

On the Series Expansion Method for Computing Incomplete Elliptic Integrals of the First and Second Kinds

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Abstract. In the present paper an attempt is made to improve the series expansion method for computing the incomplete integrals $F(\phi, k)$ and $E(\phi, k)$. Therefore the following three pairs of series covering the region $-1 \leq k \leq 1, 0 \leq \phi < \pi/2$ are used: series obtained by a straightforward binomial expansion of the integrands, series valid for $k'^2 \tan^2 \phi < 1$, and new series which converge for $\phi > \pi/4$ and for all values of k . Terms of the last two pairs of series can be generated by means of the same recurrence relations, so that the coding of the whole is not longer than that for similar methods using only two pairs of series. Any degree of accuracy can be obtained. In general the method is a little bit slower than Bulirsch' calculation procedures which are based on the Landen transformation, but it works more quickly in case of large values of k^2 and/or ϕ . The new series introduced are also represented in trigonometric form, and the double passage to the limit $k^2 \rightarrow 1, \phi \rightarrow \pi/2$ is discussed. ■

1. Introduction. The incomplete elliptic integrals of the first and second kinds are defined by

$$(1.1) \quad F(\phi, k) = \int_0^\phi (1 - k^2 \sin^2 \theta)^{-1/2} d\theta,$$

$$(1.2) \quad E(\phi, k) = \int_0^\phi (1 - k^2 \sin^2 \theta)^{1/2} d\theta,$$

respectively. We assume that the modulus k and the amplitude ϕ take values in the ranges $0 \leq k^2 \leq 1, 0 \leq \phi < \pi/2$. The comodulus k' is given by $k' = (1 - k^2)^{1/2}$. If $k^2 = 1$, (1.1) and (1.2) reduce to the well-known formulas

$$(1.3) \quad F(\phi, \pm 1) = \log \tan (\pi/4 + \phi/2) = \log ((1 + \sin \phi)/\cos \phi),$$

$$(1.4) \quad E(\phi, \pm 1) = \sin \phi.$$

For the numerical calculation of these integrals two methods are in common use. The first is based on the well-known Landen transformation and seems to have been used first by A. M. Legendre [1] for the construction of tables. The second is the series expansion method. A binomial expansion of the integrands in (1.1) and (1.2) provides series which are suitable for machine computation if $k^2 \sin^2 \phi$ is sufficiently small (see Section 2). We will call them the classical series. If $k^2 \sin^2 \phi$ is large, other series must be used. E. L. Kaplan [2] derived series valid when k^2 and $\sin^2 \phi$ are both close to 1; they are in powers of k'/k^2 and served for interpolation in tables of incomplete elliptic integrals. A. R. DiDonato and A. V. Hershey [3] rejected these

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series as tools for numerical calculation with the argument that the rate of convergence deteriorates rapidly as k'/k^2 approaches unity. Instead both authors derived new series; though very complicated, they can be coded by means of relatively simple recurrence relationships. A combination of the classical series, applied when $k^2 \sin^2 \phi \leq \frac{1}{2}$, and these new series, applied in the complementary region, turned out to be a more accurate and a faster computation method for $F(\phi, k)$ and $E(\phi, k)$ than Legendre's method. Later on G. E. Lee-Whiting [4] showed that the rate of convergence of Kaplan's series is independent of the value of k'/k^2 ; it was found that slightly modified forms of these series are completely satisfactory for numerical calculations in the region $k^2 \sin^2 \phi > \frac{1}{2}$. Moreover they are much simpler than the corresponding series of [3]. However, for the computation of the complete elliptic integrals $K = F(\pi/2, k)$ and $E = E(\pi/2, k)$ appearing in Kaplan's series, C. Hastings' [5] method of polynomial approximation is suggested, which has a fixed limit of accuracy. In both references [3] and [4] other series for $F(\phi, k)$ and $E(\phi, k)$ are discussed, which have been given by B. Radon [6]; these series are obtained from differential equations having $F(\phi, k)$ and $E(\phi, k)$ as their solutions. They are much more complicated than the series already mentioned and their regions of practical convergence do not cover the whole of the region $k^2 \sin^2 \phi > \frac{1}{2}$.

Returning to the method based on the Landen transformation, D. J. Hofsommer and R. P. van de Riet [7] have combined this method with Gauss' theory of the arithmetico-geometric means, with the result that more compact and faster programs for the computation of $F(\phi, k)$ and $E(\phi, k)$ were obtained, compared with those for the methods of [3] and [4]. Both authors apply the Landen transformation four times in the downward direction if $k' \geq 0.3$ and three times in the upward direction if $k' < 0.3$; the results always have a relative error of the order 10^{-12} . In its turn, Hofsommer and van de Riet's method has been improved (and extended to the cases of more general elliptic integrals) by R. Bulirsch [8]; in the calculation procedures devised by this author the number of iterative cycles is not predetermined so that any degree of accuracy can be obtained. These procedures are shorter than the corresponding ones of [7] and faster when $k' > 10^{-2}$.

The series expansion method also has the advantage that a freely chosen accuracy of results can be imposed. The main purpose of the present paper is to establish an improved version of this method (Section 2). It consists essentially in a combination of three series for each of $F(\phi, k)$ and $E(\phi, k)$ instead of two: the classical series, series valid for $k'^2 \tan^2 \phi < 1$, and new series valid for $\phi > \pi/4$. The relevant formulas are (2.1) and (2.2), (2.4) and (2.6), (2.10) and (2.13), respectively. Terms of the latter two pairs of series can be generated by the same recurrence relations (2.9), with the result that the coding of the whole is not significantly longer than that for the method of [4], while the average computation time is shorter. The regions of application of all series are discussed. The idea of using three pairs of series dates back to S. C. van Veen [9], who obtained series expansions for $F(\phi, k)$ and $E(\phi, k)$ after having first applied the Landen transformation a number of times. His formulas, however, are very complicated and their practical use is restricted to the computation of the leading terms.

The $F(\phi, k)$ and $E(\phi, k)$ series, as represented in Section 2, can be put in trigonometric form. This can be done in several ways, but preference is given to the representation as appearing in [10]. In Section 3, the list of formulas for $F(\phi, k)$ and

$E(\phi, k)$ of this reference is corrected for the $E(\phi, k)$ series valid for $k'^2 \tan^2 \phi < 1$, and complemented by our new series, for which it is shown that they can be brought into trigonometric forms similar to the other ones.

2. Numerical Calculation of $F(\phi, k)$ and $E(\phi, k)$. A binomial expansion of the integrands in (1.1) and (1.2) yields the classical series

$$(2.1) \quad F(\phi, k) = \sum_{n=0}^{\infty} \left| \binom{-\frac{1}{2}}{n} \right| k^{2n} I_n,$$

$$(2.2) \quad E(\phi, k) = - \sum_{n=0}^{\infty} \left| \binom{-\frac{1}{2}}{n} \right| \frac{k^{2n}}{2n-1} I_n,$$

where I_n stands for $\int_0^\phi \sin^{2n} \theta d\theta$. Using the recurrence relation

$$(2.3) \quad I_n = \frac{2n-1}{2n} I_{n-1} - \frac{\sin^{2n-1} \phi \cos \phi}{2n}, \quad n \geq 1,$$

these series can easily be programmed, as described in [3]. It follows from Schwarz' inequality that $I_{n+2}/I_{n+1} > I_{n+1}/I_n$. Since $I_{n+1} = \sin^2 \xi_n \cdot I_n$, where $0 < \xi_n < \phi$, we have $\xi_{n+1} > \xi_n$ and therefore $\lim_{n \rightarrow \infty} \xi_n = \phi$. Consequently, the limiting value of the ratio of adjacent terms in the series (2.1) and (2.2) is $k^2 \sin^2 \phi$; hence these series are at least as convergent as a geometric series with common ratio $k^2 \sin^2 \phi$.

Now we derive two other pairs of series, for which the rate of convergence can be determined. The first pair is valid for $k'^2 \tan^2 \phi < 1$ and can be obtained by writing the integrands in (1.1) and (1.2) in the forms $[\cos \theta(1 + k'^2 \tan^2 \theta)^{1/2}]^{-1}$ and $\cos \theta(1 + k'^2 \tan^2 \theta)^{1/2}$, respectively. Applying a binomial expansion to the first integrand we obtain

$$(2.4) \quad F(\phi, k) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} k'^{2n} J_n,$$

where J_n stands for $\int_0^\phi \sin^{2n} \theta / \cos^{2n+1} \theta d\theta$. Between the integrals J_n the following recurrence relation holds:

$$(2.5) \quad J_n = \frac{1}{2n} \frac{\sin^{2n-1} \phi}{\cos \phi} - \frac{2n-1}{2n} J_{n-1}, \quad n \geq 1.$$

A similar formula for $E(\phi, k)$ can be obtained by first performing a partial integration on the second integrand; then applying a binomial expansion yields a series in terms of J_{n+1} ; these terms can be reduced to J_n by using the relation (2.5). The result is

$$(2.6) \quad E(\phi, k) = \sin \phi \left[1 - \frac{k'^2}{1 + (1 + k'^2 \tan^2 \phi)^{1/2}} \right] + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{2n+1}{2n+2} k'^{2n+2} J_n.$$

In the same way as done for the classical series we can show that the limiting value of the ratio of adjacent terms in the series (2.4) and (2.6) is $-k'^2 \tan^2 \phi$; hence these series are at least as convergent as a geometric series with common ratio $-k'^2 \tan^2 \phi$. They can be computed as follows. We set

$$q = k'^2 \tan^2 \phi,$$

$$A_n = \left(-\frac{1}{n} \right), \quad \text{with } A_0 = 1,$$

$$B_n = q^n \operatorname{cosec} \phi, \quad \text{with } B_0 = \operatorname{cosec} \phi,$$

$$C_n = k'^{2n} J_n, \quad \text{with } C_0 = \log((1 + \sin \phi)/\cos \phi);$$

then we have

$$(2.7) \quad F(\phi, k) = C_0 + \sum_{n=1}^{\infty} A_n C_n,$$

$$(2.8) \quad E(\phi, k) = \sin \phi \left[1 - \frac{k'^2}{1 + (1 + q)^{1/2}} \right] + \frac{k'^2 C_0}{2} \\ + \sum_{n=1}^{\infty} \frac{2n + 1}{2n + 2} A_n k'^2 C_n;$$

terms in these series can, by virtue of (2.5), be generated with the recurrence relations

$$(2.9) \quad A_n = -\frac{2n-1}{2n} A_{n-1}; \quad B_n = q B_{n-1}; \quad C_n = \frac{B_n}{2n} - \frac{2n-1}{2n} k'^2 C_{n-1}.$$

Next we derive series for $F(\phi, k)$ and $E(\phi, k)$ which are valid for large values of ϕ . As in [2], [3], [4] and [9], we start from the expressions $K - F(\phi, k)$ and $E - E(\phi, k)$. They can be written in the forms

$$\int_0^{\pi/2-\phi} [\cos \theta (k'^2 + \tan^2 \theta)^{1/2}]^{-1} d\theta \quad \text{and} \quad \int_0^{\pi/2-\phi} \cos \theta (k'^2 + \tan^2 \theta)^{1/2} d\theta,$$

respectively, where k' must be > 0 for the case of $F(\phi, k)$. If $\tan^{-1} k' > \pi/2 - \phi$, thus $k' \tan \phi > 1$ and $\phi > \pi/4$, the substitution $\tan \theta = k' \tan \psi$ and a binomial expansion (which is allowed) yields

$$(2.10) \quad K - F(\phi, k) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} k'^{2n} \int_0^u \frac{\sin^{2n} \theta}{\cos^{2n+1} \theta} d\theta,$$

$$(2.11) \quad E - E(\phi, k) = \sum_{n=0}^{\infty} \binom{-3/2}{n} k'^{2n+2} \int_0^u \frac{\sin^{2n} \theta}{\cos^{2n+3} \theta} d\theta,$$

where

$$(2.12) \quad u = \cot^{-1}(k' \tan \phi).$$

If $\tan^{-1} k' \leq \pi/2 - \phi$ and hence $k' \tan \phi \leq 1$, we divide the integration interval into $[0, \tan^{-1} k']$ and $[\tan^{-1} k', \pi/2 - \phi]$; performing the substitution $\tan \theta = k' \tan \psi$ in the first subinterval and $\tan \theta = k' \cot \psi$ in the second, and applying a binomial expansion (which is allowed if $\phi > \pi/4$), we obtain again the formulas (2.10) and (2.11). Hence these series are valid for $\phi > \pi/4$ and for all positive values of k' . Formula (2.11) can still be reduced by means of the recurrence relation

$$\int_0^u \frac{\sin^{2n} \theta}{\cos^{2n+3} \theta} d\theta = \frac{1}{2n+2} \left(\frac{\sin^{2n+1} u}{\cos^{2n+2} u} + \int_0^u \frac{\sin^{2n} \theta}{\cos^{2n+1} \theta} d\theta \right),$$

obtaining

$$(2.13) \quad E - E(\phi, k) = (1 - \sin \phi)(1 + k'^2 \tan^2 \phi)^{1/2} + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{2n+1}{2n+2} k'^{2n+2} \int_0^u \frac{\sin^{2n} \theta}{\cos^{2n+1} \theta} d\theta.$$

Formulas (2.10) and (2.13) are believed to be new. Again we can show that the limiting value of the ratio of adjacent terms is $-k'^2 \tan^2 u$, that is $-\cot^2 \phi$; hence it is independent of the value of k . The rate of convergence of these series is at least as good as that of a geometric series with common ratio $-\cot^2 \phi$. Setting

$$\begin{aligned} q &= \cot^2 \phi, \\ A_n &= -\binom{-\frac{1}{2}}{n}, \quad \text{with } A_0 = -1, \\ B_n &= (1 + k'^2 \tan^2 \phi)^{1/2} \cot^{2n} \phi, \quad \text{with } B_0 = (1 + k'^2 \tan^2 \phi)^{1/2}, \\ C_n &= k'^{2n} \int_0^u \frac{\sin^{2n} \theta}{\cos^{2n+1} \theta} d\theta, \quad \text{with } C_0 = \log \frac{1 + B_0}{k' \tan \phi}, \end{aligned}$$

we get

$$(2.14) \quad F(\phi, k) = K - C_0 + \sum_{n=1}^{\infty} A_n C_n,$$

$$(2.15) \quad E(\phi, k) = E - \left(\frac{B_0 \cos^2 \phi}{1 + \sin \phi} + \frac{k'^2 C_0}{2} \right) + \sum_{n=1}^{\infty} \frac{2n+1}{2n+2} A_n k'^2 C_n,$$

and terms in the series can be generated by the recurrence relations (2.9) already obtained for the series (2.7) and (2.8).

In the new series of DiDonato and Hershey, K and E are replaced by their appropriate series, valid for large values of $|k|$. (A derivation of these series can be found in [11].) The terms of these series are included in the series for $K - F(\phi, k)$ and $E - E(\phi, k)$. This procedure is not very well applicable for the case of the modified Kaplan series or for the series (2.14) and (2.15); indeed, these series converge independently of k'^2/k^2 and k respectively; if $k^2 \approx \frac{1}{2}$ and ϕ is large their convergence is much stronger than that of the series for K and E ; when a high degree of accuracy is required, underflows may then be generated (if numbers are represented in floating-point form). Therefore K and E must be calculated separately. As remarked in [7], this calculation can be performed very quickly with the process of the arithmetico-geometric means (see for example [12]) when $k' > 0.3$; for smaller values of k' the series expansions for K and E already mentioned can be used.

From the foregoing investigations we may conclude that there are three pairs of series available for the computation of $F(\phi, k)$ and $E(\phi, k)$, of which the rates of convergence are determined by the quantities $k^2 \sin^2 \phi$, $-k'^2 \tan^2 \phi$ and $-\cot^2 \phi$, respectively. It is easy to see that in the ranges $0 \leq k^2 \leq 1.0 \leq \phi < \pi/2$, always at least one of the absolute values of these quantities is smaller than or equal to $\frac{1}{2}$. Hence for every pair of values (ϕ, k) the incomplete integrals can be computed by means of series which are at least as convergent as a geometric series with common ratio $\pm \frac{1}{2}$. The boundaries of the subregions wherein each couple of series can be applied may then be fixed by the following procedure:

series (2.1) and (2.2) if $k^2 \sin^2 \phi \leq k'^2 \tan^2 \phi$ and $k^2 \sin^2 \phi \leq \cot^2 \phi$;

series (2.7) and (2.8) if $k'^2 \tan^2 \phi < k^2 \sin^2 \phi$ and $k'^2 \tan^2 \phi \leq \cot^2 \phi$;
 series (2.14) and (2.15) if $\cot^2 \phi < k^2 \sin^2 \phi$ and $\cot^2 \phi < k'^2 \tan^2 \phi$.

If $k = 0$, it is seen that the classical series will be applied, which lead to the result $F(\phi, 0) = E(\phi, 0) = \phi$. If $k^2 = 1$, the series (2.7) and (2.8) will be used, which clearly reduce in that case to the formulas (1.3) and (1.4). The case $k^2 = 1$ does not allow the use of the modified Kaplan series [4] or the procedures of R. Bulirsch [8] without special precautions. The calculation of the difference between K and the leading term in the series (2.14) does not involve a loss of significant figures, since in its region of application the value of this leading term never exceeds 53 percent of the value of K . The same is true for the difference between E and the leading terms in Eq. (2.15); there the value of the leading terms never exceeds 31 percent of the value of E . Also the first term in the right member of Eq. (2.8) cannot cause a loss of significant figures, since $k'^2/(1 + (1 + q)^{1/2}) \leq \frac{1}{4}$.

Fortran IV double precision routines were written for the IBM-360 computer which compute $F(\phi, k)$ and $E(\phi, k)$ as explained above within an arbitrary chosen relative error (the choice of this error being restricted to the minimum value 10^{-16} , due to the fact that the double precision floating-point numbers are 16 decimal digits long) and which call subroutines for the computation of K and E . The input parameters of these routines are: (i) $\sin^{-1} k$, with $-1 \leq k \leq 1$; this permits the computation of k and k' as exact as possible; (ii) $x = \tan \phi$, with $0 \leq x^2 \leq 10^{75}$ (for the choice of x as input parameter see the remarks in [4] and [8]); (iii) the desired relative accuracy ϵ ; the series are truncated when the absolute values of the magnitudes of the last terms included in both the $F(\phi, k)$ and $E(\phi, k)$ series are less than ϵ times the appropriate partial sum (leading terms included). Computations for $\phi = 1^\circ(1)89^\circ$, $\sin^{-1} k = 1^\circ(1)89^\circ$ took a time which is 40 percent and 20 percent shorter than for the methods of [3] and [4], respectively, which were also run on this computer. The rate of convergence of the modified Kaplan series is determined by the quantity $\cos^2 \phi$ and hence is slightly better than that of the series (2.14) and (2.15). But the region of application of the former series (which has to be chosen such that $k'^2/k^2 \leq 1$) is considerably smaller than that of the latter. A combination of the classical series, the series (2.7) and (2.8), and the modified Kaplan series was also tried on the IBM-360 computer, but turned out to be inferior to our method from the standpoint of compactness and efficiency.

An overall comparison in efficiency of our $E(\phi, k)$ program with Bulirsch' el2(x, k', a, b)-procedure (which gives $E(\phi, k)$ for $a = 1$ and $b = k'^2$) showed that the latter is a little bit faster. The el1(x, k')-procedure for the computation of $F(\phi, k)$, which results from the el2-procedure by taking $a = b = 1$, constitutes a considerable simplification with the result that it is about two times faster than our $F(\phi, k)$ program. However, results are obtained more quickly with our series expansion method in some cases, i.e. for large values of $|k|$ (in [8] the Landen transformation is applied always in the downward direction, that is, $|k|$ is decreased) and/or for ϕ close to $\pi/2$, since then our second and/or third pair of series converge very strongly.

By way of example, numerical values of terms of the series (2.14) and (2.15) for $\phi = 85^\circ$ ($\cot^2 \phi = 0.00765426625$) and $\sin^{-1} k = 20^\circ, 40^\circ, 60^\circ, 80^\circ$ are given in the tables below; the relative error ϵ in the results for $F(\phi, k)$ and $E(\phi, k)$ is chosen to be $\leq 10^{-10}$; numbers in the first column refer to the leading terms (K or E included) ($n = 0$), the first term ($n = 1$), etc. in both series. It is seen that for all values of k the same number of terms is needed to obtain the imposed accuracy.

Table for $F(85^\circ, k)$

$n \backslash \sin^{-1} k$	20°	40°	60°	80°
0	1.5270563936	1.6728076300	1.9824191203	2.6687529577
1	0.0001184657	0.0001451307	0.0002212032	0.0005995196
2	-0.0000004078	-0.0000004995	-0.0000007606	-0.0000020385
3	0.0000000019	0.0000000023	0.0000000035	0.0000000092
$F(\phi, k)$	1.5271744534	1.6729522635	1.9826395664	2.6693504480

Table for $E(85^\circ, k)$

$n \backslash \sin^{-1} k$	20°	40°	60°	80°
0	1.4417036319	1.3261670261	1.1672161510	1.0243504244
1	0.0000784559	0.0000638746	0.0000414756	0.0000135583
2	-0.0000003001	-0.0000002443	-0.0000001585	-0.0000000512
3	0.0000000014	0.0000000012	0.0000000008	0.0000000002
$E(\phi, k)$	1.4417817891	1.3262306576	1.1672574689	1.0243639317

3. Trigonometric Series for $F(\phi, k)$ and $E(\phi, k)$. The classical series and the two other pairs of series discussed in the previous section can be carried out by means of their appropriate recurrence relations (2.3) and (2.5), respectively, providing series for $F(\phi, k)$ and $E(\phi, k)$ in trigonometric form. These transformations can be performed in several ways and, as to the first two pairs of series, we may (among others) refer for that purpose to [10], [13] and [14]. In the author's opinion the series mentioned in [10] seem to be the most attractive ones, in particular if ϕ and k are related to each other by means of functions of the same variable, say x , and one wishes to obtain series developments for $f_1(x) \equiv F(\phi(x), k(x))$ and $f_2(x) \equiv E(\phi(x), k(x))$ in powers of x .

The classical series in [10] are given by the formulas (8.117). The series for $F(\phi, k)$ which is valid for $k'^2 \tan^2 \phi < 1$ (formula (8.118.1)) is

$$(3.1.a) \quad F(\phi, k) = \frac{2}{\pi} K' \log \frac{1 + \sin \phi}{\cos \phi} - \frac{\tan \phi}{\cos \phi} \left(a_0' - \frac{2}{3} a_1' \tan^2 \phi + \frac{2 \cdot 4}{3 \cdot 5} a_2' \tan^4 \phi - \dots \right),$$

where

$$a_n' = \sum_{m=n+1}^{\infty} \binom{-\frac{1}{2}}{m}^2 k'^{2m}, \quad a_0' = \frac{2}{\pi} K' - 1,$$

and $K' = F(\pi/2, k')$. The corresponding formula for $E(\phi, k)$ in this reference is not correct. One can show that the correct formula is

$$(3.1.b) \quad E(\phi, k) = \frac{2}{\pi} (K' - E') \log \frac{1 + \sin \phi}{\cos \phi} + (1 + k'^2 \tan^2 \phi)^{1/2} \sin \phi - \frac{\tan \phi}{\cos \phi} \left(b_0' - \frac{2}{3} b_1' \tan^2 \phi + \frac{2 \cdot 4}{3 \cdot 5} b_2' \tan^4 \phi - \dots \right),$$

where

$$b_n' = \sum_{m=n+1}^{\infty} \left(-\frac{1}{2} \right)^2 \frac{2m}{2m-1} k'^{2m}, \quad b_0' = \frac{2}{\pi} (K' - E'),$$

and $E' = E(\pi/2, k')$. If $k^2 \rightarrow 1$ Eqs. (3.1.a) and (3.1.b) reduce to formulas (1.3) and (1.4). The limit $\phi \rightarrow \pi/2$ cannot be performed with these series.

The new series (2.10) and (2.11) can be put in forms similar to (3.1). Indeed, it follows from Eqs. (2.4) and (2.10) that for $K - F(\phi, k)$ the series expansion (3.1.a) holds, where ϕ is replaced by u (see Eq. (2.12)). Then we find

$$(3.2.a) \quad K - F(\phi, k) = \frac{2}{\pi} K' \sinh^{-1} \left(\frac{1}{k' \tan \phi} \right) - (1 + k'^2 \tan^2 \phi)^{1/2} \\ \times \cot^2 \phi \left(c_0' - \frac{2}{3} c_1' \cot^2 \phi + \frac{2 \cdot 4}{3 \cdot 5} c_2' \cot^4 \phi - \dots \right),$$

where

$$c_n' = \frac{a_n'}{k'^{2n+2}} = \sum_{m=n+1}^{\infty} \left(-\frac{1}{2} \right)^2 k'^{2m-2n-2}.$$

It also follows from Eqs. (2.6) and (2.13) that for the expression $E - E(\phi, k) + k^2 \sin^2 \phi / (1 + k'^2 \tan^2 \phi)^{1/2}$ the series expansion (3.1.b) holds, where ϕ is replaced by u . We find

$$(3.2.b) \quad E - E(\phi, k) = \frac{2}{\pi} (K' - E') \sinh^{-1} \left(\frac{1}{k' \tan \phi} \right) \\ + (1 - k^2 \sin^2 \phi)^{1/2} \cot \phi - (1 + k'^2 \tan^2 \phi)^{1/2} \\ \times \cot^2 \phi \left(d_0' - \frac{2}{3} d_1' \cot^2 \phi + \frac{2 \cdot 4}{3 \cdot 5} d_2' \cot^4 \phi - \dots \right),$$

where

$$d_n' = \frac{b_n'}{k'^{2n+2}} = \sum_{m=n+1}^{\infty} \left(-\frac{1}{2} \right)^2 \frac{2m}{2m-1} k'^{2m-2n-2}.$$

With series (3.2.a) and (3.2.b) the double limit $k^2 \rightarrow 1$, $\phi \rightarrow \pi/2$ can be performed. (The sign "lim" will denote this double limit in what follows.) It is seen that $E - E(\phi, k) \rightarrow 0$, while $K - F(\phi, k) \sim \sinh^{-1}(1/k' \tan \phi)$. Hence $\lim [K - F(\phi, k)]$ exists if $\lim (k' \tan \phi) > 0$, and tends to infinity like $\log(2/k' \tan \phi)$ if $\lim (k' \tan \phi) = 0$. (The limiting value of $k' \tan \phi$ can be determined if it is assumed that $k \equiv k(x)$, $\phi \equiv \phi(x)$ and $k^2 \rightarrow 1$, $\phi \rightarrow \pi/2$ if $x \rightarrow x_0$.) This result can also be obtained directly from Kaplan's series [2].

Series for $F(\phi, k)$ and $E(\phi, k)$ can be obtained from (3.2.a) and (3.2.b) by replacing K and E by their appropriate series expansions [11]. Then the leading terms in the expressions for $F(\phi, k)$ and $E(\phi, k)$ are (if k^2 is close to 1)

$$\log \frac{4 \tan \phi}{1 + (1 + k'^2 \tan^2 \phi)^{1/2}} \quad \text{and} \quad 1,$$

respectively. If $\lim (k' \tan \phi)$ is finite we see that $F(\phi, k)$ tends to infinity like $\log [2/(\pi/2 - \phi)]$, in accordance with the behaviour of the function in (1.3) when $\phi \rightarrow \pi/2$. If $\lim (k' \tan \phi)$ is infinite, $\lim [K - F(\phi, k)] = 0$, and $F(\phi, k)$ tends to infinity like $\log(4/k')$. In all cases $E(\phi, k)$ tends to 1, in accordance with Eq. (1.4) when $\phi \rightarrow \pi/2$.

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