Osculatory Interpolation*

By S. W. Kahng

Abstract. An explicit method of osculatory interpolation with a function of the form

$$\begin{aligned}R(x) = {} & f_{00}(a_{00} + g_0(x)f_{01}(a_{01} + g_0(x)f_{02}(a_{02} + \cdots + g_0(x) \\ & \cdot f_{0,m_0}(a_{0,m_0} + g_0(x)f_{10}(a_{10} + g_1(x)f_{11}(a_{11} \\ & + \cdots + g_1(x)f_{1,m_1}(a_{1,m_1} + g_1(x)f_{20}(a_{20} + \cdots + g_{n-1}(x) \\ & \cdot f_{n0}(a_{n0} + g_n(x)f_{n1}(a_{n1} + \cdots + g_n(x)f_{n,m_n}(a_{n,m_n})) \cdots)\end{aligned}$$

is described. Error terms for the interpolation are determined.

1. Introduction. Osculatory interpolation of a continuous function and its first m_i derivatives at base points $x_0, x_1, x_2, \cdots, x_n$ has been studied by many authors. Wendroff described an explicit method using polynomials. Salzer [4] and Thacher [5] showed, separately, the method of interpolation with a continued fraction when $m_i = 1$, and indicated that similar interpolation could be made with other classes of functions.

In this paper, we describe a class of interpolation functions and show the explicit method of osculatory interpolation with a function in the class. Also, error terms for the interpolation are determined.

2. Interpolating Functions. Interpolation of a function is made ordinarily by a polynomial or a rational function and is adequate for most purposes. However, it has been shown recently that the generalization of interpolation functions yield new results. Larkin [2] has generalized Neville-Aitken's method and Kahng [1] showed the generalization of Newton's method and applied it to the approximation problems. These generalizations extend the applicable interpolation functions from polynomials to rational functions, their transformations, and some nonlinear functions. Also, these generalizations enable us to treat the interpolation in a unified manner. Kahng has employed the interpolation function

$$Q(x) = f_0(a_0 + g_0(x)f_1(a_1 + g_1(x)f_2(a_2 + \cdots + g_{n-1}(x)f_n(a_n)) \cdots)) .$$

This function can also be expressed as $Q(x) = f_0\{D_0(x)\}$, where

$$D_i(x) = a_i + g_i(x) \cdot f_{i+1}\{D_{i+1}(x)\} , \qquad i = 0, 1, 2, \cdots, n - 1 ,$$

and $D_n(x) = a_n$.

Some of the special cases of the above interpolation functions are shown below with indices $i = 0, 1, \cdots, n$ and $j = 0, 1, 2, \cdots, n - 1$ unless otherwise noted:

(a) if $f_i(u) = u$, $g_j(x) = x - x_j$, then $Q(x)$ is the Newton's interpolation formula,

(b) if $f_i(u) = u$, $i = 0, 1, \cdots, K - 1$, $f_i(u) = 1/u$, $i = K, K + 1, \cdots, n$, and $g_j(x) = x - x_j$, then $Q(x)$ can be expanded to the rational function S_{n-m}/S_m, where $m = [(n - K + 1)/2]$ and S_m is a polynomial of degree m,

Received June 28, 1968, revised November 6, 1968.

* This work was supported by National Aeronautics and Space Administration under contract NAS 9-5384.

(c) if $f_0(u) = 1/u$ in (b), then $Q(x) = S_m/S_{n-m}$,

(d) if $f_i(u) = u$, $g_j(x) = \sin x - \sin x_i$, then $Q(x)$ is a trigonometric function and may be expanded to a finite Fourier series,

(e) if we set $g_j(x) = h_j(x) - h_j(x_j)$, and choose $f_i(x)$ and $h_j(x)$ from x, $1/x$, e^x, x^2, and $\cos x$ etc., then we have a class of interpolation functions.

The conditions on the functions f's and g's for the existence of unique parameters $a_0, a_1, \cdots, a_n$ are given below using the following notations:

Notations.

$$h(A) = \{h(x) | x \in A\}\ ,$$
$$R(h)\text{:range of } h(x)\ .$$

THEOREM [1]. *Given a function $y(x)$ continuous in a finite interval $[a, b]$ and $n + 1$ points x_i such that $a \leqq x_0 < x_1 < \cdots < x_n \leqq b$.*

Then there exists a unique set of parameters $a_0, a_1, \cdots, a_n$ for the interpolation function

$$Q(x) = f_0(a_0 + g_0(x) f_1(a_1 + \cdots + g_{n-1}(x) f_n(a_n)) \cdots)$$

satisfying $Q(x_i) = y(x_i)$, $i = 0, 1, 2, \cdots, n$ and $Q(x)$ is continuous if

(a) *f_i is continuous, strictly monotone in $(-\infty, \infty)$, and the range of $f_i(x)$ covers $(-\infty, \infty)$, $i = 1, 2, \cdots, n$,*

(b) *f_0 is continuous and its inverse function f_0^{-1} exists in $R(f_0)$, and $R(f_0) \supset y([a, b])$,*

(c) *functions $g_j(x)$, $j = 0, 1, 2, \cdots, n - 1$ are continuous in $[a, b]$ and*

$$g_j(x) = 0 \qquad x = x_j\ ,$$
$$\neq 0 \qquad x > x_j\ .$$

When above conditions are satisfied, the parameters are determined from the following equations in sequence: $a_0 = f_0^{-1}(Q(x_0))$ is determined from $Q(x_0) = f_0(a_0)$, $a_1 = f_1^{-1}((f_0^{-1}(Q(x_1)) - a_0)/g_0(x_1))$ is found from $Q(x_1) = f_0(a_0 + g_0(x_1) f_1(a_1))$, and so on. In practice a divided difference table may be constructed to determine the parameters, and the table will be a special case of Table 1.

3. Osculatory Interpolation. Consider the problem of interpolating a function $y(x)$ and its initial m_i derivatives of $y(x)$ at x_i, $i = 0, 1, 2, \cdots, n$.

In case of the Hermite interpolation one uses

$$\begin{aligned} R(x) = {} & a_{00} + (x - x_0)(a_{01} + (x - x_0)(a_{02} + \cdots + (x - x_0)(a_{0,m_0} + (x - x_0) \\ & \cdot (a_{10} + (x - x_1)(a_{11} + (x - x_1)(a_{12} + \cdots + (x - x_1)(a_{1,m_1} \\ & + (x - x_1)\ . \\ & \cdot \\ & \cdot \\ & \cdot \\ & (a_{n0} + (x - x_n)(a_{n1} + x - x_n)(a_{n2} + \cdots + (x - x_n)(a_{n,m_n})) \cdots) \end{aligned}$$

as the interpolation function. This function can be generalized in analogies to the generalization of the Newton's interpolation function as follows:

$$
R(x) = f_{00}(a_{00} + g_0(x) f_{01}(a_{01} + g_0(x) f_{02}(a_{02} + \cdots \tag{1}
$$
$$
+ g_0(x) f_{0,m_0}(a_{0,m_0} + g_0(x) f_{10}(a_{10} + g_1(x) f_{11}(a_{11} + \cdots
$$
$$
+ g_1(x) f_{1,m_1}(a_{1,m_1} + g_1(x) f_{20}(a_{20} + \cdots + g_{n-1}(x) f_{n0}(a_{n0}
$$
$$
+ g_n(x) f_{n1}(a_{n1} + \cdots + g_n(x) f_{n,m_n}(a_{n,m_n})) \cdots) .
$$

This function can also be written as $R(x) = f_{00}\{E_{00}(x)\}$, where

$$
E_{ij}(x) = a_{ij} + g_i(x) f_{i,j+1}(E_{i,j+1}(x)) \quad \text{for } i = 0, 1, 2, \cdots, n; j = 0, 1, \cdots, m_{i-1} ,
$$
$$
E_{i,m_i}(x) = a_{i,m_i} + g_i(x) f_{i+1,0}(E_{i+1,0}(x)) , \qquad i = 0, 1, \cdots, n-1
$$

and

$$
E_{n,m_n}(x) = a_{n,m_n} .
$$

$R(x)$ includes, as special cases, the following functions:

(a) Hermite interpolation function

$$
f_{ij}(x) = x , \qquad g_i(x) = x - x_i , \qquad i = 0, 1, \cdots, n , \qquad j = 0, 1, \cdots, m_i ,
$$

(b) truncated continued fraction interpolation function

$$
\begin{aligned}
f_{ij}(x) &= x \quad \text{if } i = j = 0 \\
&= 1/x \quad \text{otherwise} , \\
g_i(x) &= x - x_i .
\end{aligned}
$$

In what follows we assume certain properties for functions f's and g's and then derive a method of osculatory interpolation with $R(x)$ using the following notations:

$$
f_{ij} = f_{ij}(E_{ij}(x)) , \qquad f_{ij}^{(k)} = \frac{d^k}{dx^k} f_{ij}(x) ,
$$
$$
g_i = g_i(x) , \qquad E_{ij} = E_{ij}(x) , \qquad E_{ij}^{(k)} = \frac{d^k}{dx^k} (E_{ij}(x)) .
$$

Assume that

(a) Functions f's and g's have continuous Mth derivatives and $f'(x) \neq 0$ in $(-\infty, \infty)$, where $M = \max \{m_i | i = 0, 1, 2, \cdots, n\}$.

(b) Each function f_{ij} has its inverse function f_{ij}^{-1} in $(-\infty, \infty)$.

(c)

$$
\begin{aligned}
g_i(x_j) &= 0 \quad \text{if } i = j , \\
&\neq 0 \quad \text{if } i < j ,
\end{aligned}
$$

and

$$
g_i'(x_j) \neq 0 \quad \text{for } i, j = 0, 1, \cdots, n .
$$

By repeatedly differentiating E_{ij} and f_{ij} we have

$$
(2)\qquad
\begin{aligned}
E_{ij} &= a_{ij} + g_i f_{i,j+1}(E_{i,j+1}) ,\\
E'_{ij} &= g_i' f_{i,j+1} + g_i \frac{d}{dx} f_{i,j+1} ,\\
E''_{ij} &= g_i'' f_{i,j+1} + 2g_i' \frac{d}{dx} f_{i,j+1} + g_i \frac{d^2}{dx^2} f_{i,j+1} ,\\
&\;\;\vdots\\
E^{(m)}_{ij} &= \sum_{k=0}^{m} \binom{m}{k} g_i^{(m-k)} \frac{d^k}{dx^k} f_{i,j+1}
\end{aligned}
$$

and omitting the subscripts ij in the right-hand side:

$$
(3)\qquad
\begin{aligned}
\frac{d}{dx} f_{ij} &= f'E' ,\\
\frac{d^2}{dx^2} f_{ij} &= f''(E')^2 + f'E'' ,\\
\frac{d^3}{dx^3} f_{ij} &= f'''(E')^3 + 3f''E'E'' + f'E''' ,\\
\frac{d^4}{dx^4} f_{ij} &= f^{\mathrm{IV}}(E')^4 + 6f'''(E')^2E'' + 3f''(E'')^2 + 4f''E'E''' + f'E^{\mathrm{IV}} ,\\
\frac{d^5}{dx^5} f_{ij} &= f^{\mathrm{V}}(E')^5 + 10f^{\mathrm{IV}}(E')^3E'' + 15f'''E'(E'')^2\\
&\quad + 10f'''(E')^2E''' + 10f''E''E''' + 5f''E'E^{\mathrm{IV}} + f'E^{\mathrm{V}} ,\\
&\;\;\vdots
\end{aligned}
$$

From Eqs. (3) we have, if f'_{ij} is nonzero, by omitting the subscripts ij in the right-hand side:

$$(4.1)\qquad E_{ij} = f^{-1}(f(E_{ij})) ,$$

$$(4.2)\qquad E'_{ij} = (df/dx)/f' ,$$

$$(4.3)\qquad E''_{ij} = (d^2f/dx^2 - f''(E')^2)/f' ,$$

$$(4.4)\qquad E'''_{ij} = (d^3f/dx^3 - f'''(E')^3 - 3f''E'E'')/f' ,$$

$$(4.5)\qquad E^{\mathrm{IV}}_{ij} = (d^4f/dx^4 - f^{\mathrm{IV}}(E')^4 - 6f'''(E')^2E'' - 3f''(E'')^2 - 4f''E'E''')/f' ,$$

$$
(4.6)\qquad
\begin{aligned}
E^{\mathrm{V}}_{ij} = (d^5f/dx^5 &- f^{\mathrm{V}}(E')^5 - 10f^{\mathrm{IV}}(E')^3E'' - 15f'''E'(E'')^2\\
&- 10f'''(E'')^2E''' - 10f''E''E''' - 5f''E'E^{\mathrm{IV}})/f' ,\\
&\;\;\vdots
\end{aligned}
$$

We also have from Eqs. (2), if $g_i(x) \neq 0$,

$$(5.1)\qquad f_{i,j+1} = (E_{ij} - a_{ij})/g_i ,$$

(5.2) $$\frac{d}{dx} f_{i,j+1} = (E'_{ij} - g_i' f_{i,j+1})/g_i ,$$
$$\vdots$$

(5.3) $$\frac{d^m}{dx^m} f_{i,j+1} = \left(E_{ij}^{(m)} - \sum_{k=0}^{m-1} \binom{m}{k} g_i^{(m-k)} \frac{d^k}{dx^k} f_{i,j+1}\right)$$

and if $g_i(x) = 0$

(6.1) $$f_{i,j+1} = E'_{ij}/g_i' ,$$

(6.2) $$\frac{d}{dx} f_{i,j+1} = \frac{1}{2g_i'} (E''_{ij} - g_i'' f_{i,j+1}) ,$$
$$\vdots$$

(6.3) $$\frac{d^{m-1}}{dx^{m-1}} f_{i,j+1} = \frac{1}{mg_i'} \left(E_{ij}^m - \sum_{k=0}^{m-2} \binom{m}{k} g_i^{(m-k)} \frac{d^k}{dx^k} f_{i,j+1}\right) .$$

In the above equations, it is to be understood that if $j = m_i$, then $f_{i,j+1} = f_{i+1,0}$, and if $j = m_i + 1$, then $f_{i,j} = f_{i+1,0}$ and g_i is replaced by g_{i+1}.

With the above equations, one can determine the parameters of the interpolation function $R(x)$ for $y(x_i) = R(x_i)$ and $y^{(j)}(x_i) = R^{(j)}(x_i)$, $i = 0, 1, 2, \cdots, n$, $j = 1, 2, \cdots, m_i$. The procedure is described below using the following notation for simplicity:

$$h_{kl}^{(j)}(x_i) = \frac{d^j}{dx^j} f_{kl}(x)|_{x=x_i} .$$

We are given $y(x_0) = h_{00}^{(0)}(x_0) = f_{00}(a_{00})$. Using (4.1), we have $a_{00} = f_{00}^{-1}(h_{00}^{(0)}(x_0))$. From $y'(x_0) = h_{00}^{(1)}(x_0)$,

$$E'_{00}(x_0) = h'_{00}(x_0)/f'_{00}(a_0)$$

by (4.1),

$$h_{01}^{(0)}(x_0) = f_{01}(x_0) = E'_{00}(x_0)/g_0'(x_0)$$

by (6.1), and

$$a_{01} = E_{01}(x_0) = f_{01}^{-1}(h_{01}^{(0)}(x_0))$$

by (4.1). For $h_{00}^{(j)}(x_i)$, we apply (4. $j - 1$) and (5.3) alternately until $E_{i0}^{(j)}(x_i)$ is obtained. Then apply (6.3) and (4. $j - 1$) alternately until $a_{ij} = E_{ij}(x_i)$ is found.

The above process of determining the parameters is better described by the divided difference table shown in Table 1. This table is filled from top to bottom and from left to right. In each entry, the term in the left-hand side is computed with the equation numbered in the right-hand side of that entry.

The uniqueness of the interpolation function is dependent on the functions f_{ij} employed. If $f_{ij}(x) = x$ for all i and j, then the interpolation function is unique. In other cases the interpolation function may not be unique in the sense that there

x_0	$h_{00}(x_0)$ GIVEN	$E_{00}(x_0)$, (4-1)	$h_{01}(x_0)$, (6-1)	...	$E_{0,m_0}(x_0)$, (4-1)				
x_0	$h'_{00}(x_0)$ GIVEN	$E'_{00}(x_0)$, (4-2)	$h'_{01}(x_0)$, (6-2)						
x_0	$h''_{00}(x_0)$ GIVEN	$E''_{00}(x_0)$, (4-3)	$h''_{01}(x_0)$						
⋮	⋮	⋮	⋮						
x_0	$h_{00}^{(m_0-1)}(x_0)$ GIVEN	$E_{00}^{(m_0-1)}(x_0)$	$h_{01}^{(m_0-1)}(x_0)$, (6-3)						
x_0	$h_{00}^{(m_0)}(x_0)$ GIVEN	$E_{00}^{(m_0)}(x_0)$							
x_1	$h_{00}(x_1)$ GIVEN	$E_{00}(x_1)$, (4-1)	$h_{01}(x_1)$, (5-1)		$E_{0,m_0}(x_1)$, (4-1)	$h_{10}(x_1)$, (5-1)	$E_{10}(x_1)$, (4-1)	$h_{11}(x_1)$, (6-1)	
x_1	$h'_{00}(x_1)$ GIVEN	$E'_{00}(x_1)$, (4-2)	$h'_{01}(x_1)$, (5-2)		$E'_{0,m_0}(x_1)$, (4-2)	$h'_{10}(x_1)$, (5-2)	$E'_{10}(x_1)$, (4-2)	$h'_{11}(x_1)$, (6-2)	
x_1	$h''_{00}(x_1)$ GIVEN	$E''_{00}(x_1)$, (4-3)	$h''_{01}(x_1)$		$E''_{0,m_0}(x_1)$, (4-3)	$h''_{10}(x_1)$	$E''_{10}(x_1)$, (4-3)	$h''_{11}(x_1)$	
⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	...
x_1	$h_{00}^{(m_1-1)}(x_1)$ GIVEN	$E_{00}^{(m_1-1)}(x_1)$	$h_{01}^{(m_1-1)}(x_1)$		$E_{0,m_0}^{(m_1-1)}(x_1)$	$h_{1,0}^{(m_1-1)}(x_1)$	$E_{10}^{(m_1-1)}(x_1)$	$h_{11}^{(m_1-1)}(x_1)$, (6-3)	
x_1	$h_{00}^{(m_1)}(x_1)$ GIVEN	$E_{00}^{(m_1)}(x_1)$	$h_{01}^{(m_1)}(x_1)$, (5-3)		$E_{0,m_0}^{(m_1)}(x_1)$	$h_{10}^{(m_1)}(x_1)$, (5-3)	$E_{10}^{(m_1)}(x_1)$		
x_2	$h_{00}(x_2)$ GIVEN	$E_{00}(x_2)$, (4-1)	$h_{01}(x_2)$, (5-1)		$E_{0,m_0}(x_2)$, (4-1)	$h_{10}(x_2)$, (5-1)	$E_{10}(x_2)$, (4-1)	$h_{11}(x_2)$, (5-1)	
x_2	$h'_{00}(x_2)$ GIVEN	$E'_{00}(x_2)$, (4-2)	$h'_{01}(x_2)$, (5-2)		$E'_{0,m_0}(x_2)$, (4-2)	$h'_{10}(x_2)$, (5-2)	$E'_{10}(x_2)$, (4-2)	$h'_{11}(x_2)$, (5-2)	
⋮		⋮						⋮	...

Table 1. DIVIDED DIFFERENCE TABLE

NOTATIONS: $h_{ij}^{(k)}(x_\ell) = \left.\frac{d^{(k)}}{dx^{(k)}} f_{ij}(x)\right|_{x = x_\ell}$, $a_{ij} = E_{ij}(x_i)$

	f_{00}	f_{01}	f_{02}	f_{03}	f_{04}	f_{05}	f_{06}	a_{00}	a_{01}	a_{02}	a_{03}	a_{04}	a_{05}	a_{06}	N/D
1	u	u	u	u	u	u	u	1	1	1/2	1/6	1/24	1/120	1/720	6/0
2	u	u	u	u	u	1/u	1/u	1	1	1/2	1/6	1/24	120	-1/20	5/1
3	u	u	u	1/u	1/u	1/u	1/u	1	1	1/2	6	-2/3	-30	1/4	4/2
4	1/u	1/u	1/u	1/u	1/u	1/u	1/u	1	-1	-2	3	2	-5	-1/38	3/3
5	1/u	u	u	1/u	1/u	1/u	1/u	1	-1	1/2	-6	-2/3	30	1/4	2/4
6	1/u	u	u	u	u	1/u	1/u	1	-1	1/2	-1/6	1/24	-120	-1/20	1/5
7	1/u	u	u	u	u	u	u	1	-1	1/2	-1/6	1/24	-1/120	1/720	0/6

Table 2. OSCULATORY INTERPOLATION OF EXPONENTIAL FUNCTION

may be another interpolation function using the same functions f_{ij} and g_i that agree with $y(x)$ and its derivatives at the given points, yet these two functions may differ. However, the above scheme gives unique parameters a_{ij} for the given base points.

We note that the conditions on the functions f's and g's given in the beginning of this section may be required to hold in their domains rather than in $(-\infty, \infty)$ without restricting the results.

4. Error Term. Let $R(x)$ interpolate $y(x)$ and its first m_i derivatives at $x_0, x_1, \cdots, x_n$,

$$m = \sum_{i=0}^{n} (m_i + 1) ,$$

$$\pi(x) = \prod_{i=0}^{n} (x - x_i)^{m_i+1} ,$$

and

$$F(z) = (z - u)\{y(z) - R(z) - (y(x) - R(x))\pi(z)/\pi(x)\} ,$$

where $x_0, x_1, \cdots, x_n$, and u are all distinct.

Let I be an interval that connects $x_0, x_1, \cdots, x_n$, and x, J be an interval that connects above $n+2$ points and u, and let $y(x)$ and $R(x)$ be in $C^{m+2}(J)$. Then $F(z)$ vanishes at least $m+2$ times, counting multiplicities, in J. By repeatedly applying generalized Rolle's theorem [3], we have

$$\begin{aligned} F^{(m+1)}(z) = {} & (m+1)\{y^{(m)}(z) - R^{(m)}(z) - (y(x) - R(x))^{\pi^{(m)}(z)}/\pi(x)\} \\ & + (z-u)\{y^{(m+1)}(z) - R^{(m+1)}(z)\} , \end{aligned}$$

and $F^{(m+1)}(z)$ has at least one zero in the interior of J. Call this vanishing point ξ, which is a function of x and u. Then,

$$\begin{aligned} y(x) - R(x) = \Big(& y^{(m)}(\xi) + \frac{\xi - u}{m+1} y^{(m+1)}(\xi) - R^{(m)}(\xi) \\ & - \frac{\xi - u}{m+1} R^{(m+1)}(\xi) \Big) \frac{\pi(x)}{m!} . \end{aligned} \tag{7}$$

If we set $u = \xi$, then

$$y(x) - R(x) = (y^{(m)}(\xi) - R^{(m)}(\xi))^{\pi(x)}/m! , \tag{8}$$

where ξ is in the interior of I. On the other hand, provided $R^{(m+1)}(\xi) \neq 0$, if we choose

$$u = \xi + (m+1) \frac{R^{(m)}(\xi)}{R^{(m+1)}(\xi)} ,$$

then

$$R^{(m)}(\xi) + \frac{\xi - u}{m+1} R^{(m+1)}(\xi) = 0 ,$$

and

$$y(x) - R(x) = \left(y^{(m)}(\xi) - \frac{R^{(m)}(\xi)}{R^{(m+1)}(\xi)}\,(y^{(m+1)}(\xi))\right)\frac{\pi(x)}{m!}\,, \tag{9}$$

where ξ is in the interior of J.

5. Choice of $R(x)$. In practical applications, the choice of f's and g's may be determined by the desired form of interpolation function, e.g., polynomial, rational function of degree n with the numerator polynomial of degree l, or certain transformation of a rational function. If there is no restriction as to the form of $R(x)$, the best choice may be the interpolation function that gives the smallest error term among the functions of certain complexity. However, it is not easy to determine such a function without the process of trial and comparison.

6. Example. Interpolation of the exponential function and its first six derivatives at $x = 0$ was made with seven different types of $R(x)$. Here $g_j(x) = x$, and $f_{i0}(x)$, $i = 0, 1, \cdots, 6$, and parameters for $R(x)$ are listed in Table 2. As an illustration, Interpolation No. 3 in the table represents

$$1 + x(1 + x(1/2 + x/\{6 + x/\{-2/3 + x/\{-30 + 4x\}\}\}))\,,$$

and this can be expanded to a polynomial rational function with numerator and denominator degrees four and two, respectively.

The above interpolations may be considered as Taylor series-like expansions since finite Taylor series with m terms can be said to be the polynomial osculatory interpolation of a function with its first $m - 1$ derivatives at a point.

Lockheed Electronics Company
Houston, Texas 77058

1. S. W. Kahng, *Generalized Newton's Interpolation Functions and Their Applications to Chebyshev Approximations*, Lockheed Electronics Company Report, 1967.

2. F. M. Larkin, "Some techniques for rational interpolation," *Comput. J.*, v. 10, 1967, pp. 178–187. MR **35** #6334.

3. A. Ralston, *A First Course in Numerical Analysis*, McGraw-Hill, New York, 1965, p. 62, 74. MR **32** #8479.

4. H. E. Salzer, "Note on osculatory rational interpolation," *Math. Comp.*, v. 16, 1962, pp. 486–491. MR **26** #7133.

5. H. C. Thacher, Jr., "A recursive algorithm for rational osculatory interpolation," *SIAM Rev.*, v. 3, 1961, p. 359.

6. B. Wendroff, *Theoretical Numerical Analysis*, Academic Press, New York, 1966. MR **33** #5080.