Chebyshev Polynomial Expansion of Bose-Einstein Functions of Orders 1 to 10*

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Abstract. Chebyshev series approximations are given for the complete Bose-Einstein functions of orders 1 to 10. This paper also gives an exhaustive presentation of the relation of this function to other functions, with the emphasis that some Fermi-Dirac functions and polylogarithms are readily computable from the given approximations. The coefficients are given in 21 significant figures and the maximal relative error for function representation ranges from 2×10^{-20} to 3×10^{-19}. These expansions are fast convergent; for example, typically six terms gives an accuracy of 10^{-8}.

1. Introduction. The Bose-Einstein function occurs in a wide variety of physical problems, in many different forms. It has been used in problems of statistical physics, quantum electrodynamics, polymer structure and electrical networks [1], [2]. We shall define the most general Bose-Einstein function by its integral representation, as follows:

$$B_p(\eta, u) = \frac{1}{\Gamma(p+1)} \int_0^u \frac{x^p dx}{e^{x-\eta} - 1}, \tag{1}$$

where η and u may be complex. For $u > \eta$ the integral is to be interpreted as a principal value. In this paper we shall only investigate the complete Bose-Einstein function for η, defined as

$$B_p(\eta) \equiv \lim_{u \to \infty} B_p(\eta, u) . \tag{2}$$

The important mathematical properties of this function are discussed in Section 2. Its relations to other functions are presented in Section 3, emphasizing the fact that some functions are readily computable from $B_p(\eta)$. Our method of obtaining the Chebyshev expansions, including discussions of actual computations and accuracy are given in Section 4. The coefficients of the Chebyshev expansion are presented in the microfiche appendix of this issue.

2. Mathematical Properties. The following properties can be found in Truesdell [3], [4], and Dingle [5]:

$$B_p(\eta) = \frac{\partial}{\partial \eta} B_{p+1}(\eta) , \tag{3}$$

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(4) $$B_p(\eta) = \sum_{k=1}^{\infty} \frac{e^{k\eta}}{k^{p+1}}, \qquad \eta < 0,$$

(5) $$= \sum_{k=0}^{\infty} \frac{\eta^k}{k!} \zeta(p+1-k) - \frac{\pi(-\eta)^p}{p! \sin \pi p}, \qquad 0 < -\eta < 2\pi,\ p \neq \text{integer},$$

(6) $$= \sum_{k=0}^{\infty} \frac{\eta^k}{k!} \zeta(p+1-k) - \frac{\pi\eta^p}{p! \tan \pi p}, \qquad 0 < \eta < 2\pi,\ p \neq \text{integer},$$

(7) $$= \sum_{k=0;\ k\neq p}^{\infty} \frac{\eta^k}{k!} \zeta(p+1-k) - \frac{\eta^p}{p!} \{\ln|\eta| - \Psi(p+1) + \Psi(1)\}, \qquad |\eta| < 2\pi,\ p = \text{integer},$$

where $\Psi(p) = (1/\Gamma(p))\,(d/dp)\Gamma(p)$, and $\zeta(p)$ is Riemann's zeta function.

(8) $$B_p(\eta) = \cos \pi p B_p(-\eta) + 2 \sum_{k=0}^{[(p+1)/2]} \frac{\zeta(2k)\eta^{p+1-2k}}{(p+1-2k)!} + \frac{2 \sin \pi p}{\pi} \sum_{k=[(p+3)/2]}^{\infty} \frac{\zeta(2k)(2k-p-2)!}{\eta^{2k-2p-1}},$$

where $[a]$ denotes the largest integer contained in a. This function can be treated from different routes of approach. Dingle started from the integral representation of $B_p(\eta)$ and derived useful properties, mainly from a Mellin transform. Truesdell [3] defined the function by the series (4), with Eq. (8) providing the analytic continuation. Truesdell [4] also considered a differential-difference equation of the type (3) and derived mathematical properties subject to certain boundary conditions, e.g., in this case

(9) $$B_p(0) = \zeta(p+1), \qquad p > 0,$$

(10) $$B_{-1}(\eta) = e^\eta[1 - e^\eta]^{-1}, \qquad \eta \neq 0,$$

or

(11) $$B_0(\eta) = \ln(|1 - e^\eta|), \qquad \eta \neq 0.$$

He suggests this latter approach as a basis for a unified theory for most of the special functions of mathematical physics. We note parenthetically that the functions of negative integer orders are expressible in terms of elementary functions by the use of Eqs. (3) and (10).

3. Relation to Other Functions.

(i) To the hypergeometric function ${}_nF_k(a_1 \cdots a_n; b_1 \cdots b_k; z)$: It is evident that for p = integer the series (3) can be expressed in the form

(12) $$B_n(\eta) = \lim_{\epsilon \to 0} [{}_{n+1}F_n(\epsilon, \cdots, \epsilon; 1, 1, \cdots, 1; e^\eta) - 1]\epsilon^{-(n+1)}.$$

(ii) To the polylogarithm: Using Lewin's [1] notation, we define the polylogarithm as

(13)
$$Li_n(z) = \int_0^z \frac{Li_{n-1}(z)}{z}\, dz\,,$$
$$Li_2(z) = \int_0^z \frac{\ln(|1-z|)}{z}\, dz\,,$$
$$Li_1(z) = \ln(|1-z|)\,.$$

Then

(14)
$$Li_n(z) = B_{n-1}(\ln z)\,.$$

$Li_2(z)$ has been investigated by many mathematicians, including Euler, Abel, Kummer, and Ramanujan, and is usually labeled as the dilogarithm or Euler's dilogarithm [1]. Recently Kölbig [6] presented an algorithm to compute this function to 14 significant digits.

(iii) To the Fermi-Dirac function: Define the complete Fermi-Dirac function as

(15)
$$F_p(\eta) = \frac{1}{\Gamma(p+1)} \int_0^\infty \frac{x^p dx}{e^{x-\eta}+1}\,.$$

Dingle [5] has shown the following relations to be true for real η:

(16)
$$B_p(\eta) = -\text{Real part of } F_p(\eta + i\pi)\,,$$

(17)
$$B_p(\eta) = \sum_{k=0}^{\infty} (2^{-p})^k F_p(2^k\eta)$$

and

(18)
$$F_p(\eta) = B_p(\eta) - 2^{-p}B_p(2\eta)\,.$$

The last expression suggests that the computation of the Fermi-Dirac function is facilitated readily by the Bose-Einstein function.

(iv) To the Debye function: Define the Debye function as (see [7])

(19)
$$\Gamma(p+1)D_p(x) = \int_x^\infty \frac{t^p dt}{e^t - 1}\,;$$

we see the relation

(20)
$$D_p(x) = [\zeta(p+1) - B_p(0, x)]\,.$$

(v) To the various zeta functions: If we start with the series representation of Riemann's zeta function,

(21)
$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}\,,$$

we can generalize the function as follows:

(22)
$$\zeta(s, \alpha) = \sum_{k=0}^{\infty} (\alpha + k)^{-s}\,,$$

(23)
$$F(z, s) = \sum_{k=1}^{\infty} k^{-s}z^k\,, \qquad |z| < 1\,,$$

$$(24)\qquad \Phi(z, s, \alpha) = \sum_{k=0}^{\infty} (\alpha + k)^{-s} z^k, \qquad |z| < 1 .$$

The last three functions are known as the generalized zeta function, Jonquière's function and Lerch's transcendent, respectively [8]. The last two functions can be appropriately continued analytically beyond the indicated circle of convergence [4]. Comparing Eq. (23) to Eq. (4), we have immediately

$$(25)\qquad F(z, s) = B_{s-1}(\ln z) .$$

Notice that Jonquière's function is just a generalization of the polylogarithm.

(vi) To the exponential integral: Dingle [4] has given the following identity:

$$(26)\qquad B_p(\eta) = \sum_{k=0}^{\infty} Ei_{p+1}(2\pi k i - \eta) ,$$

where

$$Ei_p(z) = \int_1^{\infty} t^{-p} e^{-zt} dt \qquad (p \geqq 1, R(z) > 0) .$$

4. Chebyshev Polynomial Expansions: Approximating Forms and Computations. The advantages of expanding functions in Chebyshev polynomials are well known. Clenshaw [9] presents exhaustive discussions of comparison among Chebyshev series, best-fit polynomials and economized power series, and also methods for computing Chebyshev coefficients. In this paper, we present three sets of expansions as follows:

$$(27)\quad B_p(\eta) \approx e^{\eta} \sum_{k=0} a_k^{(p)} T_k^*(e^{\eta+1}) \quad \text{for } -\infty \leqq \eta \leqq -1, p = 1, 2, \cdots, 10 .$$

$$(28)\quad B_p(\eta) \approx \sum_{k=0} b_k^{(p)} T_k(\eta) - \frac{\eta^p}{p!} \ln|\eta| \quad \text{for } -1 \leqq \eta \leqq 1, p = 1, 2, \cdots, 10 .$$

$$(29)\quad B_p(\eta) \approx Q_p(\eta) + \frac{1}{2}\eta^{p+2} \sum_{k=0} c_k^{(p)} T_{2k}\left(\frac{\eta}{2}\right) - \frac{\eta^p}{p!} \ln|\eta| \quad \text{for } -2 \leqq \eta \leqq 2, p = 1, 2, \cdots, 5 .$$

For the range $(1, \infty)$, one can use expansion (27) and Eq. (8) which is simple for p = integer. In the last three equations, T_n and T_n^* are the usual Chebyshev and shifted Chebyshev polynomials, Q_p is a $(p + 1)$th degree polynomial, the coefficients of which will also be given. The expansion (27) is computed from a straightforward economization of the series (4), and the expansion (28) is obtained from the use of the orthogonal property of summation, both methods being described in [9]. The expansion (29) is computed by economizing part of the series in Eq. (7), leaving out a polynomial

$$(30)\quad Q_p(\eta) = \sum_{k=0}^{p-1} \frac{\eta^k}{k!} \zeta(p + 1 - k) - \frac{\eta^p}{p!}[\Psi(1) - \Psi(p + 1)] - \frac{1}{2}\frac{\eta^{p+1}}{(p + 1)!} .$$

For the lower order functions this breaking up is advantageous because the series

$$\sum_{k=p+2}^{\infty} \frac{\eta^k}{k!} \zeta(p + 1 - k)$$

is even, due to $\zeta(-2k) = 0$, and the polynomial Q_p is a low order one. The evenness of the last series also accounts for the fact that

$$b_{p+2k+1}^{(p)} = 0 , \qquad k = 1, 2, \cdots .$$

The above three Chebyshev expansions are rapidly convergent. For example, we need typically six terms for an accuracy of 10^{-8} and thirteen for 10^{-16}.

All computations were performed on the IBM 7094 Mod II using a package of subroutines in 70-bit (about 21 decimal digits) arithmetic, written by Dr. C. L. Lawson and associates of the Jet Propulsion Laboratory. In Tables I to X in the microfiche appendix, we present the coefficients for the expansions (27) to (29). Each expansion, with its rounded coefficients is checked by its corresponding program of function generation for 1000 pseudo-random arguments. The maximal relative error ranges from 2×10^{-20} to 3×10^{-19}. In addition, the expansions (27) and (29) were checked against each other in the cross region $-2 \leqq \eta - 1$, and the expansions (28) and (29) in $-1 \leqq \eta \leqq +1$. To further insure against gross errors, we have also used each expansion to compute the polylogarithms by Eq. (14) and spot-check against a 10-decimal table of such functions [10].

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TABLE I. COEFFICIENTS FOR $B_1(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 1 FOR X IN (-INFINITY,-1)

k		$a_k^{(1)}$
0		1.05285 82239 81766 23290
1	(-2)	5.53862 68567 09330 246
2	(-3)	2.68437 82246 98853 90
3	(-4)	1.67463 06089 39455 4
4	(-5)	1.19952 18076 28102
5	(-7)	9.37378 96647 949
6	(-8)	7.77690 37272 48
7	(-9)	6.73964 05546 2
8	(-10)	6.03774 86833
9	(-11)	5.55186 5135
10	(-12)	5.21359 045
11	(-13)	4.98145 34
12	(-14)	4.82919 9
13	(-15)	4.73966
14	(-16)	4.7014
15	(-17)	4.707
16	(-18)	4.75
17	(-19)	4.8
18	(-20)	4.9

$$\sum_{k=0}^{18} a_k^{(1)} = 1.11110\ 93516\ 05231\ 73202$$

BOSE-EINSTEIN FUNCTION OF ORDER 1 FOR X IN (-1,1)

k		$b_k^{(1)}$
0		1.51993 40668 48226 43647
1	(-1)	9.89626 31109 51687 8761
2	(-1)	-1.25
3	(-3)	-3.45077 54757 15978 48
4		0.0
5	(-6)	4.25574 58405 2858
6		0.0
7	(-8)	-1.19086 65392 68
8		0.0
9	(-11)	4.28764 4049
10		0.0
11	(-13)	-1.74985 65
12		0.0
13	(-16)	7.7113
14		0.0
15	(-18)	-3.58
16		0.0
17	(-20)	1.7

$$\sum_{k=0}^{17} b_k^{(1)} = 2.38111\ 38463\ 47556\ 60391$$

BOSE-EINSTEIN FUNCTION OF ORDER 1 FOR X IN (-2,2)

k	$c_k^{(1)}$
0	(-2) -2.75090 13908 11024 28184
2	(-4) 2.65829 22379 68585 000
4	(-6) -2.89358 20907 06378 6
6	(-8) 4.04138 48538 3962
8	(-10) -6.38838 59258 67
10	(-11) 1.08934 48843 6
12	(-13) -1.95667 2987
14	(-15) 3.65137 88
16	(-17) -7.01683
18	(-18) 1.3802
20	(-20) -2.77
22	(-22) 5.6

$$\sum_{k=0}^{11} c_{2k}^{(1)} = -2.72460\ 38480\ 69278\ 07790 \times 10^{-2}$$

COEFFICIENTS FOR THE LEADING POLYNOMIAL
BOSE-EINSTEIN FUNCTION OF ORDER 1

k	$d_k^{(1)}$
0	1.64493 40668 48226 43647
1	+1.0
2	(-1) -2.5

TABLE II. COEFFICIENTS FOR $B_2(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 2 FOR X IN (-INFINITY,-1)

k		$a_k^{(2)}$
0		1.02516 50896 98219 58167
1	(-2)	2.59438 89137 13750 436
2	(-4)	8.14112 06811 94398 3
3	(-5)	3.72894 79723 29444
4	(-6)	2.10330 79748 2893
5	(-7)	1.35324 80891 422
6	(-9)	9.53221 51443 3
7	(-10)	7.17315 11792
8	(-11)	5.67635 3470
9	(-12)	4.67305 059
10	(-13)	3.97172 78
11	(-14)	3.46544 5
12	(-15)	3.09090
13	(-16)	2.8088
14	(-17)	2.594
15	(-18)	2.43
16	(-19)	2.3
17	(-20)	2.2

$$\sum_{k=0}^{17} a_k^{(2)} \quad 1.05196\ 26293\ 27385\ 63862$$

BOSE-EINSTEIN FUNCTION OF ORDER 2 FOR X IN (-1,1)

k		$b_k^{(2)}$
0		1.57575 84100 92016 82060
1		1.58243 40668 48226 43647
2	(-1)	3.73269 27164 27211 9152
3	(-2)	-2.08333 33333 33333 333
4	(-4)	-4.31878 90269 45633 8
5		0.0
6	(-7)	3.55637 87549 344
7		0.0
8	(-10)	-7.46971 36457
9		0.0
10	(-12)	2.15257 131
11		0.0
12	(-15)	-7.32320
13		0.0
14	(-17)	2.767
15		0.0
16	(-19)	-1.1

$$\sum_{k=0}^{16} b_k^{(2)} = 3.51019\ 68912\ 39983\ 95642$$

BOSE-EINSTEIN FUNCTION OF ORDER 2 FOR X IN (-2,2)

k		$c_k^{(2)}$
0	(-3)	-6.89928 53464 23519 48047
2	(-5)	4.47871 34314 59414 643
4	(-7)	-3.67781 26398 11975 8
6	(-9)	4.12708 63024 0292
8	(-11)	-5.45439 73261 58
10	(-13)	7.99272 95144
12	(-14)	-1.25881 4724
14	(-16)	2.09167 84
16	(-18)	-3.62283
18	(-20)	6.486
20	(-21)	-1.19
22	(-23)	2.2

$$\sum_{k=0}^{11} c_{2k}^{(2)} = -6.85486\ 19200\ 43686\ 97738 \times 10^{-1}$$

COEFFICIENTS FOR THE LEADING POLYNOMIAL
BOSE-EINSTEIN FUNCTION OF ORDER 2

k		$d_k^{(2)}$
0		1.20205 69031 59594 28540
1		1.64493 40668 48226 43647
2	(-1)	+7.5
3	(-2)	-8.33333 33333 33333 33333

TABLE III. COEFFICIENTS FOR $B_3(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 3 FOR X IN (-INFINITY,-1)

k		$a_k^{(3)}$
0		1.01219 31451 29650 82949
1	(-2)	1.24382 55879 25482 627
2	(-4)	2.53265 31050 84119 9
3	(-6)	8.51332 84354 5471
4	(-7)	3.77619 33554 889
5	(-8)	1.99716 22042 57
6	(-9)	1.19257 23029 0
7	(-11)	7.78156 5102
8	(-12)	5.43233 929
9	(-13)	3.99928 81
10	(-14)	3.07321 9
11	(-15)	2.44642
12	(-16)	2.0059
13	(-17)	1.686
14	(-18)	1.45
15	(-19)	1.3
16	(-20)	1.1

$$\sum_{k=0}^{16} a_k^{(3)} = 1.02489\ 35785\ 15060\ 73359$$

BOSE-EINSTEIN FUNCTION OF ORDER 3 FOR X IN (-1,1)

k		$b_k^{(3)}$
0		1.48574 42504 23194 80063
1		1.43079 04409 37322 89151
2	(-1)	4.00816 85004 53899 4245
3	(-2)	7.61724 13979 79151 471
4	(-3)	-2.60416 66666 66666 67
5	(-5)	-4.32234 54057 00568
6		0.0
7	(-8)	2.54560 60489 86
8		0.0
9	(-11)	-4.16179 9644
10		0.0
11	(-14)	9.81770 2
12		0.0
13	(-16)	-2.8273
14		0.0
15	(-19)	9.3

$$\sum_{k=0}^{15} b_k^{(3)} \quad 3.39087\ 65906\ 79815\ 86559$$

BOSE-EINSTEIN FUNCTION OF ORDER 3 FOR X IN (-2,2)

k		$c_k^{(3)}$
0	(-3)	-1.38240 23448 56439 85359
2	(-6)	6.44479 28657 44857 16
4	(-8)	-4.13655 18361 34391
6	(-10)	3.81314 96849 471
8	(-12)	-4.27812 59257 7
10	(-14)	5.44781 6440
12	(-16)	-7.58766 77
14	(-17)	1.13022 2
16	(-19)	-1.7740
18	(-21)	2.90
20	(-23)	-4.9

$$\sum_{k=0}^{10} c_{2k}^{(3)} = -1.37599\ 85404\ 18483\ 24610 \times 10^{-3}$$

COEFFICIENTS FOR THE LEADING POLYNOMIAL
BOSE-EINSTEIN FUNCTION OF ORDER 3

k		$d_k^{(3)}$
0		1.08232 32337 11138 19152
1		1.20205 69031 59594 28540
2	(-1)	8.22467 03342 41132 18235
3	(-1)	+3.05555 55555 55555 55555
4	(-2)	-2.08333 33333 33333 33333

TABLE IV. COEFFICIENTS FOR $B_4(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 4 FOR X IN (-INFINITY,-1)

k		$a_k^{(4)}$
0		1.00597 40171 90023 41635
1	(-3)	6.05236 39813 86722 88
2	(-5)	8.02626 05972 96852
3	(-6)	1.98208 20770 2585
4	(-8)	6.91269 30600 80
5	(-9)	3.00326 51255 8
6	(-10)	1.51891 79724
7	(-12)	8.58567 853
8	(-13)	5.28269 07
9	(-14)	3.47486 1
10	(-15)	2.41228
11	(-16)	1.7507
12	(-17)	1.319
13	(-18)	1.0
14	(-20)	8.2

$$\sum_{k=0}^{14} a_k^{(4)} = 1.01210\ 86981\ 50698\ 95505$$

BOSE-EINSTEIN FUNCTION OF ORDER 4 FOR X IN (-1,1)

k		$b_k^{(4)}$
0		1.36995 79515 31983 75414
1		1.28533 58254 00499 82940
2	(-1)	3.43862 84007 27161 7753
3	(-2)	6.72368 36118 67610 152
4	(-2)	1.08290 38012 56439 838
5	(-4)	-2.60416 66666 66666 7
6	(-6)	-3.60407 58431 2463
7		0.0
8	(-9)	1.59360 49053 9
9		0.0
10	(-12)	-2.08580 867
11		0.0
12	(-15)	4.10249
13		0.0
14	(-17)	-1.013
15		0.0
16	(-20)	2.9

$$\sum_{k=0}^{16} b_k^{(4)} = 3.07695\ 84719\ 85453\ 65878$$

BOSE-EINSTEIN FUNCTION OF ORDER 4 FOR X IN (-2,2)

k	$c_k^{(4)}$
0	(-4) -2.30667 54817 24920 37306
2	(-7) 8.09726 12718 00291 68
4	(-9) -4.17468 33329 83862
6	(-11) 3.21845 01652 669
8	(-13) -3.10462 70577 1
10	(-15) 3.46844 7654
12	(-17) -4.30385 49
14	(-19) 5.78113
16	(-21) -8.26
18	(-22) 1.24
20	(-24) -1.9

$$\sum_{k=0}^{10} c_{2k}^{(4)} = 2.29861\ 96485\ 11800\ 66022 \times 10^{-4}$$

COEFFICIENTS FOR THE LEADING POLYNOMIAL
BOSE-EINSTEIN FUNCTION OF ORDER 4

k	$d_k^{(4)}$
0	1.03692 77551 43369 92633
1	1.08232 32337 11138 19152
2	(-1) 6.01028 45157 97971 42700
3	(-1) 2.74155 67780 80377 39412
4	(-2) +8.68055 55555 55555 55556
5	(-3) -4.16666 66666 66666 66666

TABLE V. COEFFICIENTS FOR $B_5(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 5 FOR X IN (-INFINITY,-1)

k		$a_k^{(5)}$
0		1.00294 78351 99330 05491
1	(-3)	2.97316 01322 40910 19
2	(-5)	2.57811 10931 93397
3	(-7)	4.68595 55007 038
4	(-8)	1.28576 48714 51
5	(-10)	4.58855 47567
6	(-11)	1.96480 9486
7	(-13)	9.61575 78
8	(-14)	5.21147 5
9	(-15)	3.06097
10	(-16)	1.9185
11	(-17)	1.269
12	(-19)	8.8
13	(-20)	6.3

$$\sum_{k=0}^{13} a_k^{(5)} = 1.00594\ 72583\ 75222\ 21147$$

BOSE-EINSTEIN FUNCTION OF ORDER 5 FOR X IN (-1,1)

k		$b_k^{(5)}$
0		1.31340 89513 17848 33677
1		1.19906 81981 62292 33203
2	(-1)	3.04524 74732 04559 3197
3	(-2)	5.60264 67010 02529 653
4	(-3)	8.43715 65981 67846 02
5	(-3)	1.18743 08755 07418 97
6	(-5)	-2.17013 88888 88889
7	(-7)	-2.57547 81771 643
8		0.0
9	(-11)	8.86494 8412
10		0.0
11	(-14)	-9.49959 6
12		0.0
13	(-16)	1.5818
14		0.0
15	(-19)	-3.4

$$\sum_{k=0}^{15} b_k^{(5)} = 2.88263\ 09924\ 36145\ 20297$$

k		$c_k^{(5)}$

BOSE-EINSTEIN FUNCTION OF ORDER 5 FOR X IN (-2,2)

0	(-5)	-3.29780 92312 88972 87786
2	(-8)	9.03060 18341 58017 59
4	(-10)	-3.82215 14626 38158
6	(-12)	2.50122 57345 576
8	(-14)	-2.09701 90808 7
10	(-16)	2.07236 2352
12	(-18)	-2.30595 90
14	(-20)	2.80800
16	(-22)	-3.671
18	(-24)	5.1
20	(-26)	-7.3

$$\sum_{k=0}^{10} c_k^{(5)} = 3.28881\ 66029\ 23391\ 06755 \times 10^{-5}$$

COEFFICIENTS FOR THE LEADING POLYNOMIAL
BOSE-EINSTEIN FUNCTION OF ORDER 5

k		$d_k^{(5)}$
0		1.01734 30619 84449 13971
1		1.03692 77551 43369 92633
2	(-1)	5.41161 61685 55690 95758
3	(-1)	2.00342 81719 32657 14233
4	(-2)	6.85389 19452 00943 48529
5	(-2)	+1.90277 77777 77777 77778
6	(-4)	-6.94444 44444 44444 44444

TABLE VI. COEFFICIENTS FOR $B_6(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 6 FOR X IN (-INFINITY,-1)

k		$a_k^{(6)}$
0		1.00146 12551 33209 59981
1	(-3)	1.46950 81305 20546 87
2	(-6)	8.36276 02678 8247
3	(-7)	1.12117 28910 808
4	(-9)	2.42291 49745 1
5	(-11)	7.10545 9157
6	(-12)	2.57596 628
7	(-13)	1.09125 85
8	(-15)	5.20774
9	(-16)	2.7301
10	(-17)	1.544
11	(-19)	9.3
12	(-20)	5.9

$$\sum_{k=0}^{12} a_k^{(6)} = 1.00293\ 92406\ 37947\ 29262$$

BOSE-EINSTEIN FUNCTION OF ORDER 6 FOR X IN (-1,1)

k		$b_k^{(6)}$
0		1.28742 61587 58135 30307
1		1.16114 65776 57620 37079
2	(-1)	2.85868 93973 25112 0332
3	(-2)	4.93479 31787 04801 432
4	(-3)	6.89828 22945 92512 47
5	(-4)	8.45885 79870 56734 9
6	(-4)	1.06207 83157 33909 1
7	(-6)	-1.55009 92063 4921
8	(-8)	-1.61022 79200 03
9		0.0
10	(-12)	4.43722 400
11		0.0
12	(-15)	-3.96476
13		0.0
14	(-18)	5.66
15		0.0
16	(-20)	-1.1

$$\sum_{k=0}^{16} b_k^{(6)} = 2.79163\ 84176\ 63134\ 18402$$

TABLE VII. COEFFICIENTS FOR $B_7(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 7 FOR X IN (-INFINITY,-1)

k		$a_k^{(7)}$
0		1.00072 65010 67949 32638
1	(-4)	7.29206 60379 37048 8
2	(-6)	2.73216 26652 5465
3	(-8)	2.70774 91505 23
4	(-10)	4.61469 37088
5	(-11)	1.11283 8213
6	(-13)	3.41666 50
7	(-14)	1.25291 5
8	(-16)	5.2641
9	(-17)	2.463
10	(-18)	1.26
11	(-20)	6.9

$$\sum_{k=0}^{11} a_k^{(7)} = 1.00145\ 84673\ 84852\ 29217$$

BOSE-EINSTEIN FUNCTION OF ORDER 7 FOR X IN (-1,1)

k		$b_k^{(7)}$
0		1.27603 49784 56404 36794
1		1.14450 71898 83943 19347
2	(-1)	2.77949 66146 76430 8912
3	(-2)	4.65044 10168 22454 371
4	(-3)	6.06275 57485 42792 60
5	(-4)	6.82307 64471 46105 7
6	(-5)	7.06196 58159 33522
7	(-6)	8.03030 93341 4798
8	(-8)	-9.68812 00396 83
9	(-10)	-8.94817 57911
10		0.0
11	(-13)	2.01872 22
12		0.0
13	(-16)	-1.5271
14		0.0
15	(-19)	1.9

$$\sum_{k=0}^{15} b_k^{(7)} = 2.75181\ 98555\ 61149\ 81837$$

TABLE VIII. COEFFICIENTS FOR $B_8(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 8 FOR X IN (-INFINITY,-1)

k		$a_k^{(8)}$
0		1.00036 18977 66052 47394
1	(-4)	3.62788 56887 19901 7
2	(-7)	8.97303 38446 989
3	(-9)	6.58751 01698 7
4	(-11)	8.86604 8496
5	(-12)	1.75964 099
6	(-14)	4.57739 3
7	(-15)	1.45330
8	(-17)	5.376
9	(-18)	2.24
10	(-19)	1.0

$$\sum_{k=0}^{10} a_k^{(8)} = 1.00072\ 55903\ 16286\ 51315$$

BOSE-EINSTEIN FUNCTION OF ORDER 8 FOR X IN (-1,1)

k		$a_k^{(8)}$
0		1.27083 78604 33006 10151
1		1.13706 01477 22582 82338
2	(-1)	2.74502 05126 57352 1799
3	(-2)	4.53144 84286 51671 609
4	(-3)	5.72844 09838 41519 42
5	(-4)	5.99213 60903 83457 4
6	(-5)	5.63835 40349 16603
7	(-6)	5.05118 13828 3800
8	(-7)	5.26170 55958 177
9	(-9)	-5.38228 89109 3
10	(-11)	-4.47509 7257
11		0.0
12	(-15)	8.41771
13		0.0
14	(-18)	-5.46
15		0.0
16	(-21)	5.9

$$\sum_{k=0}^{16} b_k^{(8)} = 2.73410\ 41537\ 65980\ 83868$$

TABLE IX. COEFFICIENTS FOR $B_9(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER 9 FOR X IN (-INFINITY,-1)

k		$a_k^{(9)}$
0		1.00018 05034 81616 47885
1	(-4)	1.80797 71737 99062 9
2	(-7)	2.95830 72770 769
3	(-9)	1.61184 55884 8
4	(-11)	1.71557 7700
5	(-13)	2.80489 74
6	(-15)	6.18572
7	(-16)	1.7009
8	(-18)	5.54
9	(-19)	2.1

$$\sum_{k=0}^{9} a_k^{(9)} = 1.00036\ 15986\ 59012\ 30961$$

BOSE-EINSTEIN FUNCTION OF ORDER 9 FOR X IN (-1,1)

k		$b_k^{(9)}$
0		1.26839 07793 83443 84499
1		1.13358 69855 04227 99868
2	(-1)	2.72936 41585 90165 2682
3	(-2)	4.47957 02183 04195 388
4	(-3)	5.58940 88346 84796 29
5	(-4)	5.67248 80266 05228 2
6	(-5)	4.95135 35637 95898
7	(-6)	4.00057 67056 4932
8	(-7)	3.16035 22948 431
9	(-8)	3.04302 48122 12
10	(-10)	-2.69114 44555
11	(-12)	-2.03451 774
12		0.0
13	(-16)	3.2397
14		0.0
15	(-19)	-1.8

$$\sum_{k=0}^{15} b_k^{(9)} = 2.72592\ 04008\ 73748\ 21871$$

TABLE X. COEFFICIENTS FOR $B_{10}(\eta)$

BOSE-EINSTEIN FUNCTION OF ORDER10 FOR X IN (-INFINITY,-1)

k		$a_k^{(10)}$
0		1.00009 01046 22926 52571
1	(-5)	9.02020 38901 80247
2	(-8)	9.78088 48593 67
3	(-10)	3.96168 16909
4	(-12)	3.33904 503
5	(-14)	4.50155 2
6	(-16)	8.4218
7	(-17)	2.007
8	(-19)	5.8
9	(-20)	1.9

$$\sum_{k=0}^{9} a_k^{(10)} = 1.00018\ 04048\ 70230\ 01434$$

BOSE-EINSTEIN FUNCTION OF ORDER 10 FOR X IN (-1,1)

k		$b_k^{(10)}$
0		1.26721 01502 83260 56398
1		1.13192 25714 53935 58158
2	(-1)	2.72197 83213 31032 2417
3	(-2)	4.45578 34504 05528 842
4	(-3)	5.52856 31312 94372 00
5	(-4)	5.53989 52990 46837 3
6	(-5)	4.69397 74192 91605
7	(-6)	3.51410 71720 3391
8	(-7)	2.48672 38248 654
9	(-8)	1.75724 63551 66
10	(-9)	1.57543 70211 0
11	(-11)	-1.22324 7480
12	(-14)	-8.47850 7
13		0.0
14	(-17)	1.158
15		0.0
16	(-21)	-5.8

$$\sum_{k=0}^{16} b_k^{(10)} = 2.72202\ 16627\ 24884\ 47484$$

GAUSS QUADRATURE RULES FOR THE EVALUATION OF

$$2\pi^{-1/2} \int_0^\infty \exp(-x^2) f(x)\,dx$$

BY

DAVID GALANT

Gauss Quadrature Rules for the Evaluation of

$$2\pi^{-1/2}\int_0^{\infty} \exp(-x^2) f(x)\,dx$$

by David Galant

Table I is a tabulation of 20S values of the parameters of the three-term recurrence relation

$$p_j(x) = (x - b_j)p_{j-1}(x) - g_j p_{j-2}(x)$$

with

$$p_0(x) = 1 \quad \text{and} \quad p_{-1}(x) = 0$$

for the first twenty monic orthogonal polynomials associated with the weight function $\exp(-x^2)$ on $(0,\infty)$. These parameters were calculated from the moments using the QD algorithm (1) and 50S arithmetic. The least accurate parameters ($j = 20$) had about 23S.

Table II is a tabulation, also to 20S, of the nodes and weights of the Gauss quadrature rules

$$G_n(f) = \sum_{j=1}^{n} w_{jn} f(x_{jn}) = 2\pi^{-1/2}\int_0^{\infty} \exp(-x^2) f(x)\,dx + E_n(f)$$

where

$$E_n(x^k) = 0 \quad \text{for} \quad k = 0(1)2n - 1$$

for $n = 1(1)20$. The nodes and weights of each rule were calculated from the recurrence relation parameters by applying the QR algorithm to determine