# On the Lattice Constant for $|x^3 + y^3 + z^3| \leqq 1$

By W. G. Spohn

**Abstract.** There has been no published work on this intractable problem in the Geometry of Numbers since 1946. In 1944 and 1946 L. J. Mordell and H. Davenport gave bounds for the lattice constant in the *Journal of the London Mathematical Society*. The present attack stems from considering natural lattices with 9 points on the boundary of the region. The points of these lattices which are interior to the region are removed in the most efficient way by applying a convergent linear programming process. Apparently an infinite number of points must be removed in an infinite number of stages. A conjecture is made about the critical lattices for the region and the conjectured value .948754. . . . is given for the lattice constant. ■

**1. Introduction.** Let

$$(1.1) \quad x = \xi_1 u + \xi_2 v + \xi_3 w , \qquad y = \eta_1 u + \eta_2 v + \eta_3 w , \qquad z = \zeta_1 u + \zeta_2 v + \zeta_3 w$$

for real $\xi_i$, $\eta_i$, $\zeta_i$ with

$$(1.2) \qquad D = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix} \neq 0 .$$

Then when $u$, $v$, $w$ take on all possible integer values the points $(x, y, z)$ are said to generate a lattice $\Lambda$ of determinant $D$ with basis vectors $(\xi_1, \eta_1, \zeta_1)$, $(\xi_2, \eta_2, \zeta_2)$, and $(\xi_3, \eta_3, \zeta_3)$.

Let $R$ be the region

$$(1.3) \qquad |x^3 + y^3 + z^3| \leqq 1 .$$

If a lattice $\Lambda$ has no points in the interior of $R$ except the origin, it is said to be $R$-admissible. Let

$$(1.4) \qquad \Delta(R) = \text{g.l.b. } |D|$$

for all $R$-admissible lattices. $\Delta(R)$ is said to be the lattice constant for the region $R$ (see Cassels [1]). It can then be stated that any lattice with $|D| < \Delta(R)$ has a point in the interior of $R$ in addition to the origin; in fact, at least two, since both the lattice and $R$ are symmetric in the origin. An $R$-admissible lattice for which $D = \Delta(R)$ is called a critical lattice. Let $\Delta(R) = 1/M$, then from homogeneity considerations it follows that the region $S$ defined by

$$(1.5) \qquad |x^3 + y^3 + z^3| \leqq M$$

has $\Delta(S) = 1$.

An algebraic statement of this result would be the following: Let $x$, $y$, $z$ be linear forms in $u$, $v$, $w$ with real coefficients and determinant 1. Let $m$ denote the greatest lower bound of the values assumed by $|x^3 + y^3 + z^3|$ when $u$, $v$, $w$ take all integral

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values other than 0, 0, 0. Then $m \leqq M$ and the value $M$ is the smallest possible for all sets of forms satisfying the prescribed conditions. Estimates of $M$ were given some time ago by Mordell [2] and Davenport [3], [4]. The best known bounds, given by Davenport, are

$$.815 \leqq M \leqq 1.157. \tag{1.6}$$

(All decimal numbers in this report are truncated.) The equivalent bounds for $\Delta(R)$ are

$$.864 \leqq \Delta(R) \leqq 1.226\,. \tag{1.7}$$

The bound for $\Delta(R)$ from below was obtained by applying Blichfeldt's extension of the Fundamental Theorem of the Geometry of Numbers to $R$. The bound from above is given by the specific form

$$x^3 + y^3 + z^3 = u^3 + 2v^3 + w^3 - u^2 w \tag{1.8}$$

or

$$\begin{aligned} x &= 1.000535u - .346873w\,, \\ y &= 1.259920v\,, \\ z &= -.117148u + 1.013722w\,. \end{aligned} \tag{1.9}$$

The points (1.000535, 0, $-.117148$), (0, 1.259920, 0) and ($-.346873$, 0, 1.013722) in $xyz$-space form a basis of an $R$-admissible lattice of determinant $D = 1.226697$ which has many points on the boundary of $R$, but none inside except (0, 0, 0).

Let the norm of a lattice point $(u, v, w)$ be defined as the maximum of the absolute values of the three integer coordinates. By restricting lattices to consist of points with bounded norm, an analog of $\Delta(R)$ was found on the IBM 7094 computer. Letting the bound approach infinity (250 was all that was practical) the following conjecture was achieved:

$$M = 1.054013\,, \qquad \Delta(R) = .948754\,. \tag{1.10}$$

**2. Background.** Mordell [5] exhibits the critical lattices for the region $T$:

$$|x^3 + y^3| \leqq 1\,. \tag{2.1}$$

The boundary of $T$ consists of the two curves

$$x^3 + y^3 = 1\,, \tag{2.2}$$

$$x^3 + y^3 = -1\,. \tag{2.3}$$

Let

$$x = \xi_1 u + \xi_2 v\,, \qquad y = \eta_1 u + \eta_2 v\,, \tag{2.4}$$

generate a lattice with basis vectors $(\xi_1, \eta_1)$, $(\xi_2, \eta_2)$ where $u$ and $v$ are integers. In this case the critical lattices can be generated by forcing the lattice points (1, 0), (0, 1), (1, 1) and (1, $-1$) given in $uv$-coordinates to fall on the portion of the boundary given by (2.2). The basis is then determined by the equations

$$\begin{aligned} \xi_1^3 + \eta_1^3 = \xi_2^3 + \eta_2^3 &= (\xi_1 + \xi_2)^3 + (\eta_1 + \eta_2)^3 \\ &= (\xi_1 - \xi_2)^3 + (\eta_1 - \eta_2)^3 = 1\,. \end{aligned} \tag{2.5}$$

There are two real solutions. One can be derived from the other by permuting coordinates in the basis vectors. $|D| = |\xi_1\eta_2 - \xi_2\eta_1|$ gives the value of the lattice constant for the region $T$. We note that in (2.5) the permuting of numbers only changes the sign of the fourth expression. Thus if the roles of the basis vectors are interchanged, the critical lattices are seen to be also generated by forcing the points (1, 0), (0, 1), (1, 1) and $(-1, 1)$ to fall on (2.2), or by forcing the points (1, 0), (0, 1) and (1, 1) to fall on (2.2) while $(1, -1)$ is forced to fall on (2.3).

One might speculate about the above choice of four lattice points. They might be described as the "simplest" lattice points or the "closest" to the origin, their coordinates being restricted to $-1$, 0, and 1. There are eight points involved if we consider the image points in the origin, obtained by changing all signs. There is no loss in assuming that (1, 0) and (0, 1) lie on (2.2), rather than $(-1, 0)$ or $(0, -1)$, since this is equivalent to a change in sign of the basis vectors which is unimportant. However, once this selection is made geometry requires that (1, 1) be taken and not $(-1, -1)$. The fourth point can be either $(1, -1)$ or $(-1, 1)$.

**3. The Generalization.** This suggests a natural generalization to $R$ given by formula (1.3). The two boundary surfaces are given by

$$x^3 + y^3 + z^3 = 1\,, \tag{3.1}$$

$$x^3 + y^3 + z^3 = -1\,. \tag{3.2}$$

There are 27 points in 3-space having the coordinates $-1$, 0, and 1. After eliminating the origin and the "remote" points having no zero coordinate, there remain 18 of which half are image points in the origin. This gives 9 points to put on the surface (3.1) to determine the 9 coordinates of the basis $(\xi_1, \eta_1, \zeta_1)$, $(\xi_2, \eta_2, \zeta_2)$, $(\xi_3, \eta_3, \zeta_3)$. There is no loss in putting (1, 0, 0), (0, 1, 0) and (0, 0, 1) on the surface. Geometry forces the selection of (1, 1, 0), (1, 0, 1) and (0, 1, 1). The 3 remaining points $(1, -1, 0)$, $(1, 0, -1)$ and $(0, -1, 1)$ can be on either surface (3.1) or (3.2), giving rise to 8 possible cases. This is more, incidentally, than the 6 equivalent systems obtained by permuting basis vectors. Two prototypes appear for the 8 cases. The selection of $(1, -1, 0)$, $(1, 0, -1)$ and $(0, -1, 1)$ for the last 3 points covers 6 cases; the selection of $(-1, 1, 0)$, $(1, 0, -1)$ and $(0, -1, 1)$ covers 2 cases. The 9 points by the two selections will be referred to as type 1 and type 2, respectively. Solution of the 9 cubics in 9 unknowns analogous to (2.5) should, hopefully, give the critical lattices for $R$. These were solved on the 7094 computer using the Newton-Raphson method. Unhappily, points of the lattice fell inside $R$ for lattices of both types. For instance, the value of $x^3 + y^3 + z^3$ at (1, 1, 1) was .934 for type 1 and .601 for type 2. For type 1 the basis vectors were (1.000, $-.101$, $-.116$), ($-.339$, 1.020, $-.286$) and ($-.346$, $-.041$, 1.013) of determinant $D = .935$ and $1/D = 1.068$. The corresponding $uvw$-form was

$$u^3 + v^3 + w^3 - u^2v - u^2w - vw^2 + .934uvw\,. \tag{3.3}$$

For type 2, the basis vectors were (1.011, $-.101$, $-.325$), ($-.325$, 1.011, $-.101$) and ($-.101$, $-.325$, 1.011) with $D = .899$ and $1/D = 1.111$. The corresponding $uvw$-form was

$$u^3 + v^3 + w^3 - u^2w - uv^2 - vw^2 + .601uvw\,. \tag{3.4}$$

**4. Groping Ahead.** For type 1, some of the points of low norm that were inside $R$ were (1, 1, 1), (1, −2, −2), (1, −1, −2), (1, 1, −1), (1, −9, 7), (2, 3, −2), (3, −6, −4) and (6, −7, 4). Incidentally, the points (0, 4, −3), (−3, 0, 4) and (−3, 4, 0) were on the surface (3.1), in addition to the 9 selected points. One wonders if there is not some efficient way of removing the points from the interior of $R$ without letting the 9 selected points slip into $R$.

Let (1, 1, 1) replace each of the selected points in turn and solve the system of equations. When (1, 1, 1) replaced (1, 0, 0), the point (1, 0, 0) fell inside $R$ in the solution, so that, this replacement had to be rejected. The other eight replacements were acceptable. The best replacement was for (0, 1, 1) in that $D$ had its smallest value .942. The replacement by (−1, −1, −1) was impossible. In this process the value at (−1, −1, 1) decreased from .934 to .911, showing that (−1, −1, 1) was driven more deeply into $R$. Next (−1, −1, 1) and (1, 1, −1) replace each of the preceding points in turn. Of the 18 cases the best was to let (−1, −1, 1) replace (0, −1, 1). The value of $D$ was .944. One must check that the replaced point (0, −1, 1) is not in $R$ as well as the previously replaced point (0, 1, 1). The question arises whether this was the most efficient way to remove the two points (1, 1, 1) and (1, 1, −1). Should they be removed in reverse order or must we replace the original points two at a time, requiring many combinations. Proceeding with one replacement at a time all points of norm 4 or less were cleared after about 8 stages. However, it was not clear which point to remove at each stage. Do you remove the one of lowest norm or the one most deeply imbedded or the one that is most costly to remove? In fact, with different strategies different results were achieved. Sometimes all possible replacements had to be rejected because they pulled previously replaced points into $R$. Then one would backtrack. This attack was abandoned.

**5. A New Viewpoint.** In the preceding section when (1, 1, 1) was tested as the replacement for a point, say, (0, −1, 1), rather than think of the 9 points as being on the surface (3.1), 8 points of type 1 and (1, 1, 1), think of the value of $x^3 + y^3 + z^3$ at the 9 points of type 1 as being coordinates. The value for the first 8 is 1 and for the 9th is 1.871. Thus (1, 1, 1, 1, 1, 1, 1, 1, 1.871) is a possible set of coordinates for which (1, 1, 1) is on (3.1). The totality of such coordinates will define a hypersurface in 9-dimensional space for positions of (1, 1, 1) on (3.1). One side of this surface corresponds to (1, 1, 1) having value $\geqq 1$; all of these positions are acceptable. The other side corresponds to (1, 1, 1) having value $< 1$ and all of these must be rejected. The hypersurface can be generated for the general lattice point $(d, e, f)$. The rejected side may contain points for which the value is $\leqq -1$. These may be reclaimed by developing the surface for which $(d, e, f)$ has the value $-1$ and accepting the side for which the value is $\leqq -1$ or what is the same thing, develop the surface for $(-d, -e, -f)$ having value 1 and accepting the side for which the value is $\geqq 1$. Hypersurfaces could also be developed for the value of $D$ equal a constant. It was noticed that the hypersurfaces were strikingly linear and that by taking the excess over 1 as coordinates the problem would fit a linear programming framework. By using a convergence process, the assumption of linearity could be removed.

Let $(a_i, b_i, c_i)$ represent the $uvw$-coordinates of each of the 9 special lattice points of type 1 or type 2. Recall that the points of type 1 are (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, −1, 0), (1, 0, −1) and (0, −1, 1). Type 2 differs only in that the 7th point has coordinates $(-1, 1, 0)$. Let

$$(5.1) \qquad \begin{aligned} t_i = {} & (a_i\xi_1 + b_i\xi_2 + c_i\xi_3)^3 + (a_i\eta_1 + b_i\eta_2 + c_i\eta_3)^3 \\ & + (a_i\zeta_1 + b_i\zeta_2 + c_i\zeta_3)^3 - 1 . \end{aligned}$$

Then the $t_i \geqq 0$ constitute the coordinates in question. For a given initial set of coordinate values $\{t_i^0\}$, a basis $(\xi_1^0, \eta_1^0, \zeta_1^0)$, $(\xi_2^0, \eta_2^0, \zeta_2^0)$, $(\xi_3^0, \eta_3^0, \zeta_3^0)$ can be found by the Newton-Raphson method for the system (5.1). By taking partial derivatives and solving 9 sets of 9 linear equations in 9 unknowns, the values of $(\partial\xi_j/\partial t_i)^0$, $(\partial\eta_j/\partial t_i)^0$, $(\partial\zeta_j/\partial t_i)^0$, $(j = 1, 2, 3)$, can be found. Then two terms of the Taylor series for $D$ given by (1.2) can be found about these initial values, namely,

$$(5.2) \qquad D \approx D^* + \sum_{i=1}^{9} (\partial D/\partial t_i)^0 t_i ,$$

where

$$(5.3) \qquad D^* = D^0 - \sum_{i=1}^{9} (\partial D/\partial t_i)^0 t_i^0 .$$

Let

$$(5.4) \qquad \begin{aligned} [d, e, f] = {} & (d\xi_1 + e\xi_2 + f\xi_3)^3 + (d\eta_1 + e\eta_2 + f\eta_3)^3 \\ & + (d\zeta_1 + e\zeta_2 + f\zeta_3)^3 - 1 , \end{aligned}$$

where $d, e, f$ are integers. Then the condition that $(d, e, f)$ is not an interior point of $R$ is expressed by either

$$(5.5) \qquad [d, e, f] \geqq 0 \quad \text{or} \quad [-d, -e, -f] \geqq 0 .$$

Just as above, one gets $[d, e, f]$ as a linear function of the $t_i$ for the initial set $\{t_i^0\}$. The problem can now be formulated as a linear programming one: minimize $D$, given in (5.2), subject to constraints $t_i \geqq 0$ and various constraints of the sort given in (5.5). For $n$ points $(d, e, f)$, one of the conditions in (5.5) would be selected for each point, requiring a total of $2^n$ linear programming problems. Each of these problems must be treated as a convergence process. For an initial set $\{t_i^0\}$ the problem is solved. The resulting $t$'s are designated $\{t_i^1\}$ and the process repeats. The over-all problem is made tractable by a judicious choice of lattice points and the rapidity of convergence of the process.

**6. The Results.** For types 1 and 2 the condition $[-1, -1, -1] \geqq 0$ is geometrically impossible. For type 1, $t_i^0 = 0$ and $[1, 1, 1] \geqq 0$ one finds

$$(6.1) \qquad \begin{aligned} D \approx {} & .935 + .134t_1 + .236t_2 + .196t_3 + .109t_4 + .085t_5 \\ & + .076t_6 + .041t_7 + .025t_8 + .029t_9 \end{aligned}$$

and

$$(6.2) \qquad \begin{aligned} [1, 1, 1] \approx {} & -.065 - .944t_1 + .020t_2 + .008t_3 + .523t_4 \\ & + .107t_5 + .735t_6 + .265t_7 + .161t_8 + .056t_9 . \end{aligned}$$

The result is $D \approx .9420$, all $t_i^1 = 0$, except $t_6^1 = .0895$. The process converges to $D = .9419$, all $t_i = 0$, except $t_6 = .0891$. This much is the same as the initial stage described in Section 4. For type 2, $[1, 1, 1] \geqq 0$, the process yields $D = .951$, $t_i = 0$, except $t_4 = .867$. (The result is not unique because of the symmetries for type 2, where $t_4$, $t_5$, $t_6$ enter symmetrically.) Since the value of $D$ already exceeds the conjectured value of $\Delta(R)$ in (1.10), type 2 will no longer be considered. No further cases were found geometrically impossible. Cases were rejected solely because $D$ exceeded the conjectured value.

The condition $[1, 1, -1] \geqq 0$ gave an acceptable $D$ by itself, but in conjunction with $[1, 1, 1] \geqq 0$, proved unacceptable. The condition $[-1, -1, 1] \geqq 0$ is acceptable, $[1, 1, -1] \geqq 0$ is unacceptable. By finding only one of a pair $[d, e, f] \geqq 0$, $[-d, -e, -f] \geqq 0$ acceptable, one not only limits possibilities, but one can insist on the acceptable case as being required in all further investigations. Proceeding systematically in this way it was found that to achieve an acceptable $D$ the conditions

$$(6.3) \quad \begin{aligned} &[1, 1, 1], \quad [4, -5, 3], \quad [4, 3, -3], \quad [-1, 2, 1], \quad [-1, -3, 2], \\ &\qquad [-4, -4, 3], \quad [-1, -1, 1], \quad [1, -2, -2] \geqq 0 \end{aligned}$$

must be met, simultaneously. Linear programming was applied giving

$$(6.4) \quad \begin{aligned} &D = .948694, \quad t_1 = t_2 = t_3 = t_4 = 0, \quad t_5 = .000133, \quad t_6 = .026670, \\ &t_7 = .069379, \quad t_8 = .097608, \quad t_9 = .172127. \end{aligned}$$

The first five of the constraints (6.3) had values equal to zero; the others were greater than zero. From the viewpoint of Section 4, the lattice could be generated by placing the 9 lattice points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1), (4, −5, 3), (4, 3, −3), (−1, 2, 1) and (−1, −3, 2) on the surface (3.1). This lattice is the one of smallest determinant for which all lattice points of norm 5 or less except (0, 0, 0) are not interior to $R$. In fact all points through norm 11 clear $R$. The points (2, −12, 9) and (12, 7, −10) of norm 12 are interior to $R$. Incidentally, this case nears the limits of single-precision for the 7094 computer and in fact double-precision was used for the entire investigation. Instead of checking all lattice points $(d, e, f)$ of norm $\leqq N$, say, to be sure they are not interior points of $R$, it was sufficient to satisfy the conditions $0 \leqq d \leqq N$, $-N \leqq f \leqq N$ and then give $e$ integer values near the roots of certain obvious cubics.

The next stage had

$$(6.5) \quad \begin{aligned} &[1, 1, 1], \quad [-1, 2, 1], \quad [4, -5, 3], \quad [11, 3, -12] = 0 \\ &\quad D = .948749, \quad t_1 = t_2 = t_3 = t_4 = t_5 = 0, \quad t_6 = .026306, \\ &\quad t_7 = .068642, \quad t_8 = .099733, \quad t_9 = .174126. \end{aligned}$$

This gives the lattice of smallest determinant for which all points of norm 12 or less clear $R$. In fact all points through norm 23 clear $R$. It is interesting to note that (11, 3, −12) was not in trouble at the end of the first stage, but that in satisfying the conditions (6.3) plus the 4 pairs associated with (2, −12, 9) and (12, 7, −10), either the conjectured value was exceeded or (11, 3, −12) was in trouble.

The third stage had

$$(6.6) \quad \begin{aligned} &[1, 1, 1], \quad [11, 3, -12], \quad [-11, 24, 22], \quad [21, -27, 17], \quad [27, 10, -27] = 0 \\ &D = .948754211. \end{aligned}$$

All points through norm 38 clear $R$.

The fourth stage had

$$(6.7)\quad \begin{gathered}[1, 1, 1], \quad [11, 3, -12], \quad [-11, 24, 22], \quad [21, -27, 17], \\ [49, 30, -40] = 0 \\ D = .948754397256 .\end{gathered}$$

All points through norm 105 clear $R$.

The fifth stage had

$$(6.8)\quad \begin{gathered}[1, 1, 1], \quad [11, 3, -12], \quad [-11, 24, 22], \quad [21, -27, 17], \\ [106, 93, -78] = 0 \\ D = .948754397726 .\end{gathered}$$

All points through norm 244 clear $R$.

The sixth stage had

$$\begin{gathered}[1, 1, 1], \quad [-11, 24, 22], \quad [21, -27, 17], \quad [106, 93, -78], \\ [66, -245, 187] = 0\end{gathered}$$

$$(6.9)\quad \begin{gathered}D = .948754399505 , \quad 1/D = 1.054013557692 \\ t_1 = t_2 = t_3 = t_4 = 0 , \quad t_5 = .000157614028 , \quad t_6 = .026325663739 , \\ t_7 = .068670755978 , \quad t_8 = .099638725715 , \quad t_9 = .173887066166 .\end{gathered}$$

The basis vectors were

$$(6.10)\quad \begin{gathered}(1.000752724248, -.096532558854, -.110802150937) \\ (-.350065977785, 1.023026346629, -.302871084371) \\ (-.361698926716, -.014773819857, 1.015531810412) .\end{gathered}$$

The corresponding $uvw$-form was

$$(6.11)\quad \begin{gathered}u^3 + v^3 + w^3 - 1.034335377989u^2v - 1.049740555843u^2w \\ + .100106364953v^2w + .034335377989v^2u + .049898169871w^2u \\ -1.073780701213w^2v + .973516722231uvw .\end{gathered}$$

It is only known that points through norm 250 clear $R$ for this case.

The stages have to be viewed as arbitrary, yet there was a certain naturalness about them and the range of norms which they clear grows by a factor of the order of 2. However, a case could be made for accepting only one of the stages 3, 4, 5 since they differ in only one constraint with coefficients of like parity. The presence of these stages suggests that $R$ is not boundedly reducible.

Between the fifth and sixth stages all entries above differ in from the 8th to the 12th decimal place, that is, each entry differs from the corresponding one by less than $10^{-7}$. Since the differences of entries between the stages decrease, essentially, geometrically taken as a whole, it is believed that many of the above digits will be preserved in the limit as the norm of cleared points goes to infinity. Arbitrarily picking six decimal digits gives rise to the main conjecture (1.10). A weaker from of this conjecture, namely,

$$(6.12)\quad M \leqq 1.054013 , \qquad \Delta(R) \geqq .948754$$

is even more strongly supported by this investigation. Six decimal digits could also be picked for other ultimate quantities, in particular, for the basis vectors. Note, that the points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), and (1, 1, 1) were on the boundary in every stage. It is more demanding to expect these to be points of the ultimate critical lattice, since $t_1$, $t_2$, $t_3$, $t_4$ can have the first decimal places zero without actually being zero. Nonetheless, a pattern is established in these few stages of these five points being on the boundary of $R$ for every lattice, while the remaining four points are receding to infinity. This suggests that the critical lattice has just the 5 points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0) and (1, 1, 1) on (3.1). Six critical lattices are to be expected, just as 6 lattices with the value of $D$ given in (6.9) can be derived by permuting the coordinates of the basis vectors (6.10).

It is hoped that these results will provide a basis for conjectures which can be attacked mathematically; however, the difficulty of this problem and its "infinite" nature may well put it outside the scope of rigorous mathematics.

**7. Reservations.** The various stages occur in "holes," not at the frontier of the set of rejected "slabs." There is the possibility that a "hole" will be covered and that one would have to backtrack to make a fresh start giving rise to "discontinuities" in clearing higher and higher norms. In backtracking one may strike a stage that clears all the way to infinity; however, all elementary stages examined seemed to have the same density of problems that stage 1 had. If the "holes" kept getting covered, one might be forced all the way to the Davenport case (1.9). In this regard it is hard to conceive of a form similar to (6.11) with noninteger coefficients not taking any values between $-1$ and $+1$ except the value 0 when $(u, v, w) = (0, 0, 0)$. For stage 1 the surface $[5, -6, 2] = 0$ seems to bound the "hole." This is further constricted by $[-12, 18, -13] = 0$. However, since the successive stages needed so little extra room, since backtracking was never required, and since the first four coordinates were never needed, it is felt that the closing of "holes" is not a matter for serious concern.

Computer error or programming error is not considered to be a factor in the computation. FORTRAN subroutines "Simultaneous Linear Equation Solver" by Louis G. Kelly of the Applied Physics Laboratory as well as "Linear Programming" by John J. Jarvis of the Johns Hopkins University were used. These had been well checked out. The first two stages had been achieved independently of the linear programming, providing a check for that routine. Some points in stage 6 were also tested. All cases were run at least twice.

The author was led naturally to the linear programming approach from a graphic approach in two coordinates. There was no evidence of inflection points and practically none of curvature. Furthermore, the convergence of the linear programming process strongly supports that a local minimum is achieved. The independence of the process from various initial coordinate settings suggests an absolute minimum. The impossibility of the cases $[-1, -1, 0]$, $[-1, 0, -1]$, $[0, -1, -1]$, $[-1, -1, -1] = 0$ seemed plausible in comparison with the binary cubic and gained further support from computer runs. Even if possible, the lattices would be so distorted that other difficulties would no doubt arise. This study was carried on near $t_1 = \cdots = t_9 = 0$ by analogy to the binary cubic. The "monotoneity" of $D$ can be lost for large $t$, but then "distortion" forces other basic lattice points to be inside $R$.

The Johns Hopkins University
Silver Spring, Maryland 20910

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