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# On Generating Infinitely Many Solutions of the Diophantine Equation $A^6 + B^6 + C^6 = D^6 + E^6 + F^6$

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**Abstract.** A method of generating infinitely many solutions of the Diophantine equation $A^6 + B^6 + C^6 = D^6 + E^6 + F^6$ is presented. The technique is to reduce the equation to one of fourth degree and to use the known recursive solutions to the fourth-order equations.

This work presents the first known method for generating new solutions of the

$$A^6 + B^6 + C^6 = D^6 + E^6 + F^6 \tag{1}$$

in a recursive way from a given solution.

The technique is to use a special substitution which reduces the above sixth degree equation to one of fourth degree. Since it is known how to generate an infinite number of solutions of the fourth degree equation obtained, one is able to generate an infinite number of solutions to the equation of sixth degree.

It is easily verified that an equation in $\cos^2\beta$ of the following form:

$$\begin{aligned}(\sin\alpha\sin\beta)^6 + (\cos\alpha - \cos\beta\sin\alpha)^6 + (\cos\alpha + \cos\beta\sin\alpha)^6 \\ = 1^6 + (\cos\alpha)^6 + (\cos\beta\sin\alpha)^6\end{aligned} \tag{2}$$

has only one positive rational solution:

$$\cos^2\beta = (1 + 9\cos^2\alpha)^{-1}. \tag{3}$$

Therefore, if we let:

$$\begin{aligned}&A = \sin\alpha\sin\beta, \quad B = \cos\alpha - \cos\beta\sin\alpha, \quad C = \cos\alpha + \cos\beta\sin\alpha, \\ &D = 1, \quad E = \cos\alpha, \quad \text{and} \quad F = \cos\beta\sin\alpha,\end{aligned} \tag{4}$$

then Eq. (1) will be satisfied with rational numbers provided that all six of the expressions of Eq. (4) are rational and Eq. (3) is satisfied.

Define:

$$\begin{aligned}\cos\alpha &= 2xy/(x^2 + y^2), \quad &\sin\alpha &= (x^2 - y^2)/(x^2 + y^2), \\ \cos\beta &= (x^2 + y^2)/z, \quad &\sin\beta &= 6xy/z,\end{aligned} \tag{5}$$

where $x$, $y$, and $z$ satisfy the relation

$$x^4 + 38x^2y^2 + y^4 = z^2. \tag{6}$$

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The method for generating solutions to Eq. (6) is given by Lebesgue [1]. (See also Dickson [2].) From a known solution one can generate an infinite number of new solutions by using the relation:

$$x_2 = x_1^4 - y_1^4, \qquad y_2 = 2x_1y_1z_1. \tag{7}$$

Beginning with $x_1 = 1$, $y_1 = 2$, which corresponds to the numerical example given by L. J. Lander, T. R. Parkin and J. L. Selfridge [3],

$$36^6 \qquad 7^6 + 67^6 = 52^6 + 15^6 + 65^6,$$

we get

$$x_2 = -15, \qquad y_2 = 52 \tag{8}$$

which corresponds to:

$$11601720^6 + 15873751^6 + 1351769^6 = 8612760^6 + 7260991^6 + 16171009^6.$$

By substituting the parameters of (8) into (7), one can continue to generate solutions to (1).

Another way of obtaining new solutions to (2) in terms of old ones has been given by A. Desboves [4].

If $x_1$, $y_1$ is a solution to (2)

$$x_2 = x_1(4y_1^4z_1^2 - g^2), \qquad y_2 = y_1(4x_1^4z_1^2 - g^2), \tag{9}$$

where $g = x_1^4 - y_1^4$.

From $x_1 = 1$, $y_1 = 2$ one obtains the following numerical example:

$$143138845171047056^6 + 242057392197821 2^6 + 125812856485881456^6$$

Again one can use (9) to generate new solutions to (1).

Both examples have been checked on a pdp$^6$ computer.

Obviously, this technique finds solutions with very large integers. However, it is interesting that in principle an infinity of solutions does exist.

One should note that the original $x$, $y$, need not be rational. For example: let

$$x_1 = 17 + 7(5)^{1/2} \quad \text{and} \quad y_1 = 17 - 7(5)^{1/2},$$

from which we obtain

$$10947^6 + 10591^6 + 902^6 = 2618^6 + 11493^6 + 9689^6.$$

Solutions found by this method also automatically satisfy the following equation

$$A^2 + B^2 + C^2 = D^2 + E^2 + F^2.$$

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