Chebyshev Approximations for the Exponential Integral Ei(x)

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Abstract. The computation of the exponential integral Ei(x), $x > 0$, using rational Chebyshev approximations is discussed. The necessary approximations are presented in well-conditioned forms for the intervals (0, 6], [6, 12], [12, 24] and [24, ∞). Maximal relative errors are as low as from 8×10^{-19} to 2×10^{-21}. In addition, the value of the zero of Ei(x) is presented to 30 decimal places. ■

1. Introduction. The classical exponential integral is defined by

$$\text{Ei}(x) \equiv \int_{-\infty}^{x} \frac{e^t}{t}\, dt = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt \qquad (x > 0) \tag{1.1}$$

where the integral is to be interpreted as the Cauchy principal value. Except for the sign, it represents the natural extension of the function

$$E_1(z) \equiv \int_z^{\infty} \frac{e^{-t}}{t}\, dt = -\text{Ei}(-z) \qquad (|\arg z| < \pi)$$

to the negative real axis.

The exponential integral was studied extensively throughout the 19th century, in the form of the integral logarithm

$$\text{li}(x) \equiv \text{Ei}(\ln x) = \int_0^x \frac{t}{\ln t}\, dt$$

which plays a significant role in number and probability theory as well as in a variety of physical applications. Recent applications in the theories of molecular structure and of the solid state have produced a need for methods for high-precision evaluation of the function using automatic computers.

To meet this need, Harris [1], and Miller and Hurst [2] published tables of the function to 18S and 16S respectively, with useful interpolation aids, for the region between that where the Maclaurin series is convenient and that where the asymptotic series gives acceptable accuracy. Clenshaw [3] has published 20D tables of the coefficients for an expansion in Chebyshev polynomials valid for $x \leqq 4$. Unfortunately, this series converges very slowly (27 terms for 20D) and gives poor relative accuracy where Ei(x) is small. More recently, G. F. Miller has computed (but not yet published) coefficients for Chebyshev expansions for the intervals $4 \leqq x \leqq 16$ and $16 \leqq x \leqq \infty$. Both of these series require 39 terms to attain 20D accuracy, again exhibiting slow convergence.

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The present work presents nearly minimax rational approximations for $\mathrm{Ei}(x)$, $x > 0$, with accuracies up to 18S to 20S. The approximations presented here, and in a previously published companion paper [4], thus allow efficient high-precision computation of $\mathrm{Ei}(x)$ for all real nonzero x.

2. Functional Properties. The classical Maclaurin series

$$\mathrm{Ei}(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{k!k} \tag{2.1}$$

($\gamma = 0.57721\cdots$, Euler's constant) is satisfactory for computation for most small x. It is apparent from this series that, except for the logarithmic branch point at the origin, $\mathrm{Ei}(x)$ has no singularities in the finite complex plane, but does have a simple zero (to which an accurate approximation will be given later) in the interval (0, 1).

In the neighborhood of that zero, significance losses make (2.1) unsatisfactory for accurate computation, and a Taylor series expansion about the zero is to be preferred. Differentiating (1.1) and using Leibnitz' rule, we find

$$\mathrm{Ei}(x + h) = \mathrm{Ei}(x) + \frac{e^x h}{x} \sum_{n=0}^{\infty} c_n(x) h^n \qquad (|h| < x) \tag{2.2}$$

where $c_n(x)$ are the successive derivatives of e^x/x up to the factor $xe^{-x}/(n+1)!$. These are readily computable from the recurrence

$$c_{n+1}(x) = -\frac{1}{n+2}\left[\frac{n+1}{x} c_n(x) - \frac{1}{(n+1)!}\right],$$

with $c_0 = 1$, which Gautschi [5] has shown to be stable unless x is large.

For large x the Maclaurin series requires an excessive number of terms and other methods of computation are preferable. Letting

$$C_0(x) = xe^{-x}\,\mathrm{Ei}(x) = \int_0^{\infty} \frac{e^{-u}}{1 - u/x}\, du,$$

and using the identity

$$\frac{1}{1 - u/x} = \sum_{k=0}^{n-1} \left(\frac{u}{x}\right)^k + \frac{(u/x)^n}{1 - u/x}$$

we obtain

$$C_0(x) = \sum_{k=0}^{n-1} \frac{k!}{x^k} + \frac{n!}{x^n} C_n(x) \tag{2.3}$$

where

$$C_n(x) = \frac{1}{n!} \int_0^{\infty} \frac{u^n e^{-u}}{1 - u/x}\, du. \tag{2.4}$$

This is the asymptotic series expansion for the exponential integral with an error term. For large enough x the series without the error term is computationally useful, particularly if economized by Chebyshev polynomials. It can be shown that for $x > n$ the error in neglecting the remainder in (2.3) is positive and less than $n!/n^n$.

TABLE I

$$\mathcal{E}_{\ell m} = -100 \log_{10} \max \left| \frac{Ei(x) - E_{\ell m}(x)}{Ei(x)} \right|$$

$0 \leq X \leq 6$

m \ ℓ	0	1	2	3	4	5	6	7	8	9
0	14†	45†	93	151	215	286	363	444	530	619
1	78	141	211	286	365					
2	215†	234	321	409	499					
3	235		430	528	628					
4		372			753					
5						994				
6							1248			
7								1515		
8									1791	
9										2076

$6 \leq X \leq 12$

m \ ℓ	0	1	2	3	4	5	6	7	8	9
0	113	240	320	394	480	577	626	742	805	862
1	284†	309	369	446	544					
2	309		474	577	669					
3	417	455	556	656	721					
4	462		687	735	877					
5						1010				
6							1271			
7								1514		
8									1763	
9										1921

$12 \leq X \leq 24$

m \ ℓ	0	1	2	3	4	5	6	7	8	9
0	157	303	421	542	643	690	834	855	996	1015
1	325†	454	612†	623	681					
2	438	594†	623		785					
3	564	624		806	880					
4	632		796	876	945					
5						1167				
6							1273			
7								1503		
8									1715	
9										1922

$24 \leq X$

m \ ℓ	0	1	2	3	4	5	6	7	8	9
0		462	592	707	809	899	978	1054	1139	1223
1	505	644	764	868	958					
2	631	788	896	991	1096					
3	740	891	1007	1132†	1151					
4	834	979	1127	1151	1158					
5						1357				
6							1425			
7								1585		
8									1655	
9										1811

†Nonstandard error curve

TABLE II

$$Ei(x) \approx \ell n(x/x_o) + (x-x_o) \sum_{j=o}^{n}{}' p_j T_j^*(x/6) \Big/ \sum_{j=o}^{n}{}' q_j T_j^*(x/6), \qquad 0 < x \leq 6$$

x_o = .37250 74107 81366 63446 19918 66580

n	j	p_j		q_j	
2	0	1.98741	(02)	7.12499	(01)
	1	2.68621	(01)	-3.11537	(01)
	2	6.33466	(00)	4.50000	(00)
3	0	-2.24094 38	(03)	-8.56095 57	(02)
	1	-1.83267 70	(02)	4.09708 70	(02)
	2	-9.00537 23	(01)	-8.30561 91	(01)
	3	-5.67968 91	(00)	6.75000 00	(00)
4	0	3.36028 1229	(04)	1.33702 5876	(04)
	1	1.52125 3684	(03)	-6.73793 6555	(03)
	2	1.54047 2598	(03)	1.58366 6607	(03)
	3	7.55450 3527	(01)	-1.92119 3866	(02)
	4	6.21549 4703	(00)	1.01250 0000	(01)
5	0	-6.39839 24018 3	(05)	-2.61223 81131 4	(05)
	1	-1.37944 42831 0	(04)	1.35780 40545 3	(05)
	2	-3.22559 91481 2	(04)	-3.46327 13188 4	(04)
	3	-1.05347 89222 3	(03)	5.02160 27342 9	(03)
	4	-2.29486 86245 7	(02)	-4.10184 41329 0	(02)
	5	-6.43324 74215 4	(00)	1.51875 00000 0	(01)
6	0	1.45747 32174 3484	(07)	6.06704 88720 8133	(06)
	1	4.71761 89314 3542	(04)	-3.22441 10952 8198	(06)
	2	7.84689 68750 8871	(05)	8.67868 68083 9173	(05)
	3	1.35036 01118 4167	(04)	-1.39638 47118 9123	(05)
	4	7.35420 51394 7427	(03)	1.39851 27716 1255	(04)
	5	1.80339 13851 6268	(02)	-8.27775 60914 1545	(02)
	6	7.75745 08332 8328	(00)	2.27812 50000 0000	(01)

TABLE II—*Continued*

n	j	p_j		q_j	
7	0	-3.91546 07380 90955 48	(08)	-1.65254 29972 52109 11	(08)
	1	3.89280 42131 12014 05	(06)	8.91925 76757 56121 09	(07)
	2	-2.21744 62775 88453 78	(07)	-2.49033 37574 05403 27	(07)
	3	-1.19623 66934 92468 68	(05)	4.28559 62461 17490 39	(06)
	4	-2.49301 39345 86475 88	(05)	-4.83547 43616 21635 09	(05)
	5	-4.21001 61535 70699 34	(03)	3.57300 29805 85081 06	(04)
	6	-5.49142 26552 10851 45	(02)	-1.60708 92658 72208 52	(03)
	7	-8.66937 33995 10695 61	(00)	3.41718 75000 00000 00	(01)
8	0	1.19913 98937 84740 0194	(10)	5.11852 99521 52327 0008	(09)
	1	-2.50389 99488 63513 6166	(08)	-2.79673 35112 29845 9145	(09)
	2	7.05921 60959 00567 4746	(08)	8.02827 78294 69565 0668	(08)
	3	-3.36899 56420 15919 0124	(06)	-1.44980 71439 30238 8259	(08)
	4	8.98683 29164 37583 1259	(06)	1.77158 30801 07998 8368	(07)
	5	7.37147 79018 46574 4328	(04)	-1.49575 45720 25592 1776	(06)
	6	2.85446 88181 36470 1528	(04)	8.53771 00018 07490 9691	(04)
	7	4.12626 66724 89119 3944	(02)	-3.02523 68223 82274 1010	(03)
	8	1.10639 54724 16395 7954	(01)	5.12578 12500 00000 0000	(01)
9	0	-4.16580 81333 60499 42418 79	(11)	-1.79347 49837 15100 97233 71	(11)
	1	1.21776 98136 19959 46775 80	(10)	9.89009 34262 48174 94398 86	(10)
	2	-2.53018 23984 59901 93488 58	(10)	-2.89862 72696 55449 53426 58	(10)
	3	3.19843 54235 23773 85110 48	(08)	5.42296 17984 47295 50118 62	(09)
	4	-3.53778 09694 43113 34848 00	(08)	-7.01085 68774 21595 40653 76	(08)
	5	-3.13986 60864 24726 58620 50	(05)	6.46988 30956 57642 85876 53	(07)
	6	-1.42998 41572 09161 03800 64	(06)	-4.26484 34812 17716 14054 83	(06)
	7	-1.42870 72500 19700 57773 76	(04)	1.94184 69440 75988 03614 15	(05)
	8	-1.28312 20659 26200 06781 55	(03)	-5.56484 70543 36908 28468 19	(03)
	9	-1.29637 02602 47483 00285 90	(01)	7.68867 18750 00000 00000 00	(01)

TABLE III

$$Ei(x) \approx \frac{e^x}{x}\left\{\alpha_0 + \frac{\beta_0}{\alpha_1 + x}\Big/ + \frac{\beta_1}{\alpha_2 + x}\Big/ + \dots + \frac{\beta_{n-1}}{\alpha_n + x}\right\}, \qquad 6 \le x \le 12$$

n	j	α_j		β_j	
1	0	9.79202	(-01)	1.24646	(00)
	1	-1.85524	(00)		
2	0	1.01650 3	(00)	6.04321 6	(-01)
	1	-6.41053 4	(00)	1.72881 8	(01)
	2	3.70085 5	(-01)		
3	0	9.78519 613	(-01)	1.94508 723	(00)
	1	6.63893 533	(00)	-8.18691 421	(00)
	2	-3.42179 900	(01)	9.59300 574	(02)
	3	2.64676 525	(01)		
4	0	1.00215 31213	(00)	8.66166 40160	(-01)
	1	-5.47993 90464	(00)	3.95099 33391	(01)
	2	3.38793 31871	(00)	1.63646 40449	(01)
	3	-1.39678 75234	(01)	9.42027 52121	(01)
	4	2.71140 23688	(00)		
5	0	1.00408 79659 51	(00)	7.75115 98424 09	(-01)
	1	-7.87045 94914 24	(00)	7.55008 28963 43	(01)
	2	5.87816 53535 19	(00)	8.32012 30239 95	(00)
	3	-2.11942 36509 82	(01)	2.80230 09305 44	(02)
	4	1.03286 53064 36	(01)	-1.28806 26193 96	(-02)
	5	-3.65713 74791 48	(00)		
6	0	9.96627 29010 9757	(-01)	1.25917 50341 8513	(00)
	1	4.31734 71888 1208	(00)	-5.13240 15587 7071	(01)
	2	-1.28120 00103 6246	(01)	2.78557 34098 3813	(02)
	3	9.25088 39280 0013	(00)	-5.23104 40813 5384	(00)
	4	2.55301 26581 7553	(01)	1.02294 83159 2118	(03)
	5	-3.70581 88139 5856	(01)	1.99226 73152 4517	(00)
	6	-2.29581 32306 8056	(00)		

TABLE III—*Continued*

n	j	α_j		β_j	
7	0	1.00443 10922 80779 10	(00)	5.27468 85196 29078 54	(-01)
	1	-4.32531 13287 81345 77	(01)	2.73624 11988 93280 58	(03)
	2	6.01217 99083 00804 78	(01)	1.43256 73812 19376 00	(01)
	3	-3.31842 53199 72211 18	(01)	1.00367 43951 67257 73	(03)
	4	2.50762 81129 35598 30	(01)	-6.25041 16167 18755 43	(00)
	5	9.30816 38566 21651 46	(00)	3.00892 64837 29151 99	(02)
	6	-2.19010 23385 48806 88	(01)	3.93707 70185 27150 01	(00)
	7	-2.18086 38152 07237 06	(00)		
8	0	9.98957 66651 65517 0439	(-01)	1.14625 25324 90161 9143	(00)
	1	5.73116 70574 45080 1824	(00)	-1.99149 60023 12351 6360	(02)
	2	4.18102 42256 28566 2231	(00)	3.41365 21252 43755 3905	(02)
	3	5.88658 24075 32811 1118	(00)	5.23165 56873 45586 1379	(01)
	4	-1.94132 96751 44307 0171	(01)	3.17279 48925 43693 2786	(02)
	5	7.89472 20929 44572 2122	(00)	-8.38767 08418 96407 0656	(00)
	6	2.32730 23383 90391 4097	(01)	9.65405 21742 92803 0312	(02)
	7	-3.67783 11347 83114 5794	(01)	2.63983 00731 80245 9334	(00)
	8	-2.46940 98344 83612 6512	(00)		
9	0	9.98119 37875 37396 41321 9	(-01)	1.24988 48227 12447 89144 0	(00)
	1	9.56513 45919 78630 77421 7	(00)	-2.36921 02356 36181 00166 1	(02)
	2	-3.98885 07303 90541 05791 2	(00)	4.73109 71878 16050 25296 7	(02)
	3	1.12001 10242 27297 45152 3	(01)	2.85239 75481 19248 70014 7	(01)
	4	-3.01576 18638 40593 35916 5	(01)	7.60819 45090 86645 76312 3	(02)
	5	1.94560 37795 39281 81043 9	(01)	-8.79140 10548 75438 92502 9	(00)
	6	1.05297 63924 59015 15542 2	(01)	3.69741 22997 72985 94078 5	(02)
	7	-2.42110 69569 80653 51155 0	(01)	4.64418 59325 83286 94265 0	(00)
	8	-2.37837 28828 15725 24412 4	(00)	1.59851 79577 04779 35647 9	(-04)
	9	-2.64567 77930 77147 23780 6	(00)		

TABLE IV

$$Ei(x) \simeq \frac{e^x}{x}\left\{\alpha_0 + \frac{\beta_0}{\alpha_1 + x}\Big/ + \frac{\beta_1}{\alpha_2 + x}\Big/ + \dots + \frac{\beta_{n-1}}{\alpha_n + x}\right\}, \qquad 12 \le x \le 24$$

n	j	α_j		β_j	
1	0	1.00156 5	(00)	9.37833 0	(-01)
	1	-2.74910 7	(00)		
2	0	9.99561 812	(-01)	1.02615 357	(00)
	1	-1.46701 586	(00)	-6.02051 888	(00)
	2	-2.13276 495	(00)		
3	0	1.00010 69063	(00)	9.92038 27336	(-01)
	1	-2.21382 10608	(00)	1.95881 36863	(-01)
	2	2.56787 41018	(01)	1.12456 74486	(03)
	3	-4.07945 74283	(01)		
4	0	9.99617 70436 5	(-01)	1.05815 22064 9	(00)
	1	2.35611 09271 4	(00)	-2.46985 54336 2	(02)
	2	4.87660 50923 7	(01)	1.18689 73787 9	(01)
	3	-1.21480 21625 0	(00)	7.90865 04753 9	(01)
	4	-1.50704 42689 0	(01)		
5	0	1.00000 64520 954	(00)	9.99133 43252 930	(-01)
	1	-2.04553 50969 394	(00)	-8.77003 40170 023	(-01)
	2	-1.87087 73289 805	(01)	2.14962 78340 847	(02)
	3	6.88954 76097 851	(00)	-2.44359 72849 184	(00)
	4	4.24877 41132 104	(01)	2.87410 26247 185	(03)
	5	-6.36860 16650 368	(01)		
6	0	1.00001 16665 3056	(00)	9.98587 13453 5538	(-01)
	1	-2.06740 90098 5715	(00)	-5.00443 22143 6300	(-01)
	2	-3.38582 82273 1869	(01)	8.43507 08211 8844	(02)
	3	2.21704 72244 4619	(01)	-3.69223 29295 7639	(00)
	4	7.62285 76580 0217	(00)	4.01238 84826 5616	(02)
	5	-2.91605 84181 1188	(01)	9.75393 36994 2080	(-03)
	6	-9.09686 10692 6462	(00)		

TABLE IV—*Continued*

n	j	α_j		β_j	
7	0	9.99994 29607 47082 85	(-01)	1.00083 86740 26391 22	(00)
	1	-1.95022 32128 96598 17	(00)	-3.43942 26689 98699 69	(00)
	2	1.75656 31546 96144 22	(00)	2.89516 72792 51350 50	(01)
	3	1.79601 68876 92516 42	(01)	7.60761 14800 77345 83	(02)
	4	-3.23467 33030 54034 59	(01)	2.57776 38423 84398 67	(01)
	5	-8.28561 99414 06413 15	(00)	5.72837 19383 73237 21	(01)
	6	-1.86545 45488 33988 35	(01)	6.95000 65588 74339 78	(01)
	7	-3.48334 65360 28526 13	(00)		
8	0	1.00000 51738 33111 7204	(00)	9.99053 85353 46275 3131	(-01)
	1	-2.07309 31825 50626 1150	(00)	9.96552 64231 07191 1439	(-01)
	2	6.68163 35208 51786 0394	(01)	4.26758 15993 50395 0849	(03)
	3	-6.18811 14583 72674 2610	(01)	1.15264 05585 74517 3857	(01)
	4	1.36153 34713 98465 7547	(01)	6.88273 50646 68918 8421	(02)
	5	-3.24607 70029 93746 3678	(01)	2.75492 95584 62189 5224	(01)
	6	-8.71637 35593 96335 4058	(00)	7.76976 93140 15107 4176	(01)
	7	-1.78083 11603 69779 9966	(01)	5.06939 12820 57973 5193	(01)
	8	-4.08888 38379 36219 6702	(00)		
9	0	9.99993 31061 60568 73909 1	(-01)	1.00153 38520 45342 69781 8	(00)
	1	-1.84508 62323 91278 67452 4	(00)	-1.09355 61953 91091 24392 4	(01)
	2	2.65257 58184 52799 81985 5	(01)	1.99100 44708 17742 47072 6	(02)
	3	2.49548 77304 02059 44062 6	(01)	1.19283 24239 68601 00698 5	(03)
	4	-3.32361 25793 43962 28433 3	(01)	4.42941 31783 37928 40116 1	(01)
	5	-9.13483 56999 98742 55243 2	(-01)	2.53881 93156 30708 03171 3	(02)
	6	-2.10574 07995 48040 45039 4	(01)	5.99493 23256 67407 35525 5	(01)
	7	-1.00064 19139 89284 82996 1	(01)	6.40380 04053 52415 55132 4	(01)
	8	-1.86009 21217 26437 58225 3	(01)	9.79240 35992 17290 29684 0	(01)
	9	-1.64772 11724 63463 14004 2	(00)		

TABLE V

$$Ei(x) \simeq \frac{e^x}{x}\left\{1 + \frac{1}{x}\left[\alpha_0 + \frac{\beta_0}{\alpha_1 + x}\Big/ + \frac{\beta_1}{\alpha_2 + x}\Big/ + \dots + \frac{\beta_{n-1}}{\alpha_n + x}\right]\right\}, \qquad 24 \le x$$

n	j	α_j		β_j	
1	0	1.00015 389	(00)	1.98096 735	(00)
	1	-3.32539 397	(00)		
2	0	1.00000 09509	(00)	1.99974 97038	(00)
	1	-3.01030 27254	(00)	-2.65923 65676	(00)
	2	-6.76665 60450	(00)		
3	0	1.00000 00074 241	(00)	1.99999 64513 647	(00)
	1	-3.00027 77365 852	(00)	-2.98064 01538 776	(00)
	2	-5.22606 50476 386	(00)	-4.35679 76837 196	(00)
	3	-1.22820 81736 176	(01)		
5	0	1.00000 00000 70443	(00)	1.99999 99428 26009	(00)
	1	-3.00000 77799 35772	(00)	-2.99901 18065 26193	(00)
	2	-5.02233 17461 85109	(00)	-7.18975 18395 04450	(00)
	3	-9.14830 08216 73641	(00)	2.72761 00778 77917	(00)
	4	-1.01047 90815 76032	(01)	1.22399 93926 82269	(02)
	5	-2.77809 28934 43810	(01)		
6	0	9.99999 99997 90809 6	(-01)	2.00000 00245 00921 8	(00)
	1	-2.99999 50425 41715 1	(00)	-3.00099 33150 98587 5	(00)
	2	-4.96095 06919 91292 0	(00)	-1.09844 41603 21675 0	(01)
	3	8.29029 78997 47356 2	(00)	-4.76825 47158 68940 9	(02)
	4	1.64009 32440 03647 9	(01)	8.56008 33460 69529 7	(00)
	5	-1.21528 49306 41334 6	(01)	1.10000 77721 38291 8	(02)
	6	-2.74650 43100 92359 1	(01)		

TABLE V—*Continued*

n	j	α_j		β_j	
7	0	1.00000 00000 00583 93	(00)	1.99999 99992 41308 92	(00)
	1	-3.00000 01678 20851 77	(00)	-2.99996 43294 44464 54	(00)
	2	-5.00140 34551 59243 47	(00)	-7.90404 99229 89255 13	(00)
	3	-7.49289 16779 28844 28	(00)	-4.31325 83614 66279 61	(00)
	4	-3.08336 26905 17627 02	(01)	2.95999 39948 68313 26	(02)
	5	-1.39381 36036 44050 73	(00)	-6.74704 58046 58324 31	(00)
	6	8.91263 82257 37077 52	(00)	1.04745 36265 24683 01	(03)
	7	-5.31686 62349 44816 19	(01)		
8	0	9.99999 99999 98429 664	(-01)	2.00000 00002 75138 624	(00)
	1	-2.99999 99164 99449 771	(00)	-3.00002 51613 16953 333	(00)
	2	-4.99852 16489 02883 666	(00)	-8.16569 56642 07823 553	(00)
	3	-5.42548 34248 64505 529	(00)	-1.01407 31779 52169 808	(02)
	4	2.91374 12138 03809 072	(01)	-6.86068 28717 22917 564	(01)
	5	-3.45514 82712 11662 211	(01)	4.49171 53500 89152 693	(02)
	6	1.71450 82495 07687 053	(00)	-8.82822 46243 43449 652	(00)
	7	6.10290 79315 33490 078	(-01)	6.49474 49258 23696 291	(02)
	8	-4.65261 50211 31943 298	(01)		
9	0	1.00000 00000 00004 85503	(00)	1.99999 99999 90481 04167	(00)
	1	-3.00000 00032 09812 65753	(00)	-2.99999 89404 03249 59612	(00)
	2	-5.00006 64041 31310 02475	(00)	-7.99243 59577 63397 41065	(00)
	3	-7.06810 97789 50293 58836	(00)	-1.20187 76354 71547 43238	(01)
	4	-1.52856 62363 69296 36839	(01)	7.04831 84718 04246 75988	(01)
	5	-7.63147 70162 02536 30855	(00)	1.17179 22050 20864 55287	(02)
	6	-2.79798 52862 43053 89340	(01)	1.37790 39023 57479 98793	(02)
	7	-1.81949 66492 98689 06455	(01)	3.97277 10910 04145 18365	(00)
	8	-2.23127 67077 76324 09550	(02)	3.97845 97716 74147 20840	(04)
	9	1.75338 80126 54659 72390	(02)		

3. Converging Factors. The quantities $C_n(x)$ given by (2.4) are known as converging factors. Tables and extensive theoretical discussions of these quantities are to be found in Dingle [6], [7] and in Murnaghan and Wrench [8]. Additional expressions for the $C_n(x)$ can be obtained from the formulations of Wadsworth [9] and Hitotumatu [10].

Although not of primary importance in our work, expansions of the converging factors about an arbitrary x are convenient for many purposes. First we note that the converging factors, for each $k = 1, 2, \cdots$, obey the relation

$$C_n(x) = \sum_{j=0}^{k-1} \frac{(n+j)!}{n!x^j} + \frac{(n+k)!}{n!x^k} C_{n+k}(x) , \tag{3.1}$$

which follows from (2.3). Letting $t = xu/(x+h)$ in (2.4) we find

$$C_n(x+h) = (1 + h/x)^{n+1} \frac{1}{n!} \int_0^\infty \frac{t^n e^{-t} e^{-th/x}}{1 - t/x} \, dt .$$

Expanding the second exponential in a Maclaurin series and using (3.1), this becomes

$$C_n(x+h) = \left(1 + \frac{h}{x}\right)^{n+1} \left[e^{-h} C_n(x) - \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \frac{(n+j)!}{n!x^j} \frac{(-h)^k}{k!} \right] .$$

Reversing the order of summation we have

$$C_n(x+h) = \left(1 + \frac{h}{x}\right)^{n+1} \left[e^{-h} C_n(x) - \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \frac{(-h)^k}{k!} \frac{(n+j)!}{n!x^j} \right] , \tag{3.2}$$

in which the double series can readily be summed by an obvious extension of Horner's algorithm. Equation (3.2) can be shown to include for $n = 0$ the recurrences given by Harris [1] and by Miller and Hurst [2].

In addition to its utility for interpolating in a preexisting table of $C_n(x)$, (3.2) can also be used to generate such a table from a single initial value. Repeated application of the expansion with positive h is numerically stable and tends to damp out errors in the initial value.

4. Generation of the Approximations. The approximation forms and corresponding intervals used are

$$\begin{aligned} E_{lm}(x) &= \ln(x/x_0) + (x - x_0) R_{lm}(x) , \qquad 0 < x \leqq 6 , \\ &= (e^x/x) R_{lm}(1/x) , \qquad 6 \leqq x \leqq 12, 12 \leqq x \leqq 24 , \\ &= (e^x/x)[1 + (1/x) R_{lm}(1/x)] , \qquad 24 \leqq x , \end{aligned}$$

where

$$x_0 = .37250\ 74107\ 81366\ 63446\ 19918\ 66580$$

is the zero of $\mathrm{Ei}(x)$, and the $R_{lm}(z)$ are rational functions of degree l in the numerator and m in the denominator. The combination of forms and intervals was the best of many such combinations tried.

The approximations were computed using standard versions of the Remez algorithm for rational Chebyshev approximation [11], [12] in 25S arithmetic on a CDC 3600. All error curves were levelled to at least 3S. Function values were computed as needed using a variety of methods. For small x the standard Maclaurin series (2.1) was used except in the vicinity of x_0, where a Taylor series of the form (2.2) was used. Harris' scheme [1] was used for $5 \leqq x \leqq 55$ and an economized asymptotic series for $x > 55$. The value of x_0 was determined by Newton's method applied to (2.1) in 40S arithmetic. The same arithmetic was used to generate the Taylor series coefficients, and the values Ei(n) necessary for Harris' scheme.

The master function routines were extensively checked for accuracy by comparing calculations based on two different methods wherever possible. Additional comparisons were made against tables, especially those of Murnaghan and Wrench [8], and against 40S computations with the Maclaurin series. We believe that the master function routines were accurate to at least 22S, except for $x > 55$, where only 20.5S was achieved.

As is indicated in our tabulated results, a number of the approximations obtained had nonstandard error curves or gave computational difficulty because of near-degeneracy. (See [13] for a description of this phenomenon.) The situation for the present choice of approximation intervals is not nearly as bad, however, as the situation when the form

$$(e^x/x)[1 + (1/x)R_{lm}(1/x)]$$

was tried for the intervals [8, ∞) and [16, ∞). In each case a "barrier" which made high-accuracy approximations difficult to obtain was encountered. This barrier was signalled by almost an entire counter-diagonal of cases in the Walsh array with nonstandard error curves. Beyond this barrier, increases in the number of coefficients gave only minor increases in accuracy. For example, the barrier for the [16, ∞) interval occurred for $l + m = 4$, with the relative error for R_{22} about 2.6×10^{-8} and that for R_{66} only 1×10^{-12}. On the present [24, ∞) interval, the vestige of the barrier may be evident in the vicinity of the counter-diagonal for $l + m = 6$ (see Table I).

5. Results. Table I lists the values of

$$\mathcal{E}_{lm} = -100 \log_{10} \max \left| \frac{\mathrm{Ei}(x) - E_{lm}(x)}{\mathrm{Ei}(x)} \right|,$$

where the maximum is taken over the appropriate interval, for the initial segments of the various L_∞ Walsh arrays. Tables II–V present coefficients for all approximations along the main diagonals of these arrays except for R_{44} for $24 \leqq x$, which has essentially the same accuracy as R_{33}.

All coefficients are given to accuracies greater than that justified by the maximal errors, but reasonable additional rounding should not greatly affect the overall accuracies. Each approximation listed, with the coefficients just as they appear here, was tested against the master function routines with 5000 pseudo-random arguments. In all cases the maximal error agreed in magnitude and location with the values given by the Remez algorithm.

6. Use of the Coefficients. An attempt has been made to present the various rational functions in a well-conditioned form. Thus, the rationals used for the interval $0 < x \leqq 6$ were found to lose significance during evaluation due to subtraction of nearly equal quantities unless expressed as ratios of finite sums of shifted Chebyshev polynomials (see [3])

$$R_{nn}(x) = \sum_{j=0}^{n}{}' p_j T_j^*(x/6) \Big/ \sum_{j=0}^{n}{}' q_j T_j^*(x/6) .$$

These sums are most conveniently evaluated by noting that $T_i^*(x) = T_i(2x-1)$ and then using the Clenshaw-Rice algorithm [14]. Similarly, the remaining approximations presented in this paper are found to be well-conditioned in the J-fraction form

$$R_{nn}(1/x) = \alpha_0 + \frac{\beta_0}{\alpha_1 + x} \Big/ + \frac{\beta_1}{\alpha_2 + x} \Big/ + \cdots + \frac{\beta_{n-1}}{\alpha_2 + x} .$$

For use on computers with prohibitively large divide-times, the corresponding ratio of polynomials form may be used with losses of about 2S or 3S in some cases.

The main remaining source of avoidable error in the implementation of these approximations into computer subroutines is in the handling of x_0. If it is desired to maintain good relative accuracy, i.e. essentially machine precision, in the vicinity of x_0, the quantity $(x - x_0)$ should be computed to higher than machine precision to preserve the low order bits of x_0. This can be readily accomplished by breaking x_0 into two parts, call them x_1 and x_2, such that, to the precision desired, $x_0 \equiv x_1 + x_2$ and the floating point exponent on x_2 is much less than that on x_1. This breakup is most easily accomplished by examining the octal or hexadecimal representation

$$\begin{aligned} x_0 &= .27656\ 24522\ 55132\ 77417\ 11446\ 06004\ 16157_8 \\ &= .5F5CA\ 54AD2\ D7F0F\ 264C3_{16} . \end{aligned}$$

Then $(x-x_0)$ is computed as $(x-x_0) = (x-x_1) - x_2$. Additional precautions will have to be made to compute $\ln(x/x_0)$ for $x \simeq x_0$. We suggest

$$\ln(x/x_0) = \ln(1 + (x - x_0)/x_0)$$

coupled with a special computation of $\ln(1 + y)$ for small y (usually a few terms in the Taylor series will suffice). Note that the computation of $(x-x_0)/x_0$ can be carried out in normal precision once $(x-x_0)$ has been determined as above.

Subroutines based on the coefficients and techniques given here and in [4] have been written at Argonne National Laboratory for the CDC 3600 (single precision) and the IBM System/360 (short and long precision) computers. Essentially machine accuracy was obtained in all cases.

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