A Generalization of a Class of Test Matrices

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Abstract. We consider matrices of the following form: $G_n(a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_n) = (\beta_{i,j})$, $1 \leqq i, j \leqq n$, where $a_1, \cdots, a_{n-1}, b_1, \cdots, b_n$ are constants and

$$\beta_{i,j} = b_j\,, \quad j \geqq i; \qquad \beta_{ij} = a_j\,, \quad j < i.$$

We deduce in analytic form the determinant, inverse matrix, characteristic equation, and eigenvectors of G_n. Knowing these properties enables us to generate valuable test matrices by appropriately selecting the order and elements of G_n.

1. Introduction. In [1], matrices of the following form were considered:

$$K_n \equiv K_n(a_1, a_2, \cdots, a_{n-1}) = (\alpha_{i,j})\,, \qquad 1 \leqq i, j \leqq n\,, \tag{1}$$

where $a_1, a_2, \cdots, a_{n-1}$ were constants and

$$\alpha_{i,j} = 1\,, \qquad j \geqq i\,; \qquad \alpha_{ij} = a_j\,, \qquad j < i\,.$$

Therefore,

$$K_n = \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 \\ a_1 & 1 & \cdot & \cdot & \cdot & 1 & 1 \\ a_1 & a_2 & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot & 1 & 1 \\ a_1 & a_2 & \cdot & \cdot & \cdot & a_{n-1} & 1 \end{bmatrix}.$$

The determinant, inverse matrix, characteristic equation, and eigenvectors were deduced in analytic form. Consequently, these matrices are valuable test matrices for evaluating the accuracy and efficiency of computational procedures. We shall consider the class of matrices of the following more general form:

$$G_n \equiv G_n(a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_n) = (\beta_{i,j})\,, \qquad 1 \leqq i, j \leqq n, \tag{2}$$

where $a_1, \cdots, a_{n-1}, b_1, \cdots, b_n$ are constants and

$$\beta_{i,j} = b_j\,, \qquad j \geqq i\,; \qquad \beta_{ij} = a_j\,, \qquad j < i\,.$$

Thus

$$G_n = \begin{bmatrix} b_1 & b_2 & \cdot & \cdot & \cdot & b_{n-1} & b_n \\ a_1 & b_2 & \cdot & \cdot & \cdot & b_{n-1} & b_n \\ a_1 & a_2 & \cdot & \cdot & \cdot & b_{n-1} & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot & b_{n-1} & b_n \\ a_1 & a_2 & \cdot & \cdot & \cdot & a_{n-1} & b_n \end{bmatrix}.$$

We will assume $b_n \neq 0$. Obviously (1) is a special case of (2). We shall deduce the determinant, inverse matrix, characteristic equation and eigenvectors of G_n in the same way these properties of K_n were derived in [1].

Received February 3, 1969.

2. The Properties of G_n. The determinant of G_n in (2) is $b_n \prod_{j=1}^{n-1} (b_j - a_j)$. This can be seen by subtracting a_j/b_n times the last column of G_n from the jth column for $1 \leqq j \leqq n-1$ to obtain

$$\begin{bmatrix} (b_1 - a_1) & (b_2 - a_2) & \cdots & (b_{n-1} - a_{n-1}) & b_n \\ 0 & (b_2 - a_2) & \cdots & (b_{n-1} - a_{n-1}) & b_n \\ 0 & 0 & \cdots & (b_{n-1} - a_{n-1}) & b_n \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & (b_{n-1} - a_{n-1}) & b_n \\ 0 & 0 & \cdots & 0 & b_n \end{bmatrix}.$$

This matrix has the same determinant as G_n, namely $b_n \prod_{j=1}^{n-1} (b_j - a_j)$.

Since we are assuming $b_n \neq 0$, we see that G_n is a singular matrix if and only if $b_j = a_j$ for some j, $1 \leqq j \leqq n-1$.

By inspection we can verify that for $n \geqq 2$, the inverse of G_n in (2) is

$$G_n^{-1} = (\delta_{i,j}) = \begin{bmatrix} \frac{1}{c_1} & \frac{-1}{c_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{c_2} & \frac{-1}{c_2} & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{c_3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{-1}{c_{n-2}} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{c_{n-1}} & \frac{-1}{c_{n-1}} \\ \frac{-a_1}{c_1 b_n} & \frac{d_1}{c_1 c_2 b_n} & \frac{d_2}{c_2 c_3 b_n} & \cdots & \frac{d_{n-2}}{c_{n-2} c_{n-1} b_n} & \frac{b_{n-1}}{c_{n-1} b_n} \end{bmatrix},$$

where $c_i = (b_i - a_i)$, $1 \leqq i \leqq n-1$ and $d_i = a_i b_{i+1} - a_{i+1} b_i$, $1 \leqq i \leqq n-2$; that is

$$\begin{aligned} \delta_{i,j} &= \frac{1}{(b_i - a_i)}, \qquad j = i, i \neq n, \\ &= \frac{-1}{(b_i - a_i)}, \qquad j = i+1, i \neq n, \\ &= \frac{a_{j-1} b_j - a_j b_{j-1}}{(b_j - a_j)(b_{j-1} - a_{j-1}) b_n}, \qquad j \neq 1, n; i = n, \\ &= \frac{-a_1}{(b_1 - a_1) b_n}, \qquad j = 1, i = n, \\ &= \frac{b_{n-1}}{(b_{n-1} - a_{n-1}) b_n}, \qquad j = n, i = n, \\ &= 0, \qquad j > i+1, i < n-1, \\ &= 0, \qquad j < i, i \neq n. \end{aligned}$$

If $n = 1$, $G_n^{-1} = 1/b_1$.

Assume λ is a nonzero eigenvalue of G_n. If $\mathbf{x}$ is the corresponding eigenvector in the form $\mathbf{x} = (1, x_2, x_3, \cdots, x_n)^T$, then $G_n\mathbf{x} = \lambda\mathbf{x}$, which we can write as

$$\begin{aligned}
b_1 + b_2x_2 + \cdots + b_{n-1}x_{n-1} + b_nx_n &= \lambda\,, \\
a_1 + b_2x_2 + \cdots + b_{n-1}x_{n-1} + b_nx_n &= \lambda x_2\,, \\
a_1 + a_2x_2 + \cdots + b_{n-1}x_{n-1} + b_nx_n &= \lambda x_3\,, \\
\cdot \qquad\qquad \cdot \qquad &\cdot \quad\ \cdot \\
a_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + b_nx_n &= \lambda x_n\,.
\end{aligned} \tag{3}$$

By subtracting the first equation from the second in (3), we obtain

$$x_2 = (\lambda - b_1 + a_1)/\lambda\,. \tag{4}$$

By subtracting the second equation from the third in (3), and using (4), we obtain

$$x_3 = (\lambda - b_2 + a_2)(\lambda - b_1 + a_1)/\lambda^2\,.$$

Continuing in this fashion, we find that

$$x_j = \frac{\prod_{k=1}^{j-1} (\lambda - b_k + a_k)}{\lambda^{j-1}}\,, \qquad 2 \leqq j \leqq n\,. \tag{5}$$

This determines the eigenvector for $\lambda \neq 0$. Note that if $\lambda \neq 0$, no matter what the multiplicity of λ is, there is only one eigenvector corresponding to it.

In order to obtain the characteristic equation of G_n, we substitute the values of x_j in (5) into the first equation of (3) and multiply both sides of this equation by λ^{n-1}. This yields

$$\lambda^n - b_1\lambda^{n-1} - \sum_{j=1}^{n-1} \left\{ b_{j+1}\lambda^{n-j-1} \prod_{k=1}^{j} (\lambda - b_k + a_k) \right\} = 0\,, \tag{6}$$

which is the characteristic equation. In the notation of [1], if we let

$$S_{k,j} = \sum (b_{\alpha_1} - a_{\alpha_1})(b_{\alpha_2} - a_{\alpha_2}) \cdots (b_{\alpha_k} - a_{\alpha_k})\,,$$

with the sum extending over all possible products of the j terms $(b_i - a_i)$, $1 \leqq i \leqq j$, taken in combinations k at a time, then because

$$\prod_{k=1}^{j} (\lambda - b_k + a_k) = \lambda^j - S_{1,j}\lambda^{j-1} + S_{2,j}\lambda^{j-2} - \cdots + (-1)^j S_{j,j}\,,$$

the characteristic equation in (6) can be written in the form

$$\begin{aligned}
\lambda^n - \left(\sum_{j=1}^{n} b_j\right)\lambda^{n-1} &+ (S_{1,1} + S_{1,2} + \cdots + S_{1,n-1})\lambda^{n-2} \\
&- (S_{2,2} + S_{2,3} + \cdots + S_{2,n-1})\lambda^{n-3} \\
&+ (S_{3,3} + S_{3,4} + \cdots + S_{3,n-1})\lambda^{n-4} \\
&\cdots \\
&(-1)^{n-1}(S_{n-2,n-2} + S_{n-2,n-1})\lambda + (-1)^n S_{n-1,n-1} = 0\,.
\end{aligned} \tag{7}$$

If zero is an eigenvalue of G_n, then $b_i = a_i$, for at least one i, $1 \leqq i \leqq n$, and the ith column is a multiple of the nth column. As in [1], in calculating the characteristic equation and the eigenvectors we can simply ignore the columns (and the corresponding rows) which are multiples of the last column and proceed as before with the smaller matrix. We then multiply the resulting characteristic equation by the proper power of λ and add to the eigenvectors a basis set for the null space of G_n.

3. Test Matrices. The test matrix given in [1] corresponds to G_n in (2) with $b_i = 1$, $1 \leqq i \leqq n$, and $a_i = 10^i - 1/10^i$, $1 \leqq i \leqq n - 1$. Another interesting and useful test matrix is obtained by letting $b_i = 1/2i$, $1 \leqq i \leqq n$, and $a_i = 1/2i + 1$, $1 \leqq i \leqq n - 1$, in (2). This yields the matrix

$$J_n = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n-2} & \frac{1}{2n} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n-2} & \frac{1}{2n} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{2n-2} & \frac{1}{2n} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n-2} & \frac{1}{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n-2} & \frac{1}{2n} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n-1} & \frac{1}{2n} \end{bmatrix}.$$

By the results in Section 2, the determinant of J_n is $1/(2n)!$ and

$$(J_n)^{-1} = \begin{bmatrix} 6 & -6 & 0 & \cdots & 0 & 0 \\ 0 & 20 & -20 & \cdots & 0 & 0 \\ 0 & 0 & 42 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -(2n-4)(2n-3) & 0 \\ 0 & 0 & 0 & \cdots & (2n-2)(2n-1) & -(2n-2)(2n-1) \\ -4n & -4n & -4n & \cdots & -4n & (2n)(2n-1) \end{bmatrix}.$$

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1. H. W. Milnes, "A note concerning the properties of a certain class of matrices," *Math. Comp.*, v. 22, 1968, pp. 827–832.