Symmetric Elliptic Integrals of the Third Kind*

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Abstract. Legendre's incomplete elliptic integral of the third kind can be replaced by an integral which possesses permutation symmetry instead of a set of linear transformations. Two such symmetric integrals are discussed, and direct proofs are given of properties corresponding to the following parts of the Legendre theory: change of parameter, Landen and Gauss transformations, interchange of argument and parameter, relation of the complete integral to integrals of the first and second kinds, and addition theorem. The theory of the symmetric integrals offers gains in simplicity and unity, as well as some new generalizations and some inequalities.

1. Introduction. Present practices in tabulating and computing elliptic integrals are influenced more than one might suppose by the choice of standard integrals which Legendre made near the end of the eighteenth century. In the light of later developments, especially Weierstrass' theory of elliptic functions and Appell's double hypergeometric series, it appears that Legendre's choice conceals and even runs against the grain of an underlying permutation symmetry which offers important simplifications in practice as well as theory. Whether these simplifications outweigh the familiarity of Legendre's integrals is a partly subjective question, but it seems important at least to determine the price of adhering to tradition. A modern choice of standard elliptic integrals of the first and second kinds was investigated theoretically in [3] and later applied to practical questions of computation [4] and tabulation [10]. The present paper extends the theoretical background to integrals of the third kind.

Permutation symmetry has a bearing even on Legendre's complete integral of the first kind, $K(k)$, for the quantity

$$\frac{2}{\pi y^{1/2}} K\left[\left(1 - \frac{x}{y}\right)^{1/2}\right] = \frac{2}{\pi} \int_0^{\pi/2} (x \cos^2 \theta + y \sin^2 \theta)^{-1/2} d\theta \tag{1.1}$$

is seen to be symmetric in x and y by putting $\theta = \pi/2 - \psi$. The right-hand side of (1.1), after being identified by Gauss with the reciprocal of the arithmetic-geometric mean of $x^{1/2}$ and $y^{1/2}$, formed the starting point of his study of elliptic functions; one may therefore say that Gauss initiated the use of homogeneous symmetric elliptic integrals in the complete case. The consequences of permutation symmetry are more extensive in the incomplete integral $F(\phi, k)$, for the symmetry in x, y, z of the quantity

$$(z - x)^{-1/2} F\left[\cos^{-1}\left(\frac{x}{z}\right)^{1/2}, \left(\frac{z - y}{z - x}\right)^{1/2}\right] \tag{1.2}$$

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expresses succinctly the content of the five linear transformations [11, p. 210] of F, one for each nontrivial permutation of x, y, z. A notation using x, y, z as variables in place of ϕ, k eliminates the linear transformations and also the excess quadratic transformations which result from combining quadratic with linear transformations. The change of variables is not a sufficient remedy for Legendre's integrals of the second and third kinds, for these must be combined with F to get symmetric quantities. Symmetry or lack of it can be made conspicuous by using the hypergeometric R-function, $R(a;\, b_1,\, b_2,\, \cdots,\, b_k;\, z_1,\, z_2,\, \cdots,\, z_k)$, which is unchanged by permutation of the subscripts $1, 2, \cdots, k$ and hence is symmetric in any set of z-variables whose corresponding b-parameters are equal [2]. For example the standard symmetric integrals of the first kind, incomplete and complete, are taken in [3] to be

$$R_F(x, y, z) = R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x, y, z)\,, \qquad R_K(x, y) = R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; x, y)\,, \tag{1.3}$$

and these are exactly the quantities (1.2) and (1.1). As illustrated by these examples, the R-function is homogeneous of degree $-a$ in the variables.

Let a' be defined by

$$a' = b_1 + b_2 + \cdots + b_k - a\,. \tag{1.4}$$

The class of elliptic integrals is the class of R-functions for which exactly four of the parameters a, a', $b_1, \cdots, b_k$ are half-odd-integers and the rest are integers. The complete elliptic integrals are the subclass for which a and a' are both half-odd. If a and a' have positive real parts, the R-function has the integral representation

$$B(a, a')R(a; b_1, \cdots, b_k; z_1, \cdots, z_k) = \int_0^\infty t^{a'-1} \prod_{i=1}^{k} (t + z_i)^{-b_i} dt\,, \tag{1.5}$$

where B is the beta function, $\arg t = 0$, $|\arg z_i| < \pi$, $|\arg (t + z_i)| < \pi$, $(i = 1, \cdots, k)$. The integrand contains the square root of a cubic or quartic polynomial in the elliptic case. Because of the restriction on a and a', this representation does not exist (at least not in the simple form shown here) for many important cases, including those chosen in [3] as standard integrals of the second kind,

$$R_G(x, y, z) = R(-\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x, y, z)\,, \qquad R_E(x, y) = R(-\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; x, y)\,. \tag{1.6}$$

However, if the b-parameters have positive real parts, R can be written as a $(k - 1)$-fold integral [7] representing a weighted average of z^{-a} over the convex hull of $\{z_1, \cdots, z_k\}$, the weight function being a Dirichlet distribution with parameters $b_1, \cdots, b_k$. That R_G is represented by a double integral is no hindrance to practical work [10] and is a minor disadvantage by comparison with the five linear transformations of Legendre's $E(\phi, k)$ [11, p. 210], which are replaced by the symmetry of R_G.

In this paper we shall consider two functions, $R_H(x, y, z, \rho)$ and $R_J(x, y, z, \rho)$, either of which can serve to replace Legendre's normal integral of the third kind. Both are completely symmetric in x, y, z but perforce not in ρ. For numerical purposes R_H has some distinct advantages, but certain relations and theorems are simpler in terms of R_J. (There is a similar rivalry between Legendre's integral and Jacobi's integral.) Fortunately it is easy to express one in terms of the other, and we shall use whichever is more convenient for the purpose at hand. Proving directly the properties of R_H or R_J is usually quicker than deducing them from the corresponding properties of Legendre's integral. The proofs, which seldom run parallel to corresponding proofs given by Cayley [8], are often more economical because of permutation symmetry. As expected from experience with the integrals of the first two kinds, symmetry not only replaces the linear transformations of Legendre's third integral but also gives its Landen and Gauss transformations a unified form. Moreover, it puts the two change-of-parameter relations in a unified form along with a third and apparently new relation which is a combination of the first two. The Landen transformation is shown to be a special case of a new quadratic transformation of a multiple hypergeometric function with one free parameter, and a similar generalization is found for the change-of-parameter relations. The form taken by the interchange theorem is more symmetrical than for Legendre's integral but still rather complicated; this theorem seems to find its natural expression in terms of Jacobi's integral of the third kind treated as a function of two integrals of the first kind [8]. A new proof of the addition theorem is given for all three kinds of elliptic integral. Finally, because R_H and R_J are R-functions with positive b-parameters, they satisfy some conveniently simple inequalities.

2. Elliptic Integrals of the Third Kind. Legendre's standard elliptic integral of the third kind is

$$\Pi(\phi, \alpha^2, k) = \int_0^\phi (1 - \alpha^2 \sin^2\theta)^{-1}(1 - k^2\sin^2\theta)^{-1/2} d\theta . \tag{2.1}$$

As a rule this integral is used in numerical and other practical work, whereas the forms of Weierstrass and Jacobi are used in theoretical considerations. Jacobi's integral with angles as the first two arguments is

$$\begin{aligned}\Pi^*(\phi, \psi, k) &= k^2 \sin\psi \cos\psi (1 - k^2 \sin^2\psi)^{1/2} \\ &\quad \cdot \int_0^\phi \sin^2\theta (1 - k^2\sin^2\psi\sin^2\theta)^{-1}(1 - k^2\sin^2\theta)^{-1/2} d\theta .\end{aligned} \tag{2.2}$$

This integral is usually taken to be a function of two integrals of the first kind, u and a, related to ϕ and ψ by $\sin\phi = \operatorname{sn} u$ and $\sin\psi = \operatorname{sn} a$.

To reduce the above integrals to R-functions, we note the result [2, p. 470]

$$\begin{aligned}&\int_0^\phi (\sin\theta)^{2a-1}(\sin^2\phi - \sin^2\theta)^{a'-} (\cos\theta)^{1-2b_1}(1 - k^2\sin^2\theta)^{-b_2} \\ &\qquad\qquad \cdot (1 - \alpha^2\sin^2\theta)^{-b_3} d\theta \\ &\quad = \tfrac{1}{2} B(a, a')(\sin\phi)^{2c-2} R(a; b_1, b_2, b_3, b_4; \cos^2\phi, \Delta^2, 1 - \alpha^2\sin^2\phi, 1) ,\end{aligned} \tag{2.3}$$

where

$c = a + a' = b_1 + b_2 + b_3 + b_4$, $\Delta^2 = 1 - k^2\sin^2\phi$, and $\operatorname{Re} a > 0$, $\operatorname{Re} a' > 0$.

It follows that

$$\Pi(\phi, \alpha^2, k) = \sin\phi\, R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, 1; \cos^2\phi, \Delta^2, 1, 1 - \alpha^2 \sin^2\phi)\,,$$

$$\text{(2.4)}\quad \Pi^*(\phi, \psi, k) = \frac{k^2}{3} \sin^3\phi \sin\psi \cos\psi (1 - k^2 \sin^2\psi)^{1/2} \cdot R(\tfrac{3}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1; \cos^2\phi, \Delta^2, 1, 1 - k^2 \sin^2\phi \sin^2\psi)\,.$$

Using relations between associated functions [2, p. 458], we can write Π in terms of functions which are symmetric in their first three arguments:

$$\text{(2.5)}\quad \begin{aligned} \Pi(\phi, \alpha^2, k) &= -\frac{2}{3} \frac{\alpha^2 \sin^3\phi}{1 - \alpha^2 \sin^2\phi} R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1; \cos^2\phi, \Delta^2, 1, 1 - \alpha^2 \sin^2\phi) \\ &\quad + \frac{\sin\phi}{1 - \alpha^2 \sin^2\phi} R_F(\cos^2\phi, \Delta^2, 1) \\ &= \frac{\alpha^2}{3} \sin^3\phi\, R(\tfrac{3}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1; \cos^2\phi, \Delta^2, 1, 1 - \alpha^2 \sin^2\phi) \\ &\quad + \sin\phi\, R_F(\cos^2\phi, \Delta^2, 1)\,. \end{aligned}$$

This suggests that a normal elliptic integral of the third kind, possessing as much symmetry as possible, could be either

$$\text{(2.6)}\quad \begin{aligned} R_H(x, y, z, \rho) &= R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1; x, y, z, \rho) \\ &= \frac{3}{4} \int_0^\infty [(t + x)(t + y)(t + z)]^{-1/2} (t + \rho)^{-1} t\, dt \end{aligned}$$

or

$$\text{(2.7)}\quad \begin{aligned} R_J(x, y, z, \rho) &= R(\tfrac{3}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1; x, y, z, \rho) \\ &= \frac{3}{2} \int_0^\infty [(t + x)(t + y)(t + z)]^{-1/2} (t + \rho)^{-1} dt\,. \end{aligned}$$

If the a-parameter were chosen to be $5/2$, $7/2$, $\cdots$, the function would be algebraic, and the choices $-1/2$, $-3/2$, $\cdots$ prove to be less convenient than either $1/2$ or $3/2$, partly because a negative a-parameter excludes the representation (1.5). Although R_J is a multiple of Jacobi's integral, R_H is not a multiple of Legendre's. If $\rho = 0$, R_H reduces to $\frac{3}{2} R_F$ while R_J has the disadvantage (for some practical purposes) of a logarithmic singularity. The constants multiplying the integral representations have the effect that

$$R_H = R_J = 1 \quad \text{if } x = y = z = \rho = 1.$$

If $\rho < 0$ we take the Cauchy principal value of each integral.

We define the respective complete integrals to be

$$\text{(2.8)}\quad \begin{aligned} R_L(x, y, \rho) &= R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, 1; x, y, \rho)\,, \\ R_M(x, y, \rho) &= R(\tfrac{3}{2}; \tfrac{1}{2}, \tfrac{1}{2}, 1; x, y, \rho)\,. \end{aligned}$$

A complete integral is a constant multiple of an incomplete integral with a zero argument. From (1.5) or [2, p. 457] we find

$$
\begin{aligned}
R_H(x, y, 0, \rho) &= (3\pi/8)R_L(x, y, \rho) , \\
R_J(x, y, 0, \rho) &= (3\pi/4)R_M(x, y, \rho) , \\
R_L(x, y, 0) &= 2R_K(x, y) , \\
R_H(x, y, z, 0) &= \tfrac{3}{2} R_F(x, y, z) , \\
R_H(x, y, 0, 0) &= (3\pi/4)R_K(x, y) , \\
\lim_{\rho\to\infty} \rho^{1/2}R_H(x, y, z, \rho) &= \frac{3\pi}{4} , \\
\lim_{\rho\to\infty} \rho R_J(x, y, z, \rho) &= 3R_F(x, y, z) .
\end{aligned} \tag{2.9}
$$

Legendre's integral (2.1) is said to be complete when $\phi = \pi/2$. From (2.5) and (2.9) we obtain

$$
\begin{aligned}
\frac{4}{\pi}\Pi\left(\frac{\pi}{2}, \alpha^2, k\right) &= \frac{\alpha^2}{\alpha^2 - 1} R_L(1 - k^2, 1, 1 - \alpha^2) + \frac{2}{1 - \alpha^2} R_K(1 - k^2, 1) \\
&= \alpha^2 R_M(1 - k^2, 1, 1 - \alpha^2) + 2R_K(1 - k^2, 1) .
\end{aligned}
$$

An important connection between R_H and R_J is easily obtained from (2.6) and (2.7):

$$2R_H(x, y, z, \rho) + \rho R_J(x, y, z, \rho) = 3R_F(x, y, z) . \tag{2.10}$$

Putting $z = 0$ and using (2.9), we have

$$R_L(x, y, \rho) + \rho R_M(x, y, \rho) = 2R_K(x, y) . \tag{2.11}$$

Equations (2.10) and (2.11) allow rapid conversion from one choice of standard integrals to the other. Euler's transformation of the R-function [2, Eq. (2.8)] furnishes a second connection between R_L and R_M (cf. (4.9) below),

$$\rho R_M(x, y, \rho) = R_L(x, y, xy/\rho) . \tag{2.12}$$

3. Linear Transformations. The linear transformations of Legendre's integrals $F(\phi, k)$ and $E(\phi, k)$ are used to put the modulus and amplitude into ranges which have been tabulated. It has been shown [3] that these transformations are replaced by the symmetry of R_F and R_G in the arguments x, y, z. Similarly, the five linear transformations of $\Pi(\phi, \alpha^2, k)$ are equivalent to the five permutations of the first three arguments in R_H or R_J, and the quantities $\cos^2\phi = x/z$, $\alpha^2 = (z - \rho)/(z - x)$, and $k^2 = (z - y)/(z - x)$ are all changed by these permutations. Since $R_{H,J}$ are symmetric in x, y, z, their linear transformations are trivial. For example, if the imaginary-modulus transformation [1, p. 38] is written in terms of R_H by using (2.5) and (2.6), it reduces to $R_H(x, y, z, \rho) = R_H(x, z, y, \rho)$. Similarly, the imaginary-argument transformation reduces to interchanging x and z and the reciprocal-modulus transformation to interchanging x and y.

4. Change of Parameter. In (2.1), α^2 is called the parameter and is real (like k^2 and ϕ) in most cases of practical interest. (If $\alpha^2 \sin^2 \phi > 1$ the Cauchy principal value of the integral is to be taken.) After arranging by linear transformation (if necessary) that $0 < k^2 < 1$, the real α^2 axis is divided into the four intervals $(-\infty, 0)$, $(0, k^2)$, $(k^2, 1)$, $(1, \infty)$. The first and third intervals are designated as circular, the second and fourth as hyperbolic. There exist transformations called change-of-parameter relations [8, pp. 119–125] connecting two integrals whose parameters are related either by $\alpha^2\alpha_1^2 = k^2$ or $(1 - \alpha^2)(1 - \alpha_1^2) = k'^2$, where $k'^2 = 1 - k^2$. The former relation maps each circular interval onto itself and interchanges the two hyperbolic intervals, whereas the latter relation maps each hyperbolic interval onto itself while interchanging the circular intervals.

There are corresponding divisions of the real axis in the ρ-plane. If x, y, z are positive, as they are in most cases of interest, then by symmetry we may assume $0 < x < y < z$. The intervals $(-\infty, x)$, (x, y), (y, z), (z, ∞) of the real ρ-axis will be labeled H', C', H, C, respectively, the first and third intervals now being hyperbolic while the second and fourth are circular. The product $(\rho - x)(\rho - y)(\rho - z)$ is positive in the circular cases and negative in the hyperbolic cases.

If in (1.5) we substitute $t = s(s + f)/(s + g)$, where f and g are unequal and nonzero but otherwise arbitrary, we obtain the useful result, valid if $\operatorname{Re} a > 0$ and $\operatorname{Re} a' > 0$,

$$\begin{aligned} &B(a, a')R(a; b_1, \cdots, b_k; z_1, \cdots, z_k) \\ (4.1)\quad &= \int_0^\infty [s(s+f)]^{a'-1}(s+g)^{a-1}(s+u_0)(s+v_0)\prod_{i=1}^{k}[(s+u_i)(s+v_i)]^{-b_i}ds\,, \end{aligned}$$

where

$$(4.2)\qquad \begin{aligned} &u_0 + v_0 = 2g\,, \qquad u_0v_0 = fg\,, \\ &u_i + v_i = f + z_i\,, \qquad u_iv_i = z_ig\,, \qquad (i = 1, \cdots, k)\,. \end{aligned}$$

The choices $k = 2$ and $a' = b_2 = 1$ imply $b_1 = a$. By decomposing $(s + u_0) \cdot (s + v_0)/(s + u_2)(s + v_2)$ into partial fractions, we easily obtain a new quadratic transformation for hypergeometric functions with one free parameter:

$$(4.3)\qquad \begin{aligned} (1 + a)R(a; a, 1; \xi, \eta) = {} & (1 + a)R(a; 1 - a, a, a; x, y, z) \\ & - a(\rho - x)R(a + 1; 1 - a, a, a, 1; x, y, z, \rho) \\ & - a(\gamma - x)R(a + 1; 1 - a, a, a, 1; x, y, z, \gamma)\,, \end{aligned}$$

where g, u_1, v_1, u_2, v_2 have been relabeled x, y, z, ρ, γ and

$$(4.4)\qquad \begin{aligned} &\xi = yz/x\,, \qquad \eta = \rho\gamma/x\,, \\ &(\rho - x)(\gamma - x) = (y - x)(z - x)\,. \end{aligned}$$

These relations imply that

$$(4.5)\qquad \eta - \xi = \frac{(\rho - y)(\rho - z)}{\rho - x} = \frac{(\gamma - y)(\gamma - z)}{\gamma - x}\,.$$

The quantities $f = y + z - yz/x$ and $u_0 - x = x - v_0 = [(y - x)(z - x)]^{1/2}$ are determined by (4.2) but do not appear explicitly in (4.3). The restriction on the real part of a can be dropped by analytic continuation.

Putting $a = 1/2$ in (4.3) and permuting x, y, z, we obtain three distinct relations for elliptic integrals. If i, j, k stand for any permutation of 1, 2, 3, we have

$$(4.6)\quad (\rho - x_i)R_J(x, y, z, \rho) + (\gamma - x_i)R_J(x, y, z, \gamma) = 3R_F(x, y, z) - 3R_f(\xi_i, \eta_i)\,,$$

where

$$(4.7)\quad \begin{aligned}(x_1, x_2, x_3) &= (x, y, z)\,,\\ (\rho - x_i)(\gamma - x_i) &= (x_j - x_i)(x_k - x_i)\,,\\ \xi_i = x_j x_k/x_i, \quad \eta_i &= \rho\gamma/x_i\,,\\ \eta_i - \xi_i &= (\rho - x_j)(\rho - x_k)/(\rho - x_i)\\ &= (\gamma - x_j)(\gamma - x_k)/(\gamma - x_i)\,.\end{aligned}$$

The last equality shows that ρ and γ are both in circular intervals or both in hyperbolic intervals. The function R_f is an elementary function defined by

$$(4.8)\quad \begin{aligned}R_f(x, y) &= R_F(x, y, y)\\ &= R(\tfrac{1}{2}; \tfrac{1}{2}, 1; x, y)\\ &= (y - x)^{-1/2}\cos^{-1}(x/y)^{1/2}\,, \qquad (0 \leqq x < y)\,,\\ &= (x - y)^{-1/2}\cosh^{-1}(x/y)^{1/2}\,, \qquad (0 \leqq y < x)\,,\\ &= (x - y)^{-1/2}\sinh^{-1}(x/-y)^{1/2}\,, \qquad (y < 0 \leqq x)\,.\end{aligned}$$

When the second argument of R_f is negative, we have taken the Cauchy principal value of the integral representation (1.5); this case arises if $\rho\gamma < 0$. Real inverse circular or hyperbolic functions occur in the last term of (4.6) according as the parameters ρ and γ lie in circular or hyperbolic intervals.

The transformation of ρ into γ determined by the second of Eqs. (4.7) will be called a τ_i-transformation, although we shall use the term also to designate the corresponding case of (4.6). Assuming $0 < x_1 < x_2 < x_3$, we can easily determine for given i and ρ the interval in which γ lies. The results are summarized in Table 1; if $i = 1$, for example, $\rho \in H'$ implies $\gamma \in H'$ whereas $\rho \in C'$ implies $\gamma \in C$.

TABLE 1

	γ		
ρ	τ_1	τ_2	τ_3
H'	H'	H	H
C'	C	C	C'
H	H	H'	H'
C	C'	C'	C

The relations τ_1 and τ_3 correspond in Legendre's notation to $(1 - \alpha^2)(1 - \alpha_1^2) = k'^2$ and $\alpha^2\alpha_1^2 = k^2$, respectively. The τ_2-transformation is the product of τ_1 and τ_3 (which generate an Abelian group of order four in which each element is its own inverse) and corresponds to $(\alpha^2 - k^2)(\alpha_1^2 - k^2) = -k^2k'^2$. We have not seen this product transformation elsewhere, but the present notation brings it out on an equal footing with the other two and in a unified form. The transformation τ_3 is sometimes used in numerical work to transform an integral of type H' into one of type H, so that only the latter type needs to be tabulated, and similarly τ_1 is used to transform type C into C'; the single transformation τ_2 suffices for both purposes.

To obtain similar results for complete integrals, we need only choose $x = 0$ in each transformation and use (2.9). The results are best expressed in terms of R_L. For the τ_1-transformation we find

$$R_L(y, z, \rho) + R_L(y, z, yz/\rho) = 2R_K(y, z) , \tag{4.9}$$

in agreement with (2.11) and (2.12). The τ_3-transformation reduces to

$$\begin{aligned}(\rho - y)(\rho - z)R_L(y, z, \rho) + \rho(z - y)R_L(y, z, ((\rho - y)/(\rho - z))z) \\ = 2y(z - \rho)R_K(y, z) + 2[\rho(\rho - y)(\rho - z)]^{1/2}\end{aligned} \tag{4.10}$$

in the circular cases; in the hyperbolic cases the last term is missing because $\eta_3 < 0$ and the Cauchy principal value of R_f is zero according to (4.8). The complete case of the τ_2-transformation is the same as that of the τ_3-transformation with y and z interchanged.

5. Landen and Gauss Transformations. Equation (4.1) is the source of another set of quadratic transformations, namely the Landen and Gauss transformations. When used recursively, these provide a method for numerical computation of elliptic integrals.

If in (4.1) we choose $k = 4$, $b_1 = b_2 = 1/2$, $b_4 = 1$, $a' = 2$, and $f = (z_1z_2)^{1/2}$, $g = (z_1^{1/2} + z_2^{1/2})^2/4$, then from (4.2) it can be shown that $u_1 = v_1 = u_0 = (z_1g)^{1/2}$ and $u_2 = v_2 = v_0 = (z_2g)^{1/2}$. Expanding $(s + f)/(s + u_4)(s + v_4)$ in partial fractions and relabeling variables, we find a new quadratic transformation with one free parameter:

$$\begin{aligned}&(\sigma - \tau)R(a; 1/2, 1/2, a, 1; x, y, z, \rho) \\ &\quad = (\rho - \tau)R(a; 1 - a, a, a, 1; u, v, w, \sigma) \\ &\quad\quad + (\sigma - \rho)R(a; 1 - a, a, a, 1; u, v, w, \tau) ,\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}2u &= (xy)^{1/2} + (x + y)/2 , \\ 2v &= (xy)^{1/2} + z + (z - x)^{1/2}(z - y)^{1/2} , \\ 2w &= (xy)^{1/2} + z - (z - x)^{1/2}(z - y)^{1/2} , \\ 2\sigma &= (xy)^{1/2} + \rho + (\rho - x)^{1/2}(\rho - y)^{1/2} , \\ 2\tau &= (xy)^{1/2} + \rho - (\rho - x)^{1/2}(\rho - y)^{1/2} .\end{aligned} \tag{5.2}$$

The inverse relations are

$$\begin{aligned}x &= [u + (u - v)^{1/2}(u - w)^{1/2}]^2/u , \\ y &= [u - (u - v)^{1/2}(u - w)^{1/2}]^2/u , \\ z &= vw/u = v + w - (xy)^{1/2} , \\ \rho &= \sigma\tau/u = \sigma + \tau - (xy)^{1/2} .\end{aligned} \tag{5.3}$$

$$(\sigma - u)(\tau - u) = (v - u)(w - u) .$$

Putting $a = 1/2$ in (5.1), we have

$$(5.4)\quad (\sigma - \tau)R_H(x, y, z, \rho) = (\rho - \tau)R_H(u, v, w, \sigma) + (\sigma - \rho)R_H(u, v, w, \tau)$$

or, in terms of R_J,

$$(5.5)\quad (\sigma - \tau)\rho R_J(x, y, z, \rho) = (\rho - \tau)\sigma R_J(u, v, w, \sigma) + (\sigma - \rho)\tau R_J(u, v, w, \tau) .$$

Because $\sigma\tau = u\rho$, (5.5) reduces to

$$(5.6)\quad (\sigma - \tau)R_J(x, y, z, \rho) = (\sigma - u)R_J(u, v, w, \sigma) - (\tau - u)R_J(u, v, w, \tau) .$$

One of the R_J terms in (5.6) can be eliminated since the relation $(\sigma - u)(\tau - u) = (v - u)(w - u)$ is formally the same as the relation between ρ and γ in (4.4). Using (4.6) and recalling from [3] that $R_F(x, y, z) = R_F(u, v, w)$, we find

$$(5.7\sigma)\quad 2(\sigma - u)R_J(u, v, w, \sigma) = (\sigma - u\rho/\sigma)R_J(x, y, z, \rho) + 3R_F(x, y, z) - 3R_f(z, \rho) ,$$

$$(5.7\tau)\quad 2(\tau - u)R_J(u, v, w, \tau) = (\tau - u\rho/\tau)R_J(x, y, z, \rho) + 3R_F(x, y, z) - 3R_f(z, \rho) .$$

If we assume x, y, z nonnegative and ρ real, then (5.2) is a real transformation if and only if $(z - x)(z - y) \geqq 0$ and $(\rho - x)(\rho - y) \geqq 0$. Let the real σ-axis, and likewise the real τ-axis, be divided into circular and hyperbolic intervals $\mathcal{H}'$, $\mathcal{C}'$, $\mathcal{H}$, $\mathcal{C}$ separated by the nonnegative numbers u, v, w arranged in increasing order. The preceding condition on ρ and the identities

$$(5.8)\quad \frac{(\sigma - v)(\sigma - w)}{\sigma - u} = \frac{(\tau - v)(\tau - w)}{\tau - u} = \rho - z$$

ensure that ρ, σ, and τ are all in circular intervals or all in hyperbolic intervals. Table II shows which intervals occur in each case. For definiteness we may take $\sigma \geqq \tau$ by appropriate choice of sign of a square root, since the Landen and Gauss transformations are unaltered by interchanging σ and τ (or x and y, or v and w). Note that ρ, σ, τ lie in corresponding intervals in all but two of the real cases. These two cases are inconvenient for iterative computation; and hence, for example, if $\rho \in H'$ one would use (5.7τ) instead of (5.7σ) in making successive Gauss transformations.

TABLE II

	Landen $x, y < z$ $u < v, w$		*Gauss* $z < x, y$ $v, w < u$	
ρ	σ	τ	σ	τ
H'	$\mathcal{H}'$	$\mathcal{H}'$	$\mathcal{H}$	$\mathcal{H}'$
C'	*complex*	*complex*	$\mathcal{C}'$	$\mathcal{C}'$
H	$\mathcal{H}$	$\mathcal{H}$	*complex*	*complex*
C	$\mathcal{C}$	$\mathcal{C}'$	$\mathcal{C}$	$\mathcal{C}$

The Landen and Gauss transformations of Legendre's third integral [1, p. 39] can be found from (5.4) and (5.7) with the help of (2.5) and (5.3) (or (5.2) for the inverse transformations). Since the right-hand sides of (5.3) are symmetric in v and w, the transformation is fixed by choosing u from among the quantities $\cos^2 \phi$, Δ^2, and 1. The choice $u = \cos^2 \phi$ gives Landen's transformation, $u = \Delta^2$ gives a complex transformation (see Section 6), and $u = 1$ gives Gauss' transformation in agreement with the inequalities in Table II. Thus the unified form taken by the quadratic transformation in present notation yields the Landen or the Gauss transformation according to the relative sizes of the arguments.

To obtain the quadratic transformations of R_M we put $w = z = 0$ in (5.7σ) and find

$$(5.9)\quad \begin{aligned} &2(\sigma - u)R_M[(xy)^{1/2}, u, \sigma] \\ &\qquad = (\rho - x)^{1/2}(\rho - y)^{1/2}R_M(x, y, \rho) + 2R_K(x, y) - 2\rho^{-1/2}. \end{aligned}$$

The complete case of (5.7τ) is obtained from (5.9) by replacing σ by τ and changing the sign of the square root of $(\rho - x)$ $(\rho - y)$.

6. Conjugate Complex Arguments. When an elliptic integral is reduced to standard integrals, it may happen that two of the arguments are complex conjugates of each other, say z and $\bar{z}$, while the others are real. A quadratic transformation can be used to replace the complex arguments by real ones, as is done in [10, p. 227] for the integrals of the first two kinds. Similar results are easily found for $R_{H,J}(z, \bar{z}, x, \rho)$ from (5.4) and (5.6) or (5.7). In addition to replacing x, y, z by z, $\bar{z}$, x, we need only substitute the following values obtained from (5.2):

$$(6.1)\quad \begin{aligned} 2u &= |z| + \operatorname{Re} z\,, \\ 2v &= |z| + x + |z - x|, \\ 2w &= |z| + x - |z - x|\,, \\ 2\sigma &= |z| + \rho + |z - \rho|\,, \\ 2\tau &= |z| + \rho - |z - \rho|\,. \end{aligned}$$

Alternatively, we may replace u, v, w by x, $\bar{z}$, z and use (5.3) and (5.7).

7. The Interchange Theorem. When Jacobi's integral (2.2) is regarded as a function of two integrals of the first kind, the theorem for the interchange of amplitude and parameter takes its simplest (but not always real) form [8, p. 159]. The corresponding results for Legendre's third integral, which are given by Cayley [8, pp. 133–141] except for integrals of type H', are complicated and take different real forms in the various circular and hyperbolic intervals. However, they are useful because their complete cases allow Legendre's complete integral of the third kind to be expressed in terms of integrals of the first and second kinds [11, pp. 189–192]. Similarly the interchange theorem for R_J takes two different forms but allows the complete integral R_M (and therefore R_L) to be expressed in terms of R_F, R_G, R_K, and R_E.

Let $x = (x_1, x_2, x_3)$ and $\delta(x, \rho) = (\rho - x_1)$ $(\rho - x_2)$ $(\rho - x_3)$. From (2.7) and (1.5) we find

$$\frac{\partial}{\partial \rho} R_J(x, \rho) = -\tfrac{3}{5} R(\tfrac{5}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 2; x_1, x_2, x_3, \rho) . \tag{7.1}$$

By means of relations between associated functions [2, p. 458] the right-hand side can be expressed in terms of R_F, R_G, and R_J, with the result that

$$\begin{aligned} &2\delta(x, \rho) \frac{\partial}{\partial \rho} R_J(x, \rho) + R_J(x, \rho) \frac{\partial}{\partial \rho} \delta(x, \rho) \\ &\qquad = 3\rho R_F(x) - 6R_G(x) + \frac{3}{\rho} (x_1 x_2 x_3)^{1/2} . \end{aligned} \tag{7.2}$$

Let i, j, k be a permutation of 1, 2, 3. Putting $\rho = x_i$ in (7.2) and using the resulting equation to eliminate R_G, we have

$$\begin{aligned} &2\delta(x, \rho) \frac{\partial}{\partial \rho} R_J(x, \rho) + R_J(x, \rho) \frac{\partial}{\partial \rho} \delta(x, \rho) \\ &\quad = 3(\rho - x_i) R_F(x) + (x_i - x_j)(x_i - x_k) R_J(x, x_i) - 3 \frac{\rho - x_i}{\rho} \left(\frac{x_j x_k}{x_i}\right)^{1/2} . \end{aligned} \tag{7.3}$$

Assume now that x_i, x_j, x_k, ρ are positive and pairwise unequal and that neither x_j nor x_k lies between x_i and ρ. Let ϵ denote the sign of $-\delta(x, \rho)$ and s_i the sign of $\rho - x_i$; note that $(\rho - x_j)$ $(\rho - x_k)$ and $(x_i - x_j)$ $(x_i - x_k)$ both have the sign $-\epsilon s_i$. Then (7.3) can be written as

$$\begin{aligned} &-2\epsilon |\delta(x, \rho)|^{1/2} \frac{\partial}{\partial \rho} \{ |\delta(x, \rho)|^{1/2} R_J(x, \rho) \} = 3 s_i |\rho - x_i| R_F(x) \\ &\qquad - \epsilon s_i |x_i - x_j|\, |x_i - x_k| R_J(x, x_i) - 3 s_i \frac{|\rho - x_i|}{\rho} \left(\frac{x_j x_k}{x_i}\right)^{1/2} . \end{aligned} \tag{7.4}$$

We divide both sides by $|\delta(x, \rho)|^{1/2}$, replace ρ by t, and integrate with respect to t from x_i to ρ if $s_i = +1$ or from ρ to x_i if $s_i = -1$. By using [10, Eq. (T.1)] each integral on the right-hand side can be written as an R-function in which the first three arguments are the components of $x' = (x_1', x_2', x_3')$, where

$$\begin{aligned} x_i' &= x_i , \\ (x_j' - x_i)(x_j - x_i) &= x_i(x_i - \rho) , \\ (x_k' - x_i)(x_k - x_i) &= x_i(x_i - \rho) . \end{aligned} \tag{7.5}$$

If we define

$$V_i(x, \rho) = |\rho - x_j|^{1/2} |\rho - x_k|^{1/2} R_J(x, \rho) , \tag{7.6}$$

the resulting theorem for interchange of x and x' takes the form

$$\begin{aligned} &V_i(x, \rho) - x_i^{1/2} R_F(x') V_i(x, x_i) = \epsilon \{ V_i(x', \rho) - x_i^{1/2} R_F(x) V_i(x', x_i) \} , \\ &\qquad\qquad (i = 1, 2, 3) . \end{aligned} \tag{7.7}$$

Only the first term on each side is an integral of the third kind, for we have

$$\begin{aligned} R_J(x, x_i) &= R(\tfrac{3}{2}; \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x_i, x_j, x_k) \\ &= \frac{3}{(x_i - x_j)(x_i - x_k)} \left[x_i R_F(x) - 2R_G(x) + \left(\frac{x_j x_k}{x_i}\right)^{1/2} \right] . \end{aligned} \tag{7.8}$$

It is easy to verify that $(x_i - x_j')/(\rho - x_j') = x_i/x_j > 0$ and similarly for x_k', whence it follows that neither x_j' nor x_k' (nor x_j nor x_k, by assumption) lies between x_i and ρ, the two invariant points. Moreover, $(\rho - x_j')$ $(\rho - x_k')$ has the sign $-\epsilon s_i$, and hence ρ lies either in circular intervals with respect to both x and x' or in hyperbolic intervals with respect to both, as is shown in more detail in Table III. Since ϵ is the sign of $(x_i - \rho)$ $(x_j - \rho)$ $(x_k - \rho)$, it is $+1$ in hyperbolic cases and -1 in circular cases.

TABLE III

(x, ρ)	(x', ρ) $0 < x_i < \rho$	(x', ρ) $0 < \rho < x_i$
H'	—	H'
C'	C	C'
H	H	H
C	C'	—

The preceding method fails because of divergent integrals if we attempt to integrate from 0 to ρ or from ρ to $\pm\infty$, and hence we have no theorem yet for the case $\rho < 0$. A remedy is to substitute (2.10) in (7.2) in order to show that

$$
\begin{aligned}
&4\delta(x, \rho) \frac{\partial}{\partial\rho}\left[\frac{R_H(x, \rho)}{\rho}\right] + 2\frac{R_H(x, \rho)}{\rho}\frac{\partial}{\partial\rho}\delta(x, \rho) \\
&\quad = 3R_F(x)\left[\frac{2x_1x_2x_3}{\rho^2} - \frac{x_1x_2 + x_2x_3 + x_3x_1}{\rho}\right] + 6R_G(x) - \frac{3}{\rho}(x_1x_2x_3)^{1/2} .
\end{aligned} \tag{7.9}
$$

Assuming x_1, x_2, x_3 are positive and ρ is negative, we can now divide by $|\delta(x, \rho)|^{1/2}$, replace ρ by t, and integrate with respect to t from $-\infty$ to ρ. Since ρ is negative, $R_H(x, \rho)$ is taken to be a Cauchy principal value, and it is easy to verify by (2.6) that

$$
\lim_{\rho\to-\infty} (-\rho)^{1/2}R_H(x, \rho) = R_H(0, 0, 0, -1) = 0 , \tag{7.10}
$$

in contrast with the last but one of Eqs. (2.9). The final result is best expressed in terms of R_J and can be put in the same form as a second result obtained similarly for the case $0 < x_1, x_2, x_3 < \rho$ by an integration from ρ to $+\infty$. Letting ϵ denote as before the sign of $(x_1 - \rho)$ $(x_2 - \rho)$ $(x_3 - \rho)$ and defining

$$
\begin{aligned}
x' &= (x_1', x_2', x_3') = \epsilon(x_1 - \rho, x_2 - \rho, x_3 - \rho) , \\
\rho' &= -\epsilon\rho , \\
W(x, \rho; x', \rho') &= \tfrac{1}{3}(x_1'x_2'x_3')^{1/2}R_J(x, \rho) + R_F(x') \\
&\qquad \times [x_1R_F(x) - 2R_G(x)] + (1 - \epsilon)\pi/8 ,
\end{aligned} \tag{7.11}
$$

we have the further interchange theorem

$$
W(x, \rho; x', \rho') = \epsilon W(x', \rho'; x, \rho) . \tag{7.12}
$$

In the case $\epsilon = 1$ this theorem connects two R_J's of type H', one with a negative and one with a positive value of the fourth argument. In the case $\epsilon = -1$ both integrals are of type C. In both cases the terms proportional to x_1 and x_1' can be combined by noting that $x_1 - \epsilon x_1' = \rho$.

The complete cases of the interchange theorems have special interest. Putting $x_k = 0$ in (7.5), we observe that $x_k' = \rho$ and hence $V_i(x', \rho) = 0$ in (7.7). By (2.9), R_J reduces to R_M and we encounter

$$R_M(x_i, x_j, x_i) = R(\tfrac{3}{2}; \tfrac{3}{2}, \tfrac{1}{2}; x_i, x_j) = \frac{2}{x_i(x_i - x_j)} [x_i R_K(x_i, x_j) - R_E(x_i, x_j)] . \tag{7.13}$$

The form of (7.7) can then be simplified by introducing the function

$$\begin{aligned} R(\tfrac{1}{2}; \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}; x, y, z) &= \tfrac{1}{3}(z - x)R_J(x, y, z, x) + R_F(x, y, z) \\ &= \frac{1}{(y - x)} \left[yR_F(x, y, z) - 2R_G(x, y, z) + \left(\frac{yz}{x}\right)^{1/2} \right], \end{aligned} \tag{7.14}$$

where the last equation follows from (7.8). The resulting expression for R_M in terms of integrals of the first and second kinds is

$$\begin{aligned} \tfrac{1}{2}(x_i - x_j)(\rho x_j'/x_i)^{1/2} R_M(x_i, x_j, \rho) &= x_i R_K(x_i, x_j) R(\tfrac{1}{2}; \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}; x_i, x_j', \rho) \\ &\quad - R_E(x_i, x_j) R_F(x_i, x_j', \rho) , \end{aligned} \tag{7.15}$$

where x_i, x_j, ρ are positive, x_j does not lie between x_i and ρ, and x_j' is given by (7.5).

To obtain the complete case of (7.12) we put $x_3 = 0$ in (7.11) and observe that $x_3' = \rho'$. The result is

$$\begin{aligned} \tfrac{1}{2}(x_1' x_2' \rho')^{1/2} R_M(x_1, x_2, \rho) &= R_E(x_1, x_2) R_F(x_1', x_2', \rho') \\ &\quad + \epsilon R_K(x_1, x_2)[\rho' R_F(x_1', x_2', \rho') - 2R_G(x_1', x_2', \rho')] - \frac{1 - \epsilon}{2} , \end{aligned} \tag{7.16}$$

where x_1 and x_2 are positive and ρ is either negative or else positive and greater than x_1 and x_2. The quantities x_1', x_2', ρ' are defined by (7.11), and $\epsilon = -\rho/\rho'$ is the sign of $-\rho$. If $\epsilon = -1$ and ρ is put equal to the larger of x_1 and x_2, the left-hand side vanishes and (7.16) becomes equivalent to Legendre's relation [9, p. 320] between complete integrals of the first and second kinds.

Comparison of (7.15) and (7.16) with known results for Legendre's third integral involves some subtleties. Tricomi [11, pp. 189–192] gives formulas for integrals of types H', C', H, C in four parts of Eqs. (3.107) which are labeled (I), (II), (III), (IV), respectively. (The square brackets in (II) should be multiplied by $(k')^{-2}$. These formulas are reproduced in [9, p. 321] but with misprints in Eqs. (22) and (24).) However, the four equations have alternative forms, designated here by (Ia), (IIa), (IIIa), (IVa), which are obtained by using a special case of the addition theorems for F and E. The procedure is indicated by Tricomi preceding Eq. (3.112), which is essentially (IVa), but the other three equations are not given explicitly. The various cases of (7.15), distinguished by the relative sizes of x_i, x_j, and ρ, are equivalent for integrals of type C' to (IIa), for type H to (IIIa) or (III) according as $x_i < \rho < x_j$ or $x_j < \rho < x_i$, and for type C to (IV). For integrals of type C, (7.16) is equivalent to (IVa) and for type H' to (I), except that the imaginary term in (I) is missing because R_M is a Cauchy principal value.

8. The Addition Theorem. The cubic polynomials $(t + x)\ (t + y)\ (t + z)$ and $t(t - \lambda)\ (t - \mu)$ have equal values for exactly one value of t if and only if the quadratic polynomial

$$Q(t) = (t + x)(t + y)(t + z) - t(t - \lambda)(t - \mu) \\ = t^2(x + y + z + \lambda + \mu) + t(xy + yz + zx - \lambda\mu) + xyz \tag{8.1}$$

has a double zero. For fixed positive x, y, z we denote the discriminant of Q by $D(\lambda, \mu)$; the condition for a double zero is $D(\lambda, \mu) = 0$, i.e.,

$$(\lambda\mu - xy - yz - zx)^2 = 4xyz(x + y + z + \lambda + \mu)\,. \tag{8.2}$$

This biquadratic relation between λ and μ has a branch on which both are positive and

$$\mu = \frac{xy + yz + zx}{\lambda} + \frac{2xyz}{\lambda^2} + \frac{2}{\lambda^2}\,[xyz(\lambda + x)(\lambda + y)(\lambda + z)]^{1/2}\,, \tag{8.3}$$

where the positive square root is taken. Note that $\mu = 2(xyz)^{1/2}\lambda^{-1/2} + O(\lambda^{-1})$ as $\lambda \to +\infty$. The same relations hold with λ and μ interchanged. Since it follows from (8.3) that $\lambda\mu - xy - yz - zx > 0$, the double zero of Q is positive:

$$Q(t_0) = Q'(t_0) = 0,\ t_0 = \frac{\lambda\mu - xy - yz - zx}{2(x + y + z + \lambda + \mu)} \\ = \frac{(xyz)^{1/2}}{(x + y + z + \lambda + \mu)^{1/2}} > 0\,. \tag{8.4}$$

We denote the positive square root of Q by

$$[Q(t)]^{1/2} = \pm(x + y + z + \lambda + \mu)^{1/2}t \mp (xyz)^{1/2}\,, \tag{8.5}$$

where the upper signs are to be taken for $t > t_0$ and the lower signs for $t < t_0$. It is easily seen from (8.4) that $\lambda > t_0$ and $\mu > t_0$.

Differentiating $D(\lambda, \mu) = 0$ and noting that

$$\begin{aligned} \partial D/\partial\lambda &= 2(\lambda\mu - xy - yz - zx)\mu - 4xyz \\ &= 4(xyz)^{1/2}(x + y + z + \lambda + \mu)^{1/2}\mu - 4xyz \\ &= 4(xyz)^{1/2}[Q(\mu)]^{1/2}\,, \end{aligned}$$

we find

$$\frac{d\lambda}{[Q(\lambda)]^{1/2}} + \frac{d\mu}{[Q(\mu)]^{1/2}} = 0\,. \tag{8.6}$$

Since $Q(\lambda) = (\lambda + x)\ (\lambda + y)\ (\lambda + z)$ by (8.1), and similarly for $Q(\mu)$, the differential equation can be integrated at once in terms of elliptic integrals to give

$$R_F(x + \lambda, y + \lambda, z + \lambda) + R_F(x + \mu, y + \mu, z + \mu) = R_F(x, y, z)\,. \tag{8.7}$$

The integration constant on the right side is determined by recalling that $\mu \to 0$ as $\lambda \to \infty$ and that R_F is homogeneous of degree $-1/2$. Equation (8.7) is the addition theorem for R_F, wherein λ and μ are related by (8.2).

If ρ is any fixed positive number, it follows from (8.5) that

$$\frac{[Q(\lambda)]^{1/2}+[Q(-\rho)]^{1/2}}{\lambda+\rho}=\frac{[Q(\mu)]^{1/2}+[Q(-\rho)]^{1/2}}{\mu+\rho}. \tag{8.8}$$

Multiplying the first term of (8.6) by the left side of (8.8), and the second term by the right side, we have

$$\frac{d\lambda}{[Q(\lambda)]^{1/2}(\lambda+\rho)}+\frac{d\mu}{[Q(\mu)]^{1/2}(\mu+\rho)}=\frac{-1}{[Q(-\rho)]^{1/2}}\left(\frac{d\lambda}{\lambda+\rho}+\frac{d\mu}{\mu+\rho}\right). \tag{8.9}$$

Let $\delta = (\rho - x)\ (\rho - y)\ (\rho - z)$ and let $\omega = \omega(\lambda, \mu)$ be defined for fixed x, y, z, ρ by

$$\begin{aligned}\omega = Q(-\rho) &= \rho(\rho+\lambda)(\rho+\mu) - (\rho-x)(\rho-y)(\rho-z)\\ &= \rho(\rho+\lambda)(\rho+\mu) - \delta\,.\end{aligned} \tag{8.10}$$

Then the right-hand side of (8.9) can be rewritten as $-d\omega/\omega^{1/2}(\omega + \delta)$ and the differential equation can be integrated at once to give

$$\begin{aligned}&R_J(x+\lambda, y+\lambda, z+\lambda, \rho+\lambda) + R_J(x+\mu, y+\mu, z+\mu, \rho+\mu)\\ &\qquad = R_J(x, y, z, \rho) - 3R_f(\omega, \omega+\delta)\,,\end{aligned} \tag{8.11}$$

where R_f is an elementary function defined by (4.8). This is the addition theorem for R_J. Its proof appears to be substantially simpler than the proof for Legendre's integral [8, pp. 104–106].

If ρ is replaced by x in the last paragraph, (8.11) becomes an addition theorem for $R(3/2;\ 3/2,\ 1/2,\ 1/2;\ x,\ y,\ z)$. Expressing this function in terms of standard integrals by (7.8), we obtain the addition theorem for R_G:

$$\begin{aligned}&2R_G(x+\lambda, y+\lambda, z+\lambda) + 2R_G(x+\mu, y+\mu, z+\mu) - 2R_G(x, y, z)\\ &\quad = \lambda R_F(x+\lambda, y+\lambda, z+\lambda) + \mu R_F(x+\mu, y+\mu, z+\mu)\\ &\qquad + (x+y+z+\lambda+\mu)^{1/2}\,.\end{aligned} \tag{8.12}$$

The addition theorems (8.7), (8.11), and (8.12) become duplication theorems if we choose the particular solution of (8.2) given by

$$\lambda = \mu = (xy)^{1/2} + (yz)^{1/2} + (zx)^{1/2}\,. \tag{8.13}$$

The quantity ω in (8.11) is then determined according to (8.5) by

$$\omega^{1/2} = (x^{1/2} + y^{1/2} + z^{1/2})\rho + (xyz)^{1/2}\,. \tag{8.14}$$

Another important special case occurs if $z = 0$, so that one of the elliptic integrals in each addition theorem is complete and (8.2) reduces to $\lambda\mu = xy$. It is convenient then to replace x, y, ρ, λ, μ by $x - z, y - z, \rho - z, z, \nu + z$, respectively, where z is now positive but smaller than x and y. The addition theorem for R_F, for example, reduces to

$$\begin{aligned}&R_F(x, y, z) + R_F(x+\nu, y+\nu, z+\nu) = (\pi/2)R_K(x-z, y-z)\,,\\ &\qquad (\nu = xy/z - x - y)\,.\end{aligned} \tag{8.15}$$

9. Inequalities. Let x, y, z, ρ be positive and not all equal. We find from [5, Eqs. (2.4) and (2.7) and Theorem 2] that

$$(9.1)\quad \left(\frac{x+y+z+2\rho}{5}\right)^{-1/2} < R_H(x, y, z, \rho) < [R_J(x, y, z, \rho)]^{1/3} < (xyz\rho^2)^{-1/10},$$

$$(9.2)\qquad 2(x+y+2\rho)^{-1/2} < R_L(x, y, \rho) < [R_M(x, y, \rho)]^{1/3} < (xy\rho^2)^{-1/8}.$$

An improved lower bound for R_J is furnished by [5, Theorem 3], with the result that

$$(9.3)\qquad \left(\frac{x^{1/2}+y^{1/2}+z^{1/2}+2\rho^{1/2}}{5}\right)^{-3} < R_J(x, y, z, \rho) < (xyz\rho^2)^{-3/10}.$$

Upper bounds that are numerically better but algebraically more complex can be found by using convexity properties of the R-function [6], [12].

All these inequalities tend to be sharp when the ratios of x, y, z, ρ are close to unity. In the duplication theorem

$$(9.4)\qquad R_F(x, y, z) = 2R_F(x+\lambda, y+\lambda, z+\lambda)$$

where λ is given by (8.13), the ratios of the arguments are closer to unity on the right side than on the left. Applying [5, Theorem 2] to the right side, as suggested by W. H. Greiman, we find

$$(9.5)\qquad \frac{3}{x+y+z} < R_F(x^2, y^2, z^2) < 2[(x+y)(y+z)(z+x)]^{-1/3},$$

where x, y, z are positive and not all equal. The left-hand inequality is the same as that in [5, Eq. (4.7)], and both inequalities are sharper than those in [5, Eq. (4.5)].

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1. P. F. Byrd & M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Die Grundlehren der mathematischen Wissenschaften, Band 67, Springer-Verlag, Berlin, 1954. MR **15**, 702.
2. B. C. Carlson, "Lauricella's hypergeometric function F_D," *J. Math. Anal. Appl.*, v. 7, 1963, pp. 452–470. MR **28** #258.
3. B. C. Carlson, "Normal elliptic integrals of the first and second kinds," *Duke Math. J.*, v. 31, 1964, pp. 405–419. MR **29** #1366.
4. B. C. Carlson, "On computing elliptic integrals and functions," *J. Math. Phys.*, v. 44, 1965, pp. 36–51. MR **30** #5470.
5. B. C. Carlson, "Some inequalities for hypergeometric functions," *Proc. Amer. Math. Soc.*, v. 17, 1966, pp. 32–39. MR **32** #5935.
6. B. C. Carlson & M. D. Tobey, "A property of the hypergeometric mean value," *Proc. Amer. Math. Soc.*, v. 19, 1968, pp. 255–262. MR **36** #5401.
7. B. C. Carlson, "A connection between elementary functions and higher transcendental functions," *SIAM J. Appl. Math.*, v. 17, 1969, pp. 116–148.
8. A. Cayley, *Elliptic Functions*, Dover, New York, 1961.
9. A. Erdélyi, W. Magnus, F. Oberhettinger & F. G. Tricomi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953, Chapter 13. MR **15**, 419; MR **16**, 586.
10. W. J. Nellis & B. C. Carlson, "Reduction and evaluation of elliptic integrals," *Math. Comp.*, v. 20, 1966, pp. 223–231. MR **35** #6337.
11. F. G. Tricomi, *Funzioni Ellittiche*, Zanichelli, Bologna, 1937; German transl., Akademische Verlagsgesellschaft, Geest & Portig, Leipzig, 1948. MR **10**, 532.
12. D. G. Zill, *Elliptic Integrals of the Third Kind*, Ph. D. Thesis, Iowa State University, Ames, Iowa, 1967.
13. L. M. Milne-Thomson, "Elliptic integrals," M. Abramowitz & I. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Department of Commerce, Nat. Bur. Standards Appl. Math. Series, 55, U. S. Government Printing Office, Washington, D. C., 1964; 3rd printing, with corrections, 1965. MR **29** #4941; MR **31** #1400.