Numerical Solution of the Dirichlet Problem for Systems of Circular Conductors between Parallel Ground Lines

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Abstract. A Green's function-integral equation technique is used to obtain a numerical solution to the two-dimensional Dirichlet problem for the multiply connected region between the ground lines ($V = 0$) at $y = 0$ and $y = 1$ and exterior to the N circular conductors with arbitrarily spaced centers (x_i, y_i) and radii R_i, $i = 1, 2, \cdots, N$. If the smallest distance between each pair of conducting surfaces exceeds .1, the capacitance matrix for this system of N conductors can be calculated to 6-place accuracy at a cost of about $15\ N^2$ seconds on the IBM 7044 computer. ∎
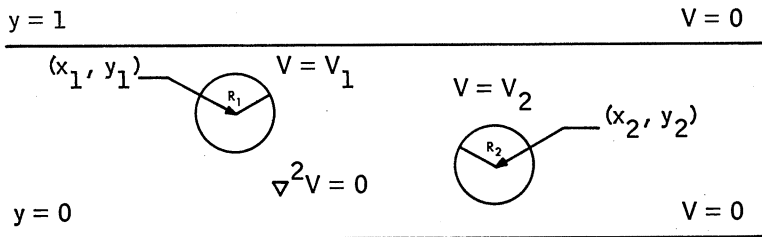
FIGURE 1. *A System of $N = 2$ Circular Conductors*

1. The Problem. A two-dimensional system of nonintersecting circular conductors is located in the strip between $y = 0$ and $y = 1$ with the ith conductor having radius R_i and center (x_i, y_i), $i = 1, 2, \cdots, N$. It is supposed that none of the conductors intersects the ground lines at $y = 0$ and $y = 1$. We wish to find a function $V(x, y)$ such that

$$(1) \quad \nabla^2 V(x, y) = 0 \quad \text{for} \begin{cases} -\infty < x < \infty, \\ 0 < y < 1, \\ (x - x_i)^2 + (y - y_i)^2 > R_i^2, i = 1, 2, \cdots, N, \end{cases}$$

$$(2) \quad \begin{aligned} &V(x, y) = 0 \quad \text{for } y = 0 \text{ and } y = 1, \\ &V(x, y) \to 0 \quad \text{as } x \to \pm\infty, \end{aligned}$$

$$(3) \quad V(x, y) = V_i \quad \text{for } (x - x_i)^2 + (y - y_i)^2 \leq R_i^2, i = 1, 2, \cdots, N.$$

Problems of a similar nature have been considered by Knight [1], Craggs [2], and Cristal [3]. We would also like to calculate the elements of the capacitance matrix $\{C_{ij}\}$ (which is independent of the boundary conditions of Eq. (3)) such that if

$$Q_i = -\epsilon \oint_{i\text{th}} \frac{\partial V}{\partial n} \, dl$$

is the charge (in MKS units) on the ith conductor, then

$$(4) \qquad\qquad\qquad Q_i = \sum_{j=1}^{N} C_{ij} V_j .$$

(The constant ϵ is the electrical permittivity constant.) We may obtain the jth column of this matrix by noting that $Q_i = C_{ij}$ for the special case where the boundary conditions of Eq. (3) are chosen such that $V_i = \delta_{ij}$ where δ_{ij} is the Dirac delta function.

   2. **The Corresponding Integral Equation.** In the usual manner we may convert the above Dirichlet problem to an integral equation [4]. Accordingly let $\sigma_i(\theta)$ be the angular charge density, i.e. $-\epsilon R_i$ times the outward normal derivative of $V(x, y)$, at the angular coordinate θ on the surface of the ith conductor. (Here θ is measured counterclockwise from the ray drawn from (x_i, y_i) to $(+\infty, y_i)$.) Then we have

$$(5) \qquad V(x, y) = \sum_{i=1}^{N} \int_{\theta=0}^{2\pi} G(x, y; x_i + R_i \cos\theta, y_i + R_i \sin\theta)\sigma_i(\theta) d\theta$$

where

$$(6) \qquad G(x, y; x', y') = -\frac{1}{4\pi\epsilon} \log \frac{\cosh \pi(x - x') - \cos \pi(y - y')}{\cosh \pi(x - x') - \cos \pi(y + y')}$$

is the Green's function for the region between the ground lines [5], i.e. $G(x, y; x', y')$ is the potential at the point (x, y) due to a unit charge at the point (x', y') in the presence of the ground lines at $y = 0, y = 1$. By construction the $V(x, y)$ of Eq. (5) satisfies Eq. (1) and the boundary conditions of Eq. (2) for any choice of the functions $\sigma_i(\theta)$; we shall select the latter in such a way as to insure that the boundary conditions of Eq. (3) are also satisfied. The boundary conditions of Eq. (3) and the integral relation of Eq. (5) imply that the functions $\sigma_i(\theta)$ have continuous derivatives of all orders, and consequently we expect the Fourier series

$$(7) \qquad\qquad \sigma_i(\theta) = a_{i0} + \sum_{m=1}^{\infty} [a_{im} \cos m\theta + b_{im} \sin m\theta]$$

to converge rapidly. After substituting Eq. (7) into Eq. (5) and interchanging the summation and integration processes, we obtain

$$(8) \qquad \begin{aligned} V(x, y) = \sum_{i=1}^{N} \Big\{ &a_{i0} A_0(x, y; x_i, y_i) \\ &+ \sum_{m=1}^{\infty} R_i{}^m [a_{im} A_m(x, y; x_i, y_i) + b_{im} B_m(x, y; x_i, y_i)] \Big\} \end{aligned}$$

where

$$(9) \qquad \begin{aligned} A_m(x, y; x_0, y_0) &\equiv R^{-m} \int_0^{2\pi} G(x, y; x_0 + R\cos\theta, y_0 + R\sin\theta) \cos m\theta\, d\theta , \\ B_m(x, y; x_0, y_0) &\equiv R^{-m} \int_0^{2\pi} G(x, y; x_0 + R\cos\theta, y_0 + R\sin\theta) \sin m\theta\, d\theta . \end{aligned}$$

(It will be seen later that the functions A_m, B_m as defined here are independent of R for $R > 0$.) It will be noted that each of the functions $A_m(x, y; x_0, y_0)$, $m = 0, 1,$

2, $\cdots$ and $B_m(x, y; x_0, y_0)$, $m = 1, 2, 3, \cdots$ satisfies the boundary conditions of Eq. (2), and each satisfies Laplace's equation at any point (x, y) with $0 < y < 1$ and $(x, y) \neq (x_0, y_0)$.

3. Numerical Determination of a_{im}, b_{im}. Upon truncating the infinite series of Eq. (8) we obtain the Mth order approximation

(10)
$$V^{(M)}(x, y) = \sum_{i=1}^{N} \left\{ a_{i0}^{(M)} A_0(x, y; x_i, y_i) \right.$$
$$\left. + \sum_{m=1}^{M} R_i{}^m [a_{im}^{(M)} A_m(x, y; x_i, y_i) + b_{im}^{(M)}(x, y; x_i, y_i)] \right\}$$

so that $V^{(M)}$ is a finite linear combination of the functions A_m, B_m and therefore satisfies the boundary conditions of Eq. (2) and satisfies Laplace's equation everywhere in the strip $0 < y < 1$ except at the points (x_i, y_i), $i = 1, 2, \cdots, N$. We shall force $V^{(M)}$ to satisfy approximately the remaining boundary conditions of Eq. (3) by choosing the Fourier coefficients $a_{im}^{(M)}$, $b_{im}^{(M)}$ such that

(11)
$$\sum_{i=1}^{N} \sum_{k=1}^{4M} \left\{ V^{(M)}(X_{ik}, Y_{ik}) - V_i \right\}^2 = \text{Minimum},$$

where

(12)
$$X_{ik} = x_i + R_i \cos(k\pi/2M),$$
$$\qquad\qquad\qquad\qquad i = 1, 2, \cdots, N; k = 1, 2, \cdots, 4M,$$
$$Y_{ik} = y_i + R_i \sin(k\pi/2M),$$

is a set of equally spaced test points chosen on the surfaces of the N conductors. The least squares condition of Eq. (11) gives rise to a system of $N \cdot (2M + 1)$ linear equations in the $N \cdot (2M + 1)$ unknowns $a_{im}^{(M)}$, $b_{im}^{(M)}$ which are thereby determined. A total of $4NM$ test points is specified by Eq. (12), since the use of approximately twice as many test points as unknown Fourier coefficients makes a near optimum use of the smoothing effect of the least squares process.

Numerically this computational procedure has been found to be very stable, and as $M \to \infty$ the Mth order potential function $V^{(M)}$ rapidly converges to the solution of the Dirichlet problem of Eqs. (1)–(3). Likewise, the Mth order charge densities $\sigma_i{}^{(M)}$ converge to their limiting forms σ_i, thus satisfying the integral equation associated with the Dirichlet problem. The Mth order estimate of the charge on the ith conductor is given by the integral of $\sigma_i{}^{(M)}$, i.e.

$$Q_i^{(M)} = \int_0^{2\pi} \left\{ a_{i0}^{(M)} + \sum_{m=1}^{M} [a_{im}^{(M)} \cos m\theta + b_{im}^{(M)} \sin m\theta] \right\} d\theta = 2\pi a_{i0}^{(M)}$$

and by using the boundary conditions $V_i = \delta_{ij}$ we may thereby obtain an Mth order estimate of the (i, j)-element of the capacitance matrix of Eq. (4).

The extent to which the approximate potential function $V^{(M)}$ satisfies the boundary condition of Eq. (3) on the selected test points (X_{ik}, Y_{ik}) gives a good indication of the degree of convergence attained for a particular value of M. Another useful test of convergence may be used when the complete capacitance matrix of Eq. (4) is to be determined for a system of $N \geq 2$ conductors. The exact matrix is symmetric [6], and the extent to which $C_{ij}^{(M)}$ agrees with $C_{ji}^{(M)}$, $i \neq j = 1, 2, \cdots, N$ gives a good indication of the accuracy of the Mth order estimates.

To give some idea of the convergence, it will be noted that if the smallest distance

between any pair of conducting surfaces (ground lines or circular conductors) exceeds about .1, then the use of $M = 6$ will give approximations to $V(x, y)$, $\sigma_i(\theta)$, and Q_i, $i = 1, 2, \cdots, N$ which are all accurate to about 6 decimal places. When one of the circular conductors is closer than about .1 to one of the ground planes or to another circular conductor having a different boundary condition, then the corresponding exact charge density $\sigma_i(\theta)$ will be quite peaked and therefore a greater number of terms of the Fourier series will be needed to obtain a 6-place approximation to $\sigma_i(\theta)$. This in turn requires the use of a correspondingly larger M to obtain 6-place approximations to $V(x, y)$ and the charges Q_i, $i = 1, 2, \cdots, N$.

4. Numerical Evaluation of A_m, B_m. The preceding analysis requires the determination of each of the functions A_m, B_m, $m = 0, 1, 2, \cdots, M$, at each of the $4N^2M$ arguments $(X_{ik}, Y_{ik}; x_j, y_j)$, $i, j = 1, 2, \cdots, N$; $k = 1, 2, \cdots, 4M$ in order to determine the $N(2M + 1)$ Fourier coefficients $a_{im}^{(M)}$, $b_{im}^{(M)}$. In addition these functions must be evaluated at the N arguments $(x, y; x_i, y_i)$, $i = 1, 2, \cdots, N$ whenever the potential $V^{(M)}(x, y)$ is calculated. The integral forms given in Eq. (9) are unsuitable for numerical computation, and consequently a more expeditious means for numerically evaluating A_m, B_m will now be presented.

We first note that the Green's function of Eq. (6) can be written in the form [5]

$$G(x, y; x', y') = -\frac{1}{4\pi\epsilon} \sum_{n=-\infty}^{\infty} \log \frac{(x - x')^2 + (y - y' - 2n)^2}{(x - x')^2 + (y + y' - 2n)^2}.$$

The two terms of the summation corresponding to n and $-n$, taken together, give the expression

$$\log \frac{[(x - x')^2 + (y - y' - 2n)^2][(x - x')^2 + (y - y' + 2n)^2]}{[(x - x')^2 + (y + y' - 2n)^2][(x - x')^2 + (y + y' + 2n)^2]}$$

$$= \log \frac{\left[1 + \left(\dfrac{x - x' + i(y - y')}{2n}\right)^2\right]\left[1 + \left(\dfrac{x - x' - i(y - y')}{2n}\right)^2\right]}{\left[1 + \left(\dfrac{x - x' + i(y + y')}{2n}\right)^2\right]\left[1 + \left(\dfrac{x - x' - i(y + y')}{2n}\right)^2\right]}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k C_k(x, y; x', y')}{k(2n)^{2k}}$$

where

$$C_k(x, y; x', y') = [(x - x') + i(y + y')]^{2k} + [(x - x') - i(y + y')]^{2k}$$
$$- [(x - x') + i(y - y')]^{2k} - [(x - x') - i(y - y')]^{2k}.$$

Since $0 < y, y' < 1$, a sufficient condition for the absolute convergence of the series is that $(x - x')^2 < 4(n^2 - 1)$. Thus for $(x - x')^2 < 140$ we have

(13)
$$G(x, y; x', y') = -\frac{1}{4\pi\epsilon} \sum_{n=-5}^{5} \log \frac{(x - x')^2 + (y - y' - 2n)^2}{(x - x')^2 + (y + y' - 2n)^2}$$

$$+ \sum_{k=1}^{\infty} \alpha_k C_k(x, y; x', y')$$

where

(14)
$$\alpha_k = \frac{(-1)^k}{k4^k} \sum_{n=6}^{\infty} n^{-2k}$$

can be computed from the Riemann Zeta Function or the Bernoulli numbers [7]. We may bound the kth term of the remaining infinite series by

$$|\alpha_k C_k(x, y; x', y')| < 4[(x - x')^2 + 4]^k |\alpha_k|$$

$$= \frac{4}{k} \left\{ \frac{(x - x')^2 + 4}{144} \right\}^k \left\{ 1 + \sum_{n=7}^{\infty} \left(\frac{6}{n} \right)^{2k} \right\}$$

$$< \frac{26.2}{k} \left\{ \frac{(x - x')^2 + 4}{144} \right\}^k$$

so that for $|x - x'| \leq 3$ the use of terms through $k = 10$ in Eq. (13) will give the Green's function to better than 10-place accuracy. We shall use this form of the Green's function to evaluate the integrals of Eq. (9).

Using [8] with a bit of reduction we obtain

$$\int_0^{2\pi} \log\left[(x - x_0 - R\cos\theta)^2 + (y \pm y_0 \pm R\sin\theta - 2n)^2\right] \cos m\theta d\theta$$

$$= \begin{cases} 2\pi \log\left[(x - x_0)^2 + (y \pm y_0 - 2n)^2\right] & \text{for } m = 0 \\ \pm 2\pi \dfrac{R^m}{m} \dfrac{\cos\{m \operatorname{Arctan}\left[(y \pm y_0 - 2n)/(x - x_0)\right]\}}{\left[(x - x_0)^2 + (y \pm y_0 - 2n)^2\right]^{m/2}} & \text{for } m = 1, 2, 3, \cdots \end{cases}$$

and

$$\int_0^{2\pi} \log\left[(x - x_0 - R\cos\theta)^2 + (y \pm y_0 \pm R\sin\theta - 2n)^2\right] \sin m\theta d\theta$$

$$= -2\pi \frac{R^m}{m} \frac{\sin\{m \operatorname{Arctan}\left[(y \pm y_0 - 2n)/(x - x_0)\right]\}}{\left[(x - x_0)^2 + (y \pm y_0 - 2n)^2\right]^{m/2}} \quad \text{for } m = 1, 2, 3, \cdots .$$

Since

$$C_k(x, y; x_0 + R\cos\theta, y_0 + R\sin\theta)$$

$$= [(x - x_0) + i(y + y_0) - Re^{-i\theta}]^{2k} + [(x - x_0) - i(y + y_0) - Re^{i\theta}]^{2k}$$

$$- [(x - x_0) + i(y - y_0) - Re^{i\theta}]^{2k} - [(x - x_0) - i(y - y_0) - Re^{-i\theta}]^{2k}$$

we also have

$$\int C_k(x, y; x_0 + R\cos\theta, y_0 + R\sin\theta)e^{im\theta}d\theta$$

$$= 2\pi(-R)^m \binom{2k}{m} \sum_{l=0}^{2k-m} i^l \binom{2k - m}{l} (x - x_0)^{2k-m-l}\{(y_0 + y)^l - (y_0 - y)^l\}$$

giving both the $\sin m\theta$ and $\cos m\theta$ integrals of C_k. Upon combining the above results, we obtain the final forms

(15) $\quad A_0(x, y; x_0, y_0) = -\dfrac{1}{2\epsilon} \log \dfrac{\cosh \pi(x - x_0) - \cos \pi(y - y_0)}{\cosh \pi(x - x_0) - \cos \pi(y + y_0)},$

$$A_m(x, y; x_0, y_0) = \frac{1}{2m\epsilon} \sum_{n=-5}^{5} \left\{ \frac{\cos m \; \text{Arctan} \; [(y - y_0 - 2n)/(x - x_0)]}{[(x - x_0)^2 + (y - y_0 - 2n)^2]^{m/2}} \right.$$

$$\left. - \frac{\cos m \; \text{Arctan} \; [(y + y_0 - 2n)/(x - x_0)]}{[(x - x_0)^2 + (y + y_0 - 2n)^2]^{m/2}} \right\}$$

(16)

$$- \frac{1}{2\epsilon} \sum_{k=1}^{\infty} \alpha_k \binom{2k}{m} \sum_{l=0(2)2k-m} (-1)^{l/2} \binom{2k - m}{l}$$

$$\times (x_0 - x)^{2k-m-l} \{(y_0 + y)^l - (y_0 - y)^l\}$$

$$\text{for } m = 1, 2, 3, \cdots,$$

$$B_m(x, y; x_0, y_0) = \frac{1}{2m\epsilon} \sum_{n=-5}^{5} \left\{ \frac{\sin m \; \text{Arctan} \; [(y - y_0 - 2n)/(x - x_0)]}{[(x - x_0)^2 + (y - y_0 - 2n)^2]^{m/2}} \right.$$

$$\left. + \frac{\sin m \; \text{Arctan} \; [(y + y_0 - 2n)/(x - x_0)]}{[(x - x_0)^2 + (y + y_0 - 2n)^2]^{m/2}} \right\}$$

(17)

$$- \frac{1}{2\epsilon} \sum_{k=1}^{\infty} \alpha_k \binom{2k}{m} \sum_{l=1(2)2k-m} (-1)^{(l+1)/2} \binom{2k - m}{l}$$

$$\times (x_0 - x)^{2k-m-l} \{(y_0 + y)^l - (y_0 - y)^l\}$$

$$\text{for } m = 1, 2, 3, \cdots.$$

These expressions are well suited to numerical computation when $|x - x_0| \leqq 3$ in which case the infinite series remaining in Eqs. (16)–(17) are rapidly convergent.
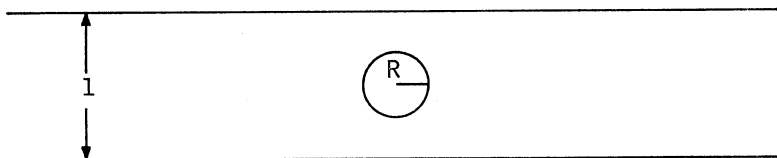

FIGURE 2. *Capacitance for a Circular Conductor Centered between Parallel Ground Lines.*

R	C/ϵ	M
.05	2.46967	1
.10	3.39473	2
.15	4.34868	2
.20	5.43684	2
.25	6.76127	4
.30	8.48134	6
.35	10.91395	6
.40	14.86438	8
.45	23.49826	12

Note. ϵ is the permittivity constant for the medium between the ground lines. For free space $\epsilon = 8.85417 \; 10^{-12}$ farad/meter (MKS) or $1/4\pi$ (Gaussian).

5. Cost of Computation. The calculational procedure which has been described provides a relatively inexpensive alternative to the finite-difference methods which

may also be used to solve boundary value problems of the type considered. Almost all of the computational time required to calculate the N^2 elements of the capacitance matrix $\{C_{ij}\}$ is spent evaluating the $4N^2M(2M+1)$ elements $A_m(X_{ik}, Y_{ik}; x_j, y_j)$, $B_m(X_{ik}, Y_{ik}; x_j, y_j)$ which are needed for the analysis. (It will be noted that these arrays are independent of the boundary conditions of Eq. (3) so that once these arrays have been computed the solution for an arbitrary choice of V_i may be obtained by solving an $N^2(2M+1)$th order system of linear equations.) Thus for a fixed M we see that the computational time is roughly proportional to N^2. From

$$1$$
$$\circ$$

$$C_{11}/\epsilon = 3.39473$$

$$1 \quad 2$$
$$\circ \quad \circ$$

$$C_{11}/\epsilon = C_{22}/\epsilon = 3.88153$$

$$C_{12}/\epsilon = C_{21}/\epsilon = -1.26037$$

$$1 \quad 2 \quad 3$$
$$\circ \quad \circ \quad \circ$$

$$C_{11}/\epsilon = C_{33}/\epsilon = 3.88184$$

$$C_{22}/\epsilon = 4.35963$$

$$C_{12}/\epsilon = C_{21}/\epsilon = C_{23}/\epsilon = C_{32}/\epsilon = -1.24832$$

$$C_{13}/\epsilon = C_{31}/\epsilon = -.03371$$

$$1 \quad 2 \quad 3 \quad 4$$
$$\circ \quad \circ \quad \circ \quad \circ$$

$$C_{11}/\epsilon = C_{44}/\epsilon = 3.88184$$

$$C_{22}/\epsilon = C_{33}/\epsilon = 4.35993$$

$$C_{12}/\epsilon = C_{21}/\epsilon = C_{34}/\epsilon = C_{43}/\epsilon = -1.24830$$

$$C_{23}/\epsilon = C_{32}/\epsilon = -1.23651$$

$$C_{13}/\epsilon = C_{31}/\epsilon = C_{24}/\epsilon = C_{42}/\epsilon = -.03304$$

$$C_{14}/\epsilon = C_{41}/\epsilon = -.00188$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5$$
$$\circ \quad \circ \quad \circ \quad \circ \quad \circ$$

$$C_{11}/\epsilon = C_{55}/\epsilon = 3.88184$$

$$C_{22}/\epsilon = C_{44}/\epsilon = 4.35993$$

$$C_{33}/\epsilon = 4.36023$$

$$C_{12}/\epsilon = C_{21}/\epsilon = C_{45}/\epsilon = C_{54}/\epsilon = -1.24830$$

$$C_{23}/\epsilon = C_{32}/\epsilon = C_{34}/\epsilon = C_{43}/\epsilon = -1.23649$$

$$C_{13}/\epsilon = C_{31}/\epsilon = C_{35}/\epsilon = C_{53}/\epsilon = -.03303$$

$$C_{24}/\epsilon = C_{42}/\epsilon = -.03238$$

$$C_{14}/\epsilon = C_{41}/\epsilon = C_{25}/\epsilon = C_{52}/\epsilon = -.00185$$

$$C_{15}/\epsilon = C_{51}/\epsilon = -.00011$$

FIGURE 3. *Capacitance Matrices for Systems of 1, 2, 3, 4, and 5 Circular Conductors*

Note. The separation of the ground lines is 1., the radius of each conductor is $R = .1$, and the center of the ith conductor is given by $(x_i, y_i) = (.4i, .5)$, $i = 1$, $\cdots, 5$.

experience with the technique, we might note that the use of $M = 6$ will give 6-place approximations to the capacitances C_{ij} (assuming that the conductors are separated by a distance of at least .1), and in this case the cost of the computation is about 15 N^2 seconds on the IBM 7044 computer.

6. Sample Results. The calculation of the capacitance of a single circular conductor of radius R with center at (0, .5) has been undertaken using the above numerical process. Figure 2 presents the capacitance as a function of R for selected values of R along with the value of M needed to achieve convergence to within $\pm 5 \cdot 10^{-6}$. These results when rounded to 4 significant figures agree precisely with those published by Knight [1]. The capacitance matrices for systems of 1, 2, 3, 4, and 5 equally spaced centered conductors having a common radius $R = .1$ and center-to-center spacing $d = .4$ are presented in Figure 3.

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