Summation of a Slowly Convergent Series Arising in Antenna Study

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Abstract. An equivalent series for the slowly convergent series

$$\sum_{n=1}^{\infty}\left[\int_{-\pi/2}^{\pi/2}\cos^{\alpha}\theta\cos(n\epsilon\sin\theta)\right]^2/n$$

which arises in antenna theory is obtained. The new form is found to consist of two rapidly convergent series for small ϵ.

A recent study of the electromagnetic radiation from cylindrical structures [1], [3] requires the evaluation of a slowly convergent series $S_1 = \pi^2 \sum_{n=0}^{\infty} [J_0(n\epsilon)]^2/n$ where $J_0(n\epsilon)$ is the zeroth order Bessel function, and ϵ is a small positive constant. The expression above is a special case of the series

$$S(\alpha) = \sum_{n=1}^{\infty} [p_n(\alpha)]^2/n \tag{1}$$

where

$$p_n(\alpha) = \int_{-\pi/2}^{\pi/2} \cos^{\alpha}\theta \cos(n\epsilon\sin\theta)d\theta\,, \quad \alpha > -1 \text{ and } 0 < \epsilon\,. \tag{2}$$

One notes that in the case $\alpha = 0$

$$p_n(0) = \int_{-\pi/2}^{\pi/2} \cos(n\epsilon\sin\theta)d\theta = \pi J_0(n\epsilon)$$

and therefore $S(0) = S_1$. The aim of this brief is to obtain a more rapidly convergent series that is equivalent to Eq. (1).

Substituting Eq. (2) into Eq. (1) and interchanging the order of summation and integration results in

$$S(\alpha) = \int_{-\pi/2}^{\pi/2} \cos^{\alpha}\theta \left\{\int_{-\pi/2}^{\pi/2} \cos^{\alpha}\theta' \left[\sum_{n=1}^{\infty} \cos(n\epsilon\sin\theta)\cos(n\epsilon\sin\theta')/n\right]d\theta'\right\}d\theta\,. \tag{3}$$

It is well known that

$$\sum_{n=1}^{\infty} \cos(n\epsilon\sin\theta)\cos(n\epsilon\sin\theta')/n = -\tfrac{1}{2}\ln 2|\cos(\epsilon\sin\theta) - \cos(\epsilon\sin\theta')|\,. \tag{4}$$

Employing the Taylor expansion for $\cos y$, the difference of two cosine functions in the vertical bars can be written as

$$\cos(\epsilon\sin\theta) - \cos(\epsilon\sin\theta') = \frac{\epsilon^2}{4}(\cos 2\theta - \cos 2\theta')(1 - A)\,, \tag{5}$$

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where

$$(6)\quad A = \frac{2\epsilon^2}{4!}(x+y) - \frac{2\epsilon^4}{6!}(x^2+xy+y^2) + \frac{2\epsilon^6}{8!}(x^3+x^2y+xy^2+y^3) - \cdots,$$

$$(7)\quad x = \sin^2\theta \quad \text{and} \quad y = \sin^2\theta' .$$

Substitution of Eq. (5) into Eq. (4) leads to

$$(8)\quad \begin{aligned}&\sum_{n=1}^{\infty} \cos(n\epsilon \sin\theta)\cos(n\epsilon\sin\theta')/n \\ &\qquad = \ln\frac{2}{\epsilon} - \frac{1}{2}\ln 2|\cos 2\theta - \cos 2\theta'| - \frac{1}{2}\ln|1-A| .\end{aligned}$$

If $|A| < 1$ we can expand the last term in the Taylor series as follows

$$(9)\quad \frac{1}{2}\ln|1-A| = \frac{1}{2}\ln(1-A) = -\frac{1}{2}\left(A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \cdots\right).$$

Inserting Eq. (9) in Eq. (8) and acknowledging that A is defined as in Eq. (6) we obtain

$$(10)\quad \begin{aligned}&\sum_{n=1}^{\infty} \cos(n\epsilon\sin\theta)\cos(n\epsilon\sin\theta')/n \\ &\qquad = \ln\frac{2}{\epsilon} + \sum_{n=1}^{\infty}\cos 2n\theta\cos 2n\theta'/n + \frac{\epsilon^2}{24}(x+y) \\ &\qquad + \frac{\epsilon^4}{2880}(x^2+6xy+y^2) + \frac{\epsilon^6}{181440}(x^3+15x^2y+15xy^2+y^3) + \cdots .\end{aligned}$$

It is desired to determine the condition for which the inequality $|A| < 1$ is satisfied. The exact form of that condition is not known. However, the upper bound of A can be obtained readily. Since the absolute value of $\sin\theta$ is always less than or equal to unity, we see from Eq. (6)

$$|A| \leqq \frac{4\epsilon^2}{4!} + \frac{6}{6!}\epsilon^4 + \frac{8}{8!}\epsilon^6 + \cdots = \frac{\sinh(\epsilon)}{\epsilon} - 1 .$$

Consequently, if the condition

$$(\sinh(\epsilon)/\epsilon) - 1 < 1$$

or

$$(11)\quad (\sinh(\epsilon)/\epsilon) < 2$$

is satisfied, then the inequality $|A| < 1$ is always true. We acknowledge that (11) is more stringent than we really need.

Substituting Eq. (10) into Eq. (3) and interchanging the order of integration and summation for the series $\sum_{n=1}^{\infty}\cos 2n\theta\cos 2n\theta'/n$ we arrive at

$$(12)\quad \begin{aligned}S(\alpha) &= C_0^2\ln(2/\epsilon) + f(\alpha) + \frac{\epsilon^2}{12}C_0C_1 + \frac{\epsilon^4}{1440}(C_0C_2 + 3C_1^2) \\ &\quad + \frac{\epsilon^6}{90720}(C_0C_3 + 15C_1C_2) + \cdots ,\end{aligned}$$

where

$$C_n = \int_{-\pi/2}^{\pi/2} \cos^\alpha\theta \sin^{2n}\theta \, d\theta \,, \qquad n = 0, 1, 2, \cdots \tag{13}$$

$$f(\alpha) = \sum_{n=1}^{\infty} \left[\int_{-\pi/2}^{\pi/2} \cos^\alpha\theta \cos 2n\theta \, d\theta\right]^2 / n \,. \tag{14}$$

Making use of the following definite integrals [2]

$$\int_0^{\pi/2} \cos^{\nu-1} z \sin^{\mu-1} z \, dz = \frac{1}{2} \frac{\Gamma(\mu/2)\Gamma(\nu/2)}{\Gamma((\mu+\nu)/2)}, \quad \operatorname{Re}\mu > 0 \,, \quad \operatorname{Re}\nu > 0 \,, \tag{15}$$

and the reflection formula for the Gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad z \neq \text{integer} \,, \tag{16}$$

and Legendre's duplication formula, it can be shown that

$$\begin{aligned} C_0 &= \sqrt{\pi}[\Gamma((1+\alpha)/2)/\Gamma(1+\alpha/2)] \,, \\ C_n &= \sqrt{\pi}\left[\Gamma\left(\frac{1+\alpha}{2}\right) \Big/ \Gamma\left(1+\frac{\alpha}{2}\right)\right] \prod_{k=0}^{n-1} \frac{2k+1}{2(n-k)+\alpha}, \qquad n = 1, 2, \cdots . \end{aligned} \tag{17}$$

Also from [2] we find

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} \cos^\alpha\theta \cos 2n\theta \, d\theta \\ &\qquad = (-1)^{n+1} \frac{2}{\sqrt{\pi}} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(1-\frac{\alpha}{2}\right) \frac{d_n(\alpha)}{2n+\alpha}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} d_1(\alpha) &= 1 \\ d_n(\alpha) &= \prod_{k=1}^{n-1} \frac{2(n-k)-\alpha}{2(n-k)+\alpha}, \qquad n \geqq 2 \,, \end{aligned} \tag{19}$$

and $\alpha > -1$.

Substituting Eq. (18) into Eq. (14) results in

$$f(\alpha) = \frac{4}{\pi}\left[\sin\frac{\alpha\pi}{2}\, \Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(1-\frac{\alpha}{2}\right)\right]^2 \sum_{n=1}^{\infty} \frac{[d_n(\alpha)]^2}{n(2n+\alpha)^2} \,. \tag{20}$$

Note that for $-1 < \alpha < 2$, $d_n(\alpha)$ is a monotonically decreasing function of both n and α, and, in particular,

$$d_n(0) = 1 \,, \qquad d_n(1) = \frac{1}{2n-1} \,;$$

therefore Eq. (12) is seen to be represented by two rapidly convergent series for small ϵ.

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1. C. F. DEN, *Admittance of a Wedge Excited Co-axial Antenna with a Plasma Sheath*, Technical Report 5825-9-T, University of Michigan, Radiation Laboratory, Ann Arbor, Mich., 1966.
2. I. S. GRADŠTEĬN & I. M. RYŽIK, *Table of Integrals, Series, and Products*, Fizmatgiz, Moscow, 1963; English transl., Academic Press, New York, 1965, pp. 369, 372. MR **28** #5198; MR **33** #5952.
3. P. J. B. CLARRICOATS & A. A. OLINER, "Improved theory of propagation through slotted circular waveguide," *Electronics Letters*, v. 3, 1967, pp. 279–282.