epsilon^{\delta-1/6}}^{\pm\infty} \left\{\sum_{k=1}^{\infty} \alpha^{1/2}|s|^3 \binom{1/2}{k} (\alpha s^6)^{-k}\right\} ds\,.$$

If we imagine (but do not carry out!) the indicated termwise integration we again obtain a negligible series. Hence, proceeding as in computation 1, we obtain

$$\int_{-s_1\epsilon^{\delta-1/6}}^{s_1\epsilon^{\delta-1/6}} (\alpha s^6+1)^{1/2} ds = \int_{-\infty}^{\infty} [(\alpha s^6+1)^{1/2} - \alpha^{1/2}|s|^3]ds$$

as the result of our neutralized calculation.

3. Our final specimen calculation shows the exceptional role played by logarithmic terms. We compute $\int_{s_1\epsilon^\delta}^1 t^{-3}\psi^{-1}dt$. As in the previous calculation we must extract the leading part of the integrand, in this case near $t = 0$. Using $\psi^{-1} \sim (\hat{\psi}^2)^{-1/2} \sim (\alpha + \beta_1 t + \beta_2 t^2 + \cdots)^{-1/2}$, it is easily established that

$$t^3\psi^{-1}(t) = \alpha^{-1/2}t^{-3} - \tfrac{1}{2}\alpha^{-3/2}\beta_1 t^{-2} - (\tfrac{1}{2}\alpha^{-3/2}\beta_2 - \tfrac{3}{8}\alpha^{-5/2}\beta_1^2)t^{-1} + \eta(t)$$

where $\eta(t)$ is infinitely differentiable. As above, we again modify the integrand by subtracting its leading part, the difference here being that the result of subtracting the t^{-1} term is not negligible. We obtain

$$\int_{s_1\epsilon^\delta}^1 t^{-3}\psi^{-1}(t)dt = \int_{s_1\epsilon^\delta}^1 \{t^{-3}\psi^{-1}(t) - \alpha^{-1/2}t^{-3} + \tfrac{1}{2}\alpha^{-3/2}\beta_1 t^{-2} + (\tfrac{1}{2}\alpha^{-3/2}\beta_2 - \tfrac{3}{8}\alpha^{-5/2}\beta_1^2)t^{-1}\}dt - (\tfrac{1}{2}\alpha^{-3/2}\beta_2 - \tfrac{3}{8}\alpha^{-5/2}\beta_1^2)\int_{s_1\epsilon^\delta}^1 t^{-1}dt\,.$$

But $\int_{s_1\epsilon^\delta}^1 t^{-1}dt = -\log s_1 - \delta\log\epsilon = -\delta\log\epsilon$, since $-\log s_1$ is negligible. Also the remaining integral $\int_{s_1\epsilon^\delta}^1$ can now be replaced by $\int_0^1$. We thus obtain

$$\int_{s_1\epsilon^\delta}^1 t^{-3}\psi^{-1}(t)dt = \int_0^1 \left\{t^{-3}\psi^{-1} - \alpha^{-1/2}t^{-3} + \frac{1}{2}\alpha^{-3/2}\beta_1 t^{-2} + \left(\frac{1}{2}\alpha^{-3/2}\beta_2 - \frac{3}{8}\alpha^{-5/2}\beta_1^2\right)t^{-1}\right\}ds + \delta\left(\frac{1}{2}\alpha^{-3/2}\beta_2 - \frac{3}{8}\alpha^{-5/2}\beta_1^2\right)\log\epsilon\,.$$

Proceeding to compute neutralized values for the integrals appearing explicitly in (13-9) in this manner we obtain the asymptotic formulas (13-8).

These formulas for the connection constants permit us to solve the following scattering problem. Since we have assumed that $\phi + \epsilon = 1$ for $|t| > 1$, the differential equation has pure exponential solutions for $t \leqq -1$ and for $t \geqq 1$. The scattering matrix relates the solution pair which has this form for positive t to the pair which has this form for negative t. Explicitly we have the following result.

Scattering Formulas. Let $E_L^{\pm}$ be the solution pair of (1-6) which has the form $\exp[\pm i\epsilon^{-1}t]$ for large negative t, and let $E_R^{\pm}$ be the pair which has the same form for large positive t. Then

$$E_R^{\pm}(t, \epsilon) = C^{\pm}(\epsilon) \exp [\pm(2/\epsilon)i] E_L^{\pm}(t, \epsilon)$$

where the $C^{\pm}(\epsilon)$ are described asymptotically by formula (13-8).

Proof. This follows immediately from $Y_R^{\pm} = \exp [\pm i((t - 1)/\epsilon)]$ for $t > 1$ and $Y_L^{\pm} = \exp [\mp i((t + 1)/\epsilon)]$ for $t < -1$. Hence $E_L^{\pm} = \exp [\mp it/\epsilon] Y_L^{\mp}$ and $E_R^{\pm} = \exp [\pm it/\epsilon] Y_R^{\pm}$. Combining these relations with $Y_R^{\pm} = C^{\pm} Y_L^{\mp}$ we obtain the asserted relation.

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