Computer Investigation of Landau's Theorem

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Abstract. Let $f(z) = a_0 + a_1 z + \cdots$ be regular for $|z| < 1$ and never take the values 0 and 1; then $|a_1|$ has a bound depending only on a_0. J. A. Jenkins gave an explicit bound (*Canad. J. Math.* **8** (1956), 423–425) $|a_1| \leqq 2|a_0| \{|\log |a_0|| + 5.94\}$. The author investigates the shapes for the curves $|a_1| \leqq L(a_0)$ for given a_0 by the aid of a computer and shows that although Jenkins' result is about right when a_0 is negative, 4.38 will be the best possible constant in his form and that a much smaller estimate should be available when a_0 is positive or complex. ■

1. Introduction. The theorem of Landau in question may be stated in the form that if the function $f(z) = a_0 + a_1 z + \cdots$ is regular for $|z| < 1$ and never takes the values 0 and 1, then $|a_1|$ has a bound depending only on a_0. Hayman [1] gave the explicit bound $|a_1| \leqq 2|a_0|\{|\log |a_0|| + 5\pi\}$ and Jenkins [2] improved it to $|a_1| \leqq 2|a_0|\{|\log |a_0|| + 5.94\}$. For a given value of a_0, there is a certain possible region of values of a_1. This region is probably not a circle $|a_1| \leqq K\ (a_0)$. This region will probably have a different shape when a_0 is near 0, 1 and ∞. In this paper, I shall show that although Jenkins' result is about right when a_0 is negative, 4.38 will be the best possible constant in his form and that a much smaller estimate should be available when a_0 is positive or complex.

2. Preliminaries. Let $\lambda(\tau)$ be an elliptic modular function,

$$\begin{aligned}\lambda(\tau) &= \theta_2^{\,4}(0)/\theta_3^{\,4}(0)\\ &= 16q(1 + q^2 + q^6 + q^{12} + \cdots)^4/(1 + 2q + 2q^4 + 2q^9 + \cdots)^4\end{aligned}$$

where $q = e^{i\pi\tau}$. By a transformation

$$\zeta = (\tau - \tau_0)/(\tau - \bar{\tau}_0)\,, \qquad I_m(\tau_0) > 0$$

we have $g(\zeta) = \lambda(\tau)$ which is regular and $g(\zeta) \neq 0$, $g(\zeta) \neq 1$ for $|\zeta| < 1$. Hence

$$a_0 = g(0) = \lambda(\tau_0)$$

and

$$a_1 = g'(0) = \lambda'(\tau_0) 2I_m(\tau_0)\,.$$

Thus, the problem of finding a better inequality in Landau's theorem may be solved by tabulating $|g'(0)|$ and $g(0)$. Hence, the matter simply depends on calculating the elliptic modular function $\lambda(\tau)$.

3. A Bound of $|a_1|$ for small $|a_0|$. When $I_m(\tau)$ is large and hence $|q|$ small we have $g(0) \simeq 16q_0$ where $q_0 = e^{i\pi\tau_0}$ and

$$g'(0) \simeq 16i\pi e^{i\pi\tau_0} 2I_m(\tau_0)\,.$$

Hence

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$$|g'(0)| \simeq \pi |g(0)| (2/\pi)| \log |g(0)/16|| .$$

Therefore

$$|g'(0)| \simeq 2|g(0)|\left\{\log \frac{1}{|g(0)|} + 2.7726\right\} .$$

Thus, for small $|a_0|$, Landau's inequality is approximately

$$|a_1| \leqq 2|a_0|\left\{\log \frac{1}{|a_0|} + 2.7726\right\} .$$

4. Computer Investigation.

4.1. *The Case of a_0 Real.* When a_0 is real, we need only to compute the value of $a_1 = \lambda'(\tau_0)\, 2I_m(\tau_0)$ against $a_0 = \lambda(\tau_0)$ which varies from 1/2 to 0 and -1 to 0. The other values can be obtained by the following transformations:

$$U = 1 - W , \qquad V = 1/U ;$$

here of course, $|U'| = |W'|$ and $|V'| = |U'|/|U|^2$. A simple computer program will give us a sufficient amount of information about the values of a_0 and a_1. I shall list only a few of them below and show the part of the curve in the attached figure. The computation is made by taking

$$\lambda(\tau) = 16q(1 + q^2 + q^6 + q^{12} + q^{20})^4/(1 + 2q + 2q^4 + 2q^9 + 2q^{16})^4.$$

TABLE

a_0	$\|a_1\|$	a_0	$\|a_1\|$	a_0	$\|a_1\|$	a_0	$\|a_1\|$	a_0	$\|a_1\|$
0.5	2.1884	−0.1	1.0744	−0.6	5.3195	−1.1	9.6336	−1.6	14.1583
0.4	2.1177	−0.2	1.9587	−0.7	6.1661	−1.2	10.5220	−1.7	15.0870
0.3	1.9020	−0.3	2.8063	−0.8	7.0203	−1.3	11.4187	−1.8	16.0234
0.2	1.5263	−0.4	3.6432	−0.9	7.8829	−1.4	12.3237	−1.9	16.9670
0.1	0.9527	−0.5	4.4793	−1.0	8.7538	−1.5	13.2371	−2.0	17.9173

4.2. $|a_1| \leqq 2|a_0| \{\log |a_0|| + 4.38\}$. From the above table, we notice that the constant $\Gamma^4(1/4)/4\pi^2 = 4.376 \cdots$ in Littlewood's result [3] at $a_0 = -1$ is very sharp and by using the inequality in 3 and the numerical tabulation of $2|a_0| \{|\log |a_0|| + 4.38\}$, we can read that 4.38 will be the best possible constant in Jenkins' form.

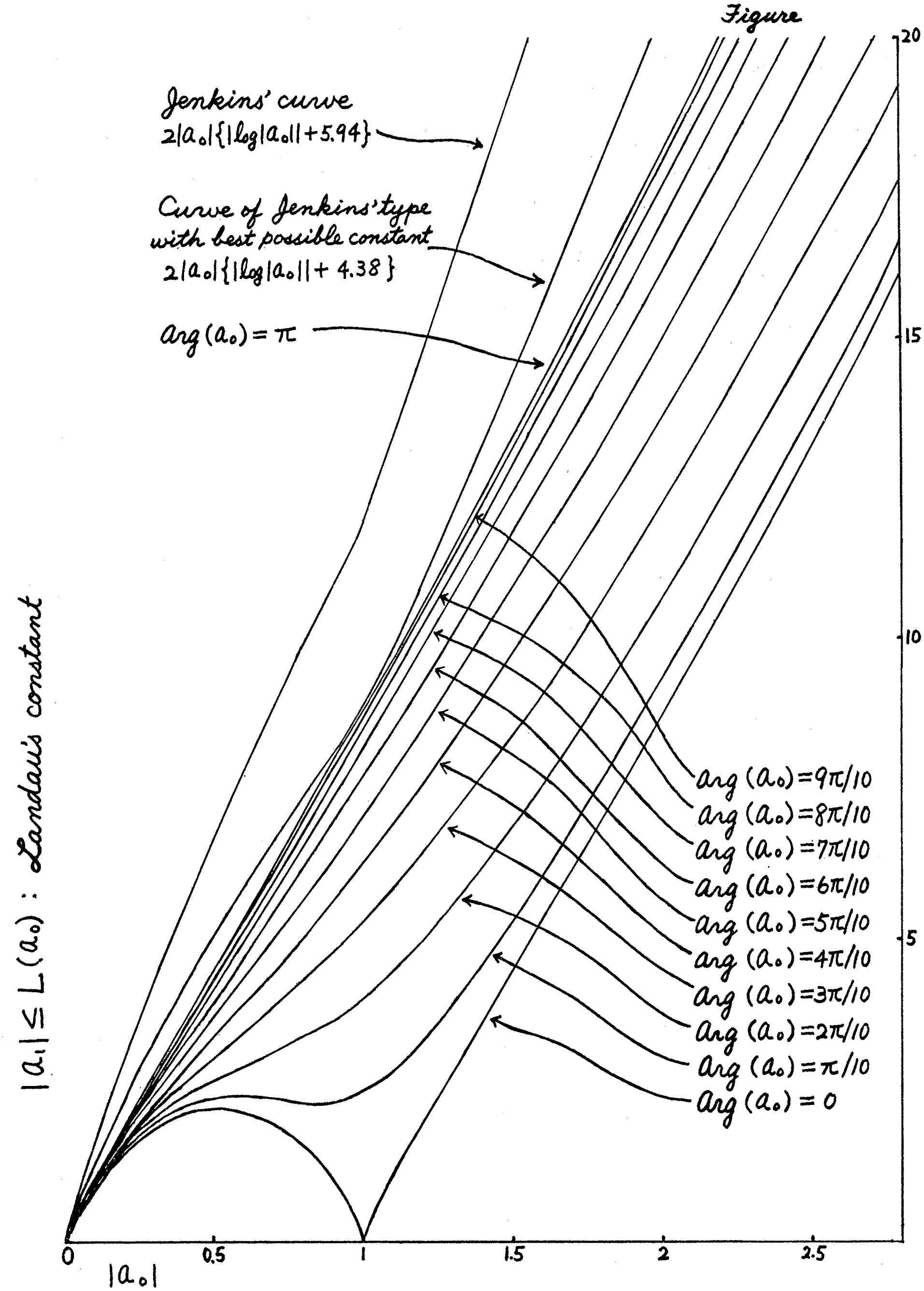
Remark 1. In fact, we have $|a_1| = 2.18843961$ and $|a_1| = 8.75375837$ for $a_0 = \lambda(i) = 0.50000000$ and $a_0 = \lambda(1 + i) = -0.99999999$ respectively. Hence, even if we consider a few more terms in $\lambda(\tau)$, almost no change in the value of $|a_1|$ can be expected.

4.3. *The Case of a_0 Complex.* I shall illustrate the best possible numerical bound of $|a_1|$ for each given a_0 with the argument $\alpha = n\pi/10, n = 1, 2, \cdots, 10$ in the figure. These curves are drawn from the values prepared by a computer by taking

$$\lambda(\tau) = 16q(1 + q^2 + q^6 + q^{12})^4/(1 + 2q + 2q^4 + 2q^9)^4 .$$

Remark 2. From the table and the figure and from the transformations $U = 1 - W$ and $V = 1/U$, we can obtain the values of a_0 and its corresponding values of $|a_1|$ which suggest the shape of a possible region of values of a_1 for a given

a_0. For instance, we may draw the contour lines of $L(a_0)$ = constant in the a_0-complex plane. It is interesting to mention that Jenkins' result would just give concentric circles in that representation.



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