Eberlein Measure and Mechanical Quadrature Formulae. II. Numerical Results

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Abstract. In a previous paper it was shown how a probability measure (Eberlein measure) on the closed unit ball of the sequence space, l_1, can be used to find the variance σ^2 of the error functional for a quadrature formula for the k-dimensional cube, regarded as a random variable. Here we give values of σ for some specific formulae.

1. Introduction. Let the function $\mathbf{x}(\mathbf{t})$, defined on the k-dimensional cube $\mathfrak{C}_k = [-1, 1]^k$, be an element of the sequence space l_1, and let

$$I(\mathbf{x}) = 2^{-k} \int_{\mathfrak{C}_k} \mathbf{x}(\mathbf{t}) d\mathbf{t}$$

be the normalized integral of $\mathbf{x}$. As an approximation to $I(\mathbf{x})$, let

$$J_N(\mathbf{x}) = \sum_{m=1}^{N} A_m \mathbf{x}(\mathbf{t}^{(m)}) \tag{1}$$

be an N-point quadrature formula with abscissae $\mathbf{t}^{(m)}$ and weights A_m. Sarma [12] showed that, with respect to the Eberlein measure, the variance of the error functional is

$$\sigma^2(I - J_N) = 3^{-1} \sum_{n=0}^{\infty} 2^n \lambda_n^{-1} S_n \tag{2}$$

where $\lambda_0 = 1$, $\lambda_n = \prod_{i=1}^{n} (c_i + 1)(c_i + 2)$, $c_i = (k + i - 1)!/(k - 1)!i!$

$$S_n = \sum_{n_1 + \cdots + n_k = n} [I(t_1^{n_1} \cdots t_k^{n_k}) - J_N(t_1^{n_1} \cdots t_k^{n_k})]^2 .$$

Chebyshev's inequality of probability theory (see, for example [6, p. 21]) states that, if we choose $\mathbf{x}(\mathbf{t})$ at random, then the probability that $|I(\mathbf{x}) - J_N(\mathbf{x})| \leqq p\sigma$ is greater than $1 - p^{-2}$ for every real $p > 1$.

We denote the 1-dimensional N-point Gauss-Legendre formula by G_N and the product of k copies of G_N for $\mathfrak{C}_k$, by G_N^k. We say that formula (1) has *degree d* if it is exact for all polynomials of degree $\leqq d$ and there is at least one polynomial of degree $d + 1$ for which it is not exact.

2. Some Formulae for $k = 1, 2, 3$. Table 1 gives values of $\sigma(I - G_N)$ for $N = 2(1)20$ and also values of the ratio $\sigma(I - G_N)/\sigma(I - G_{N-1})$. This ratio appears to approach the constant 0.1 as $N \to \infty$.

Tables 2 and 3 give σ for various known formulae for $k = 2$ and 3 respectively. For $k \geqq 2$ the series (2) converges very rapidly. For the formulae of Table 2 the first nonzero term in (2) gives σ accurate to between 3 and 4 significant figures. For

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the formulae of Table 3 the first nonzero term in (2) gives σ accurate to more than 4 significant figures.

TABLE 1.

Values of σ for Gauss-Legendre Formulae

N	$\sigma(I - G_N)$	$\sigma(I - G_N)/\sigma(I - G_{N-1})$
2	(−2)0.61788 02642	
3	(−3)0.57557 61595	0.0931533
4	(−4)0.54077 02990	0.0939529
5	(−5)0.51383 28919	0.0950187
6	(−6)0.49316 72623	0.0959781
7	(−7)0.47717 85940	0.0967580
8	(−8)0.46462 69322	0.0973696
9	(−9)0.45461 68316	0.0978456
10	(−10)0.44651 66920	0.0982182
11	(−11)0.43988 17644	0.0985141
12	(−12)0.43439 57959	0.0987529
13	(−13)0.42983 02638	0.0989490
14	(−14)0.42601 68354	0.0991128
15	(−15)0.42282 89741	0.0992517
16	(−16)0.42016 96426	0.0993711
17	(−17)0.41796 30121	0.0994748
18	(−18)0.41614 87985	0.0995659
19	(−19)0.41467 83304	0.0996466
20	(−20)0.41351 17701	0.0997186

TABLE 2.

Values of σ for Some 2-Dimensional Formulae

Formula	σ
4-point 3rd-degree, G_2^2	(−3)0.528326
7-point 5th-degree, Radon [10]	(−5)0.503273
7-point 5th-degree, Albrecht, Collatz [1]	(−5)0.463483
8-point 5th-degree, Burnside [2]	(−5)0.463685
9-point 5th-degree, G_3^2	(−5)0.427840
13-point 5th-degree, Tyler [14]	(−5)0.943847
13-point 5th-degree, Albrecht, Collatz [1]	(−5)0.491957
12-point 7th-degree, Tyler [14]	(−7)0.238278
12-point 7th-degree, Mysovskih [9]	(−7)0.440449
13-point 7th-degree, Maxwell [7]	(−7)0.220939
16-point 7th-degree, G_4^2	(−7)0.218383
21-point 7th-degree, Tyler [14]	(−7)0.666175
25-point 9th-degree, G_5^2	(−10)0.768536
36-point 11th-degree, G_6^2	(−12)0.197917
49-point 13th-degree, G_7^2	(−15)0.389280

We wish to point out that the 34-point 7th-degree formula of Hammer and Wymore [5], for $\mathfrak{C}_3$, has a slight error as given. Their values of a_3 and a_4 must be interchanged. This formula is one of a one-parameter family of 34-point 7th-degree formulas. The formula of this family with parameters

$$x_1 = 0.9317380000 \qquad a_1/8 = 0.03558180896$$
$$x_2 = 0.9167441779 \qquad a_2/8 = 0.01247892770$$
$$x_3 = 0.4086003800 \qquad a_3/8 = 0.05286772991$$
$$x_4 = 0.7398529500 \qquad a_4/8 = 0.02672752182$$
$$\sigma = (-10)0.1528581321$$

minimizes σ to 7 significant figures.

TABLE 3.

Values of σ for Some 3-Dimensional Formulae

Formula	σ
6-point 3rd-degree, Tyler [14]	(−3)0.109480
8-point 3rd-degree, G_2^3	(−4)0.560700
9-point 3rd-degree, Ewing [3]	(−3)0.163472
13-point 3rd-degree, Mustard, Lyness, Blatt [8]	(−4)0.911141
15-point 3rd-degree, Mustard, Lyness, Blatt [8]	(−4)0.841052
13-point 5th-degree, Stroud [13]	(−7)0.537794
14-point 5th-degree, Hammer, Stroud [4]	(−7)0.526443
21-point 5th-degree, Tyler [14]	(−6)0.151476
23-point 5th-degree, Mustard, Lyness, Blatt [8]	(−7)0.703028
27-point 5th-degree, G_3^3	(−7)0.434608
42-point 5th-degree, Sadowsky [11]	(−6)0.371205
27-point 7th-degree, Maxwell [7], Hammer, Stroud [4]***	(−10)0.402935 ((−10)0.511539)
34-point 7th-degree, Hammer, Wymore [5]	(−10)0.153140
64-point 7th-degree, G_4^3	(−10)0.126615
125-point 9th-degree, G_5^3	(−14)0.167686
216-point 11th-degree, G_6^3	(−18)0.114815

3. Additional Remarks. We attempted to compute some formulae which, for given N, minimize σ. We will summarize our results.

For $k = 1$ and $N = 2, 3$ we obtained by direct search formulae with σ equal to (−2)0.60322 and (−3)0.53285 respectively. For $k = 1$ and $N \geqq 4$ we tried a modified Newton's method using G_N as the initial guess; the method failed to converge.

For $k = 2$ using Newton's method and starting with known formulae with $N = 4, 7, 8, 9$ Newton's method usually converged extremely slowly and in all cases the value of σ was not reduced by more than a few units in the fourth significant figure.

The quantity

$$(3) \qquad (\gamma_k/2^k)^{1/2},$$

where γ_k was defined in [12], can be interpreted as the average of σ over all 2^k-point Monte Carlo formulae. For k large, (3) is less than $\sigma(I - G_2{}^k)$; we found by computation that $\sigma(I - G_2{}^k)$ is less than (3) for $k \leqq 107$. The first nonzero term in the series (2) gives $\sigma(I - G_2{}^k) \simeq (16/45)(k/(3\lambda_4))^{1/2}$ which is accurate to 10 significant figures for all $k \geqq 7$.

*** There are two such formulae; the value of σ given in parentheses is for the formula given in parentheses in 4].

The above computations were carried out on the CDC 6400 at the State University of New York at Buffalo. Most of the computations were done in single precision; in some cases double precision was used. In single precision this computer carries about 14.5 significant figures. We are indebted to the referee for suggestions concerning the form of this article.

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