MATHEMATICS OF COMPUTATION, VOLUME 24, NUMBER 110, APRIL, 1970

Numerical Techniques for Finding ν-Zeros of Hankel Functions

By James Alan Cochran and Judith N. Hoffspiegel

Abstract. This paper is concerned with numerical procedures for the evaluation of the zeros, with respect to order, of Hankel functions and their derivatives in cases when the arguments of these functions are held fixed. Using Olver's asymptotic expansions, two auxiliary tables have been computed, one appropriate for real and the other for purely imaginary argument. These tables, included herein, permit the calculation of rather accurate approximations to the desired ν-zeros for wide ranges of argument and index. Moreover, from the given tabular entries, the errors attendant with any approximate ν-zero so determined can be easily estimated.

1. Introduction. In numerous scattering and diffraction problems where circular or spherical boundaries are present, the zeros of Bessel functions (or combinations thereof) are of interest. Previously we have studied, from a theoretical point-of-view [1], the roots of

$$H_\nu^{(1)}(w) = 0, \tag{1.1}$$

$$(d/dw)H_\nu^{(1)}(w) = 0, \tag{1.2}$$

$$(d/dw)H_\nu^{(1)}(w) + iWH_\nu^{(1)}(w) = 0 \qquad (W = \text{constant}), \tag{1.3}$$

where these Hankel functions and their derivatives are to be considered as functions of their order ν with fixed argument w. We now focus attention on procedures which can be used to obtain numerical values for the ν-zeros of these functions. Particular consideration is given to (1.1) and (1.2) for the situation wherein the argument w is either positive real or purely imaginary. In the latter case it is well known that the ν-zeros of $H_\nu^{(1)}(w)$ and $(d/dw)H_\nu^{(1)}(w)$ are themselves purely imaginary also [6], [7].

Our point-of-departure is the uniform asymptotic expansion of Olver [2], [3] for the Hankel function of the first kind.* After reversion, it turns out that auxiliary tabular values may be computed, which allow rather accurate approximation of ν-zeros for wide ranges of argument and index. Second differences are included in the tables for purposes of interpolation, and certain limiting cases which extend the domain of applicability are also considered. In a sense, then, the work reported herein complements that already appearing in [1].

In the absence of "exact" values with which to verify the accuracy of roots of (1.1) and (1.2) computed using our procedures, the attendant error can only be estimated. We have done this in two ways: by calculation of a first-order correction term based upon Olver's representation and by comparison with results obtained using a four-

Received March 17, 1969, revised August 18, 1969.

AMS Subject Classifications. Primary 3325, 6505, 6525, 6550; Secondary 4150, 7865.

Key Words and Phrases. Special functions, cylindrical functions, Bessel functions, Hankel functions, rootfinding, numerical approximation, function evaluation, asymptotic expansions, electromagnetic scattering, and creeping-wave propagation.

*Since $H_\nu^{(2)}(\bar{w})$ is the complex conjugate of $H_{\bar{\nu}}^{(1)}(w)$, the theory we present herein carries over directly to the Hankel function of the second kind as well.

term Debye approximation to the Hankel functions [4], [5]. In the latter case we show how any inaccuracies in the Debye expression itself might be roughly ascertained.

2. Preliminaries. It should be recalled that the roots of equations (1.1), (1.2), and (1.3) are symmetric about the origin, since $e^{(1/2)\nu\pi i}H_\nu^{(1)}(w)$ and $e^{(1/2)\nu\pi i}(d/dw)H_\nu^{(1)}(w)$ are even functions of ν. We need only to concern ourselves, therefore, with ν-zeros lying, say, in the upper half-plane. If we enumerate these with the integer index s, in order of increasing magnitude, then as $s \to \infty$,

$$\nu_s = \frac{i\pi s}{i(\frac{1}{2}\pi - \arg w) + \ln(3\pi s/e|w|)}\left[1 + O\left(\frac{\ln\ln s}{\ln s}\right)\right] \tag{2.1}$$

for the roots of both (1.1) and (1.2) [1]. For positive (negative) w these ν-zeros are all located in the first (second) quadrants of the ν-plane, and for large s are such that $\arg \nu_s \approx \frac{1}{2}\pi$. If $\arg w \doteq \frac{1}{2}\pi$, then Pólya [6], [7] has shown that $\arg \nu_s = \frac{1}{2}\pi$ exactly for all s.**

The appropriate uniform representation for the Hankel function of the first kind due to Olver takes the form ([2, p. 338], [3]):

$$\begin{aligned} H_\nu^{(1)}(\nu z) \sim 2e^{-(1/3)\pi i}\left(\frac{4\zeta}{1 - z^2}\right)^{1/4} &\left\{\frac{\mathrm{Ai}(\nu^{2/3}e^{(2/3)\pi i}\zeta)}{\nu^{1/3}}\sum_{r=0}^{\infty}\frac{A_r(\zeta)}{\nu^{2r}}\right. \\ &\left. + \frac{e^{(2/3)\pi i}\mathrm{Ai}'(\nu^{2/3}e^{(2/3)\pi i}\zeta)}{\nu^{5/3}}\sum_{r=0}^{\infty}\frac{B_r(\zeta)}{\nu^{2r}}\right\} \end{aligned} \tag{2.2}$$

as $|\nu| \to \infty$, $0 \leqq \arg \nu \leqq \pi$, $|\arg z| < \pi$.*** Here the original argument w of $H_\nu^{(1)}$ has temporarily been replaced by νz and

$$\begin{aligned} B_r(\zeta) &= \tfrac{1}{2}\zeta^{-1/2}\int_0^\zeta t^{-1/2}\{f(t)A_r(t) - A_r''(t)\}\,dt, \\ A_0(\zeta) = 1, \qquad A_{r+1}(\zeta) &= -\tfrac{1}{2}B_r'(\zeta) + \tfrac{1}{2}\int f(\zeta)B_r(\zeta)\,d\zeta \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} A_{r+1}(-\infty) &= 0 \quad \text{for } r \geqq 0, \\ f(\zeta) &= \frac{5}{16\zeta^2} - \frac{\zeta z^2(z^2 + 4)}{4(1 - z^2)^3}, \end{aligned} \tag{2.4}$$

and

$$\frac{2}{3}\zeta^{3/2} \equiv \int_z^1 \frac{(1 - t^2)^{1/2}}{t}\,dt = \ln\left[\frac{1 + (1 - z^2)^{1/2}}{z}\right] - (1 - z^2)^{1/2}. \tag{2.5}$$

Ai is the well-known Airy function of the first kind (see [3, pp. 446 ff], for instance)

**These results can also be easily established using methods such as those employed in [8].

***The range $|\arg \nu| < \frac{1}{2}\pi$ which Olver gives in [2] is unnecessarily restrictive; his results actually are valid in the larger region $-\frac{1}{2}\pi < \arg \nu < (3/2)\pi$ which includes the upper half of the ν-plane as designated herein.

and is an entire function of its argument. The coefficients $A_r(\zeta)$, $B_r(\zeta)$ given by (2.3), $f(\zeta)$ from (2.4), and $z(\zeta)$ defined by (2.5), however, are only regular analytic functions in a ζ-plane cut, say, along the rays $\arg \zeta = \pm\frac{1}{3}\pi$ from $\zeta = (3\pi/2)^{2/3}e^{\pm(1/3)\pi i}$ to infinity.

TABLE 1
$(1/Z)$, Δ, *and* Δ' *as Functions of* $\lambda^{1/3}$ *for* $(\zeta^{3/2}/Z) = \lambda$

$\lambda^{1/3}$	$1/Z$	δ^2	Δ	Δ'
0.00	1.0000	0.0636	-0.0143	$-\infty$
0.20	1.0318	0.0640	-0.0140	-3.11
0.40	1.1275	0.0651	-0.0132	-0.750
0.60	1.2884	0.0670	-0.0120	-0.315
0.80	1.5162	0.0694	-0.0106	-0.165
1.00	1.8134	0.0722	-0.00923	-0.0971
1.20	2.1829	0.0754	-0.00789	-0.0617
1.40	2.6277	0.0788	-0.00669	-0.0414
1.60	3.1514	0.0824	-0.00565	-0.0290
1.80	3.7574	0.0860	-0.00477	-0.0210
2.00	4.4495	0.0897	-0.00403	-0.0156
2.00	4.4495	0.5608	-0.00403	-0.0156
2.50	6.5803	0.6189	-0.00268	-0.00820
3.00	9.3300	0.6766	-0.00184	-0.00479
3.50	12.756	0.733	-0.00130	-0.00303
4.00	16.916	0.789	-0.000952	-0.00203
4.50	21.864	0.843	-0.000713	-0.00143
5.00	27.655	0.896	-0.000546	-0.00104
5.50	34.342	0.948	-0.000427	-0.000780
6.00	41.977	0.999	-0.000340	-0.000600

3. The ν-Zeros of $H_\nu^{(1)}(w)$ for Fixed w. The expansion (2.2) shows that the ν-zeros of $H_\nu^{(1)}(\nu z)$ are given asymptotically by the ν-solutions of

$$\mathrm{Ai}(\nu^{2/3}e^{(2/3)\pi i}\zeta) = 0;$$

in other words, as $|\nu| \to \infty$ we have

$$\nu^{2/3} e^{(2/3)\pi i} \zeta \sim a_s, \tag{3.1}$$

where the a_s, $s = 1, 2, \ldots$ are the (negative real) zeros of the Airy function Ai. If we restore the original argument $w = \nu z$ of the Hankel function, it follows from (3.1) that the ν-zeros of $H_\nu^{(1)}(w)$ for fixed w should satisfy

$$w/z(\zeta) = \nu_s \sim e^{-\pi i}(a_s/\zeta)^{3/2}. \tag{3.2}$$

It is by means of this expression that ζ and z, which are originally related by (2.5), are now given as implicit functions of w and s.

The relation (3.2) can be rewritten as

$$\zeta^{3/2}(z)/z \sim i(-a_s)^{3/2}/w, \tag{3.3}$$

which for real or purely imaginary w has a right-hand side which is either purely imaginary or real. This suggests that for such arguments w, single-entry tables could be constructed, using (2.5), which would allow easy calculation of approximate ν-zeros. In fact, for given λ, the equations

$$\frac{\zeta^{3/2}(z)}{z} = \begin{cases} i\lambda \\ \lambda \end{cases} \tag{3.4}$$

may be numerically inverted to yield z (or $1/z$ for convenience). This we have done and the results are tabulated in Tables 1 and 2, to 5 significant figures, where, for simplicity in presentation of the data, $\lambda^{1/3}$ has been chosen as the independent parameter. Second central differences are given for use in interpolating intermediate values, say by Everett's formula [3, p. 880]:

$$f(x_0 + ph) = qf_0 - \frac{q(1 - q^2)}{3!}\delta_0^2 + pf_1 - \frac{p(1 - p^2)}{3!}\delta_1^2 + \cdots \tag{3.5}$$

with

$$0 \leqq p \leqq 1, \qquad q = 1 - p.$$

The tabular step h (0.20 for $0 \leqq \lambda^{1/3} \leqq 2.00$ and 0.50 for $2.00 \leqq \lambda^{1/3} \leqq 6.00$) is a compromise choice which permits interpolation accurate to within two units in the last decimal place, while at the same time keeping the size of the tables within modest bounds. The tables have been terminated at the point where $(1/z)$ can be determined from the somewhat simpler approximate relations

$$\frac{3}{2}\left(\frac{1}{z}\right)\ln\left[\frac{2}{ez}\right] = \begin{cases} i\lambda \\ \lambda \end{cases} \tag{3.6}$$

(cf. Eqs. (5.2), (5.4) of [1]) with roughly five significant figure accuracy.† Figure 1 graphically displays the real and imaginary parts of $(1/z)$ from Table 2. (Compare Figure 1 of [9].)

---

†Explicit approximate relations for $(1/z)$ as a function of λ much more elementary than Eq. (3.6) generally are considerably less accurate. For instance, the expression $z \approx 3/2\lambda \ln[4\lambda/(3e \ln(2\lambda/e))]$ (cf. Eqs. (5.3), (5.5) of [1]) still is in error by about 2% when λ is 10^6.

TABLE 2

$(1/Z)$, Δ, *and* Δ' *as Functions of* $\lambda^{1/3}$ *for* $(\zeta^{3/2}/Z) = i\lambda$

$\lambda^{1/3}$	$1/Z$		δ^2		Δ		Δ'	
	R	I	R	I	R	I	R	I
0.00	1.0000	0.0000	0.0317	0.0550	-0.0143	0.000000	$-\infty$	∞
0.20	1.0159	0.0275	0.0315	0.0554	-0.0141	0.000249	-1.54	2.73
0.40	1.0632	0.1104	0.0310	0.0564	-0.0137	0.000941	-0.355	0.680
0.60	1.1416	0.2498	0.0301	0.0582	-0.0128	0.00193	-0.138	0.299
0.80	1.2500	0.4473	0.0292	0.0605	-0.0116	0.00298	-0.0641	0.164
1.00	1.3877	0.7054	0.0282	0.0635	-0.0100	0.00386	-0.0319	0.100
1.20	1.5535	1.0270	0.0273	0.0669	-0.00836	0.00442	-0.0163	0.0655
1.40	1.7466	1.4154	0.0266	0.0706	-0.00671	0.00461	-0.00841	0.0445
1.60	1.9663	1.8744	0.0261	0.0745	-0.00525	0.00451	-0.00435	0.0312
1.80	2.2122	2.4078	0.0258	0.0785	-0.00404	0.00421	-0.00226	0.0225
2.00	2.4839	3.0198	0.0257	0.0825	-0.00309	0.00380	-0.00117	0.0166
2.00	2.4839	3.0198	0.1610	0.5157	-0.00309	0.00380	-0.00117	0.0166
2.50	3.2755	4.9195	0.1619	0.5780	-0.00158	0.00276	-0.000218	0.00854
3.00	4.2291	7.3972	0.1657	0.6384	-0.000848	0.00194	-0.000027	0.00491
3.50	5.3484	10.513	0.1706	0.696	-0.000484	0.00138	-0.000010	0.00307
4.00	6.6383	14.326	0.1762	0.753	-0.000293	0.00101	0.000014	0.00205
4.50	8.1044	18.891	0.1819	0.807	-0.000187	0.000751	0.000012	0.00143
5.00	9.7523	24.264	0.1876	0.861	-0.000125	0.000573	0.000009	0.00104
5.50	11.588	30.498	0.193	0.913	-0.000087	0.000446	0.000007	0.000780
6.00	13.617	37.644	0.199	0.964	-0.000062	0.000353	0.000005	0.000600

TABLE 3

$(-a_S)^{1/2}$ and $(-a'_S)^{1/2}$ as Functions of the Index S

S	$(-a_S)^{1/2}$	$(-a'_S)^{1/2}$
1	1.52909	1.00935
2	2.02187	1.80228
3	2.34959	2.19547
4	2.60513	2.48260
5	2.81853	2.71518
6	3.00377	2.91350
7	3.16862	3.08795
8	3.31791	3.24464
9	3.45485	3.38749
10	3.58173	3.51920
11	3.70020	3.64173
12	3.81154	3.75653
13	3.91673	3.86470
14	4.01655	3.96714
15	4.11165	4.06454
16	4.20253	4.15749
17	4.28965	4.24645
18	4.37337	4.33184
19	4.45400	4.41398
20	4.53181	4.49318
21	4.60704	4.56967
22	4.67989	4.64369
23	4.75054	4.71542
24	4.81915	4.78504
25	4.88585	4.85268
26	4.95079	4.91849
27	5.01406	4.98258
28	5.07578	5.04507
29	5.13603	5.10605
30	5.19490	5.16560
31	5.25246	5.22381
32	5.30879	5.28075
33	5.36395	5.33649
34	5.41800	5.39109
35	5 47099	5.44460
36	5.52298	5.49708
37	5.57400	5.54858
38	5.62410	5.59914
39	5.67333	5.64880
40	5.72172	5.69761
41	5.76930	5.74559
42	5.81612	5.79279
43	5.86219	5.83922
44	5.90754	5.88493
45	5.95221	5.92995
46	5.99622	5.97428
47	6.03960	6.01797
48	6.08236	6.06104
49	6.12452	6.10350
50	6.16612	6.14538

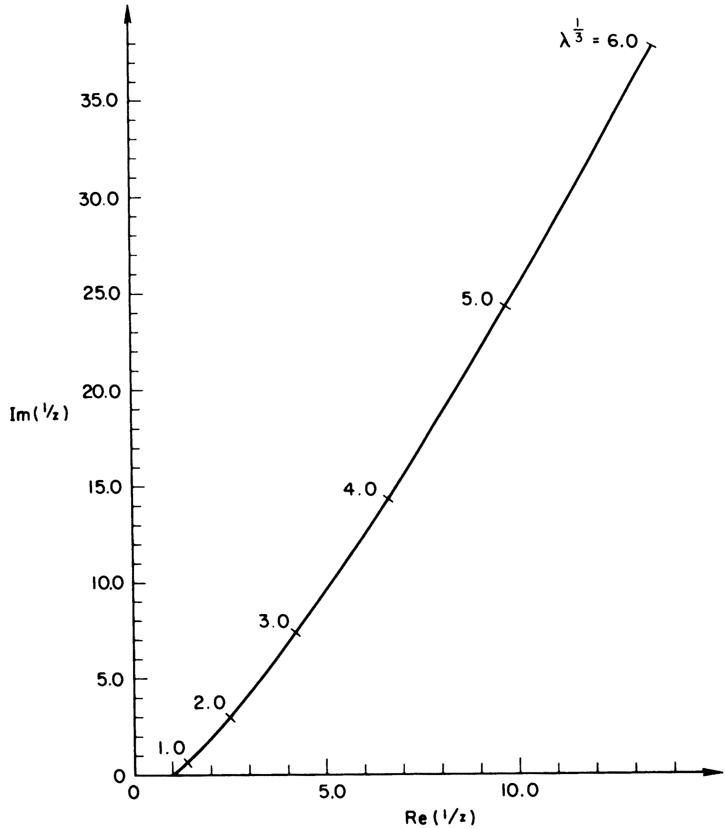

FIGURE 1. *Real and Imaginary Parts of* $(1/z)$ *from Table* 2 *for Case of* $(\zeta^{3/2}/z) = i\lambda$; *$\nu$-Zeros Given by* w/z.

4. The ν-Zeros of $(d/dw)H_\nu^{(1)}(w)$. Before proceeding to some typical calculations it is worth noting that analysis compatible with the above can be carried out for the derivatives of the Hankel function with respect to argument. The asymptotic expansion (2.2) can be differentiated term by term, with the result that the ν-zeros of $(d/dw)H^{(1)}(w)$ appear asymptotically as the ν-solutions of

$$\mathrm{Ai}'(\nu^{2/3}e^{(2/3)\pi i}\zeta) = 0.$$

The analogue of (3.2) is then

$$w/z(\zeta) = \nu_s' \sim e^{-\pi i}(a_s'/\zeta)^{3/2} \tag{4.1}$$

(as $|\nu'| \to \infty$). With the exception that the (negative real) turning-points a_s' of the Airy function Ai have here replaced the zeros a_s, the relation (4.1) is identical with that obtained earlier. Making the appropriate correlation, therefore, Tables 1 and 2 should also permit easy calculation of approximate ν'-zeros for either real or purely imaginary argument w.††

††For the smaller ν'-zeros particularly, however, the values thus obtained may be a good deal less accurate than the corresponding zeros of $H_\nu^{(1)}(w)$. This characteristic phenomenon was observed earlier by Olver [2, p. 345], with regards to ordinary Bessel function zeros. Also see Section 6.

5. Sample Calculations. Let us assume that we are interested in, say, the 5th zero of $(d/dw)H_\nu^{(1)}(w)$ for various values of w. To carry out the calculation of our approximation we need knowledge of $(-a_5')^{1/2}$. This we find from the supplementary Table 3 of values of $(-a_s)^{1/2}$ and $(-a_s')^{1/2}$, $s = 1(1)50$, to be

$$(-a_5')^{1/2} \doteq 2.7152.$$

With $w = i$, for instance, we thus obtain $\lambda^{1/3} \doteq 2.7152$, and hence interpolating from Table 1,

$$\begin{aligned}(1/z) &\doteq (0.5696)(6.5803) - (0.0641)(0.6189) + (0.4304)(9.3300) - (0.0584)(0.6766) \\ &\doteq 7.6846.\end{aligned}$$

Since $\nu = w/z$, we finally have

$$\nu_5' \doteq 7.6846i$$

which compares very favorably with the more accurate value of $7.6908i$.

For the 10th zero of $H_\nu^{(1)}(5)$ the calculation would proceed as follows:

$$\begin{aligned}(-a_{10})^{1/2} &\doteq 3.5817 \quad \text{(from Table 3)} \\ \lambda^{1/3} &\doteq 2.0946 \quad \text{(using Eqs. (3.3) and (3.4))} \\ \frac{1}{z} &\doteq 2.6213 + 3.3378i \quad \text{(interpolating using Table 2 and Eq. (3.5))} \\ \nu_{10} = w/z &\doteq 13.107 + 16.689i.\end{aligned}$$

Employing a four-term Debye approximation to the Hankel function [4], [5] we would find $\nu \approx 13.106 + 16.690i$.

6. Improved Approximations. Within the limits imposed by the accuracy of (2.2) itself, progressively better approximations to the desired ν-zeros can be obtained by incorporating higher-order terms of Olver's asymptotic expansion into our analysis. For instance, if we replace (3.1) by the more accurate relation

$$\mathrm{Ai}(\nu^{2/3}e^{(2/3)\pi i}\zeta) + \mathrm{Ai}'(\nu^{2/3}e^{(2/3)\pi i}\zeta)\frac{e^{(2/3)\pi i}}{\nu^{4/3}}B_0(\zeta) = 0,$$

ν undergoes a perturbation (to first order) in the amount

$$\Delta\nu = \Delta/w \tag{6.1}$$

where

$$\Delta \equiv -\zeta^{1/2}B_0(\zeta)z\Big/\ln\left[\frac{1 + (1 - z^2)^{1/2}}{z}\right]. \tag{6.2}$$

In these equations

$$B_0(\zeta) = -\frac{1}{24\zeta^{1/2}}\left[\frac{3}{(1 - z^2)^{1/2}} - \frac{5}{(1 - z^2)^{3/2}}\right] - \frac{5}{48\zeta^2},$$

as determined from (2.3), and z, ζ are related as in (2.5), (3.3). As an illustration, then, we can calculate the correction, given by (6.1), for the second example of the preceding section. We find

$$\Delta \doteq -0.003 + 0.004i,$$

and hence

$$\Delta\nu \doteq -0.001 + 0.001i.$$

In completely analogous fashion, the ν-zeros of the derivative of the Hankel function can be improved. For this case we obtain (again to first order)

$$\Delta\nu' = \Delta'/w$$

with

$$\Delta' \equiv -z\left[\zeta^{1/2}B_0(\zeta) + \frac{\zeta^{-3/2}}{4} - \frac{z^2}{2(1-z^2)^{3/2}}\right]\Big/\ln\left[\frac{1+(1-z^2)^{1/2}}{z}\right] \tag{6.3}$$

$$\approx -\frac{z(17+13z^2)}{150(1-z^2)} \quad \text{as } z \to 1. \tag{6.4}$$

For the first example in Section 5, we calculate

$$\Delta\nu' = -i\Delta' \doteq 0.0067i,$$

and thus

$$\nu_5' + \Delta\nu' \doteq 7.6913i.$$

The expressions (6.2) and (6.3) are functions of z and ζ; in view of (2.5), however, they can be tabulated in terms of z alone. For completeness we have included such values in Tables 1 and 2. The approximate relation (6.4) will suffice to determine Δ' in the neighborhood of the singularity at $\lambda \doteq 0$. The following expressions, moreover, which are in keeping with (3.6), will yield at least three significant figures for large λ:

$$\Delta = \frac{-z}{72\ln(2/z)}\left[\frac{11 - 6\ln(2/z)}{1-\ln(2/z)}\right],$$

$$\Delta' = \frac{z}{72\ln(2/z)}\left[\frac{1 + 6\ln(2/z)}{1-\ln(2/z)}\right].$$

7. The Overall Error. In a considerable number of numerical examples the ν-zeros obtained by the methods outlined in this paper have been compared with values computed using a Newton-Raphson procedure applied to a four-term Debye approximation to the Hankel function [4], [5]. Although we assume that these latter values are usually more accurate than even our improved approximations, they themselves do contain some inaccuracy, however. In the absence of a thorough error analysis, any such inaccuracy might be roughly ascertained as follows: Let

$$H_\nu^{(1)}(w) = Z_\nu(w) + \varepsilon$$

where Z_ν is the value of the Hankel function computed from the Debye approximation and ε the associated error. Under the assumption that ε is only a slowly varying function of ν, the identity

$$H_{\nu+1}^{(1)}(w) + H_{\nu-1}^{(1)}(w) = \frac{2\nu}{w}H_\nu^{(1)}(w)$$

permits the approximate calculation of ε as

$$\varepsilon \approx \frac{w}{2(v - w)}\left[Z_{v+1}(w) + Z_{v-1}(w) - \frac{2v}{w} Z_v(w)\right]. \tag{7.1}$$

It then follows, using a two-term Taylor series, that

$$v - v_0 \approx [Z_v(w) + \varepsilon]\bigg/\frac{\partial Z_v(w)}{\partial v}, \tag{7.2}$$

where v_0 is the desired zero of the Hankel function and v is the value computed using the Debye approximation.

For the v-zeros of the Hankel function derivatives a similar analysis leads to

$$\begin{aligned} v' - v'_0 &\approx \left[\frac{v}{w} Z_v(w) - Z_{v+1}(w) + \frac{v - w}{w}\varepsilon\right]\bigg/\frac{\partial}{\partial v}\left[\frac{v}{w} Z_v(w) - Z_{v+1}(w)\right] \\ &\approx \tfrac{1}{2}[Z_{v-1}(w) - Z_{v+1}(w)]\bigg/\left[\frac{1}{w} Z_v(w) + \frac{v}{w}\frac{\partial Z_v(w)}{\partial v} - \frac{\partial Z_{v+1}(w)}{\partial v}\right]. \end{aligned} \tag{7.3}$$

In the cases investigated it appeared that five digit accuracy generally prevailed for all but the smallest zero or two.

8. Acknowledgments. The authors are indebted to Frank W. J. Olver of the National Bureau of Standards who not only originally suggested these computations, but offered constructive criticisms as the work progressed.

Bell Telephone Laboratories
Whippany, New Jersey 07981

1. J. A. Cochran, "The zeros of Hankel functions as functions of their order," *Numer. Math.*, v. 7, 1965, pp. 238–250. MR **31** #2428.

F. W. J. Olver, "The asymptotic expansion of Bessel functions of large order," *Philos. Trans. Roy. Soc. London Ser. A*, v. 247, 1954, pp. 328–368. MR **16**, 696.

3. M. Abramowitz & I. A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Nat. Bur. Standards Appl. Math. Series, 55, Superintendent of Documents, U.S. Government Printing Office, Washington, D. C., 1964, pp. 368–369; 3rd printing, with corrections, 1965. MR **29** #4914; MR **31** #1400.

4. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, New York, 1958, pp. 262–268.

5. L. A. Berry, *Computation of Hankel Functions*, Nat. Bur. Standards Tech. Note #216, U.S. Nat. Bur. Standards, Washington, D.C., June 12, 1964.

6. G. Pólya, "On the zeros of certain trigonometric integrals," *J. London Math. Soc.*, v. 1, 1926, pp. 98–99.

7. ———, "Über trigonometrische Integrale mit nur reellen Nullstellen," *J. Reine Angew. Math.*, v. 158, 1927, pp. 6–18.

J. A. Cochran, "Remarks on the zeros of cross-product Bessel functions," *J. Soc. Indust. Appl. Math.*, v. 12, 1964, pp. 580–587. MR **31** #2427.

9. W. Streifer, "Creeping wave propagation constants for impedance boundary conditions," *IEEE Trans. Antennas and Propagation*, v. 12, 1964, pp. 764–766.