TECHNICAL NOTES AND SHORT PAPERS

On the Computation of the Eigenvalues of a Tridiagonal Matrix

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Abstract. A recent algorithm for the simultaneous approximation of all zeros of a polynomial is applied to the computation of the eigenvalues of a tridiagonal matrix. The method works in the presence of multiplicity and degeneracy and has been tested in a multitude of cases; its practical limitation on a computer is the large number of locations required for matrices of high order. ■

The efficiency of the second algorithm described in [1] for determining the zeros of a polynomial has been investigated in connection with the computation of the eigenvalues of a tridiagonal matrix, all the eigenvalues being computed at the same time together with their range of approximation.

Let A be the given matrix of order N whose coefficients (real or complex) are arranged in the following way:

$$A = \begin{bmatrix} \alpha_1 & \beta_2 & & & & \\ \gamma_2 & \alpha_2 & \beta_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot \\ & & & & \gamma_N & \alpha_N \end{bmatrix}.$$

The evaluation of the characteristic polynomial $P(z) = |A - zI|$ and of its successive derivatives $P^{(k)}(z)$, $k = 1, 2, \cdots, N$, can be carried out by means of the formulas listed below:

$$P_0^{(0)}(z) = 1\,,$$
$$P_1^{(0)}(z) = \alpha_1 - z\,,$$
$$P_{r+1}^{(0)}(z) = (\alpha_{r+1} - z)P_r^{(0)}(z) - \gamma_{r+1}\beta_{r+1}P_{r-1}^{(0)}(z)\,, \qquad r = 1, 2, \cdots, N-1\,,$$

where

$$P_N^{(0)} = P(z)\,,$$
$$P_k^{(k)} = -kP_{k-1}^{(k-1)}(z)\,,$$
$$P_{k+1}^{(k)} = (\alpha_{k+1} - z)P_k^{(k)}(z) - kP_k^{(k-1)}(z)\,,$$
$$P_{r+1}^{(k)}(z) = (\alpha_{r+1} - z)P_r^{(k)}(z) - kP_r^{(k-1)}(z) - \gamma_{r+1}\beta_{r+1}P_{r-1}^{(k)}(z)\,,$$
$$r = k+1, \cdots, N-1; k = 1, 2, \cdots, N\,,$$

where

$$P_N^{(k)}(z) = P^{(k)}(z)\,, \qquad k = 1, 2, \cdots, N\,.$$

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Since the zero-searching routine operates in the square Q_0 centered at the origin and with side 2, we first transform the matrix A into a new matrix having all the eigenvalues inside Q_0. The algorithm consists in the construction of rectangles R_i ($i = 1, 2, \cdots, I, I \leqq N$) with the properties: (1) at least one zero of $P(z)$ belongs to R_i; (2) the exact number of zeros inside R_i can be determined by applying a discrete form of the argument principle.

The given matrix can be real or complex, but in the real case, we can restrict our search to the superior or inferior half of the square Q_0. As a rectangle having a side belonging to the x-axis can have a real zero on the boundary, it is necessary to make this region symmetric with respect to the axis of abscissas before applying the argument principle.

A program was written for the IBM 360/40 in Fortran and the routine tested in some examples in which the results were known. The procedure was found to give correct results in the cases tested; however, some shortcomings have to be mentioned. First, a certain amount of calculation is required, the number of multiplications necessary to evaluate $P(z)$ and $P^{(k)}(z)$ being of the order of $5N^2$ in the real case. Secondly, the number of locations necessary to store the components of the rectangles R_i is not known a priori and increases with the order of the matrix: for $N = 30$ the average number of components to be stored is of the order of 10^3.

TABLE I

0.5533569325674574D 01	0.0	0.22152527E-12
-0.2588951455024342D 01	0.0	0.22152527E-12
-0.1164032792944693D 01	0.0	0.22152527E-12
0.5401387531459531D-01	0.0	0.22152527E-12
0.4615720899143838D 00	-0.7622959315358884D 00	0.18087463E-12
0.4615720899143838D 00	0.7622959315358884D 00	0.18087463E-12
0.1333810222560614D 01	-0.1339720129773309D 01	0.18087463E-12
0.1333810222560614D 01	0.1339720129773309D 01	0.18087463E-12
0.2030707454124239D 01	-0.1819593648996558D 01	0.28598792E-12
0.2030707454124239D 01	0.1819593648996558D 01	0.28598792E-12
0.1268321841354136D 01	0.0	0.22152527E-12
0.2021261480890686D 01	0.0	0.22152527E-12
0.2104175313526226D 01	-0.3539930113621053D 01	0.18087463E-12
0.2104175313526226D 01	0.3539930113621053D 01	0.18087463E-12
0.2528250310999125D 01	-0.4599502563845732D 00	0.18087463E-12
0.2528250310999125D 01	0.4599502563845732D 00	0.18087463E-12
0.3293146473194282D 01	0.0	0.22152527E-12
0.4665640469291146D 01	0.0	0.22152527E-12

In the following, we denote by α, β, γ the vectors of components $\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$, $\{\beta_2, \beta_3, \cdots, \beta_N\}$, $\{\gamma_2, \gamma_3, \cdots, \gamma_N\}$ respectively. A typical result of the routine is in Table I for the matrix with

$$\alpha = \{1, 2, 1, 2, 1, 1, 2, 3, 1, 2, 4, 1, 3, 2, 1, 4, 1, -2\},$$
$$\beta = \{-3, 2, 1, -3, 2, -4, -5, -5, -3, -2, 1, 2, 3, 3, 4, 1, 2\},$$
$$\gamma = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}.$$

Experimentally, regions containing a multiple zero were found to be larger than regions containing a simple root. It so happens that for the same degree of refinement of the initial square Q_0, the error bound for a multiple zero is larger than for a simple one, and increases with increasing multiplicity. As an example, we give in Table II the results for the matrix with

$$\alpha = \{1, 1, 1, -2, -2, 3\}, \beta = \{0, 1, 0, -1, 0\}, \gamma = \{-2, 0, 1, 0, 1\}.$$

Degeneracy is ignored by the algorithm. Numerical confirmation of this fact is illustrated by the computation of the eigenvalues of the matrix with $\alpha = \{0, 0, 0, 0, 0, 0\}$, $\beta = \{1, 1, 0, 1, 0\}$, $\gamma = \{0, 0, 0, 0, 0\}$, which results are in Table III.

TABLE II

0.2999999999999998D 01	0.0	0.23360153E-05
-0.1999999999999999D 01	0.0	0.46720306E-05
-0.1999999999999999D 01	0.0	0.46720306E-05
0.9999999999999980D 00	0.0	0.70080459E-05
0.9999999999999980D 00	0.0	0.70080459E-05
0.9999999999999980D 00	0.0	0.70080459E-05

TABLE III

0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05
0.0	0.0	0.40880268E-05

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