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Error Bounds for Polynomial Spline Interpolation*

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Abstract. New upper and lower bounds for the L^2 and L^∞ norms of derivatives of the error in polynomial spline interpolation are derived. These results improve corresponding results of Ahlberg, Nilson, and Walsh, cf. [1], and Schultz and Varga, cf. [5].

1. Introduction. In this paper, we derive new bounds for the L^2 and L^∞ norms of derivatives of the error in polynomial spline interpolation. These bounds improve and generalize the known error bounds, cf. [1] and [5], in the following important ways: (1) these bounds can be *explicitly calculated* and are not merely asymptotic error bounds such as those given in [1] and [5]; (2) explicit *lower* bounds are given for the error for a class of functions; (3) the degree of regularity required of the function, f, being interpolated is extended, i.e., in [1] and [5] we demand that the mth or $2m$th derivative of f be in L^2, if we are interpolating by splines of degree $2m - 1$, while here we demand only that some pth derivative of f, where $m \leqq p \leqq 2m$, be in L^2; and (4) bounds are given for high-order derivatives of the interpolation errors.

2. Notations. Let $-\infty < a < b < \infty$ and for each positive integer, m, let $K^m[a, b]$ denote the collection of all real-valued functions $u(x)$ defined on $[a, b]$ such that $u \in C^{m-1}[a, b]$ and such that $D^{m-1}u$ is absolutely continuous, with $D^m u \in L^2[a, b]$, where $Du \equiv du/dx$ denotes the derivative of u. For each nonnegative integer, M, let $\mathcal{P}_M(a, b)$ denote the set of all partitions, Δ, of $[a, b]$ of the form

$$\Delta: a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b. \tag{2.1}$$

Moreover, let $\mathcal{P}(a, b) \equiv \bigcup_{M=0}^{\infty} \mathcal{P}_M(a, b)$.

If $\Delta \in \mathcal{P}_M(a, b)$, m is a positive integer and z is an integer such that $m - 1 \leqq z \leqq 2m - 2$, we define the *spline space*, $S(2m - 1, \Delta, z)$, to be the set of all real-valued functions $s(x) \in C^z[a, b]$ such that on each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq M$, $s(x)$ is a polynomial of degree $2m - 1$. We remark that our definition is identical with the definition of deficient splines of [1]. For generalizations of this concept of spline subspace, the reader is referred to [5]. In particular, it is easy to verify that all the results of this paper remain essentially unchanged if one allows the number z to depend on the partition points, x_i, $1 \leqq i \leqq M$, in such a way that $m - 1 \leqq z(x_i) \leqq 2m - 2$ for all $1 \leqq i \leqq M$. The details are left to the reader.

Following [1] we define the interpolation mapping $\mathcal{I}_m: C^{m-1}[a, b] \to S(2m - 1, \Delta, z)$ by $\mathcal{I}_m(f) \equiv s$, where

$$D^k s(x_i) \equiv D^k f(x_i), \quad \begin{array}{ll} 0 \leqq k \leqq 2m - 2 - z, & 1 \leqq i \leqq M, \\ 0 \leqq k \leqq m - 1, & i = 0 \text{ and } M + 1. \end{array} \tag{2.2}$$

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We remark that the preceding interpolation mapping corresponds to the Type I interpolation of [1]. It is easy to modify the results of this paper for the cases in which the interpolation mapping corresponds to Types II, III, and IV interpolation of [1]. The details are left to the reader.

3. Basic L^2-Error Bounds. In this section, we obtain *explicit upper* and *lower* bounds for the quantities $\Lambda(m, p, z, j)$, $1 \leqq m$, $m \leqq p \leqq 2m$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq m$, defined by

$$\Lambda(m, p, z, j) \equiv \operatorname{Sup} \{ \| D^j(f - \mathcal{I}_m f) \|_{L^2[a,b]} / \| D^p f \|_{L^2[a,b]} \mid f \in K^p[a, b], \| D^p f \|_{L^2[a,b]} \neq 0 \}. \tag{3.1}$$

First, we recall some basic results from [1] and [5] and introduce some additional notation.

THEOREM 3.1. *The interpolation mapping given by* (2.2) *is well defined for all* $\Delta \in \mathcal{P}(a, b)$, $1 \leqq m$, *and* $m - 1 \leqq z \leqq 2m - 2$.

THEOREM 3.2 (FIRST INTEGRAL RELATION). *If* $f \in K^m[a, b]$, $1 \leqq m$, $\Delta \in \mathcal{P}(a, b)$, *and* $m - 1 \leqq z \leqq 2m - 2$,

$$\| D^m f \|^2_{L^2[a,b]} = \| D^m(f - \mathcal{I}_m f) \|^2_{L^2[a,b]} + \| D^m \mathcal{I}_m f \|^2_{L^2[a,b]}. \tag{3.2}$$

THEOREM 3.3 (SECOND INTEGRAL RELATION). *If* $f \in K^{2m}[a, b]$, $1 \leqq m$, $\Delta \in \mathcal{P}(a, b)$, *and* $m - 1 \leqq z \leqq 2m - 2$,

$$\| D^m(f - \mathcal{I}_m f) \|^2_{L^2[a,b]} = \int_a^b (f - \mathcal{I}_m f) D^{2m} f \, dx. \tag{3.3}$$

Finally, following Kolmogorov, cf. [4, p. 146], if t and d are positive integers, let $\lambda_d(t)$ denote the dth eigenvalue of the boundary value problem,

$$(-1)^t D^{2t} y(x) = \lambda y(x), \qquad a < x < b, \tag{3.4}$$

$$D^k y(a) = D^k y(b) = 0, \qquad t \leqq k \leqq 2t - 1, \tag{3.5}$$

where the λ_d are arranged in order of increasing magnitude and repeated according to their multiplicity. We remark that the problem (3.4)–(3.5) has a countably infinite number of eigenvalues, all of which are nonnegative and it may be shown that

$$\lambda_d = (\pi/(b - a))^{2t} \, d^{2t}[1 + O(d^{-1})], \quad \text{as } t < d \to \infty.$$

Using the bootstrapping technique of [1, p. 92], and letting

$$\bar{\Delta} \equiv \max_{0 \leqq i \leqq M} (x_{i+1} - x_i) \quad \text{and} \quad \underline{\Delta} \equiv \min_{0 \leqq i \leqq M} (x_{i+1} - x_i),$$

for all $\Delta \in \mathcal{P}_M(a, b)$, we have the following generalization of Theorem 7 of [5].

THEOREM 3.4.

$$\lambda_d^{-1/2}(m - j) \leqq \Lambda(m, m, z, j) \leqq K_{m,m,z,j}(\bar{\Delta})^{m-j}, \tag{3.6}$$

where

$$d \equiv (M + 1)(2m - z + 1) + z - j + 2 \tag{3.7}$$

and

$$
\begin{aligned}
K_{m,m,z,j} &= 1, && \text{if } m-1 \leqq z \leqq 2m-2,\ j=m,\\
&= (1/\pi)^{m-j}, && \text{if } m-1 = z,\ 0 \leqq j \leqq m-1,\\
&= \frac{(z+2-m)!}{\pi^{m-j}}, && \text{if } m-1 \leqq z \leqq 2m-2,\ 0 \leqq j \leqq 2m-2-z,\\
&= \frac{(z+2-m)!}{j!\,\pi^{m-j}}, && \text{if } m-1 \leqq z \leqq 2m-2,\ 2m-2-z \leqq j \leqq m-1,
\end{aligned} \tag{3.8}
$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a, b)$, $m - 1 \leqq z \leqq 2m - 2$, *and* $0 \leqq j \leqq m$.

Proof. First, we prove the right-hand inequality of (3.6). If $m - 1 \leqq z \leqq 2m - 2$ and $j = m$, the result follows directly from Theorem 3.2.

Otherwise, $D^j(f - \mathcal{S}_m f)(x_i) = 0$, $1 \leqq i \leqq M$, $0 \leqq j \leqq 2m - 2 - z$, and by the Rayleigh-Ritz inequality, cf. [3, p. 184],

$$\int_{x_i}^{x_{i+1}} (D^j(f - \mathcal{S}_m f)(x))^2\,dx \leqq \left(\frac{\bar{\Delta}}{\pi}\right)^2 \int_{x_i}^{x_{i+1}} (D^{j+1}(f - \mathcal{S}_m f)(x))^2\,dx, \tag{3.9}$$

$0 \leqq j \leqq 2m - 2 - z$. Summing both sides of (3.9) with respect to i from 0 to M, we obtain

$$\|D^j(f - \mathcal{S}_m f)\|_{L^2[a,b]} \leqq \frac{\bar{\Delta}}{\pi} \|D^{j+1}(f - \mathcal{S}_m f)\|_{L^2[a,b]}, \tag{3.10}$$

$0 \leqq j \leqq 2m - 2 - z$. Using (3.10) repeatedly we obtain

$$\|D^j(f - \mathcal{S}_m f)\|_{L^2[a,b]} \leqq \left(\frac{\bar{\Delta}}{\pi}\right)^{2m-1-z-j} \|D^{2m-1-z}(f - \mathcal{S}_m f)\|_{L^2[a,b]}. \tag{3.11}$$

Hence, if $2m - 1 - z = m$, i.e., $z = m - 1$, then

$$\|D^j(f - \mathcal{S}_m f)\|_{L^2[a,b]} \leqq \left(\frac{1}{\pi}\right)^{m-j} (\bar{\Delta})^{m-j} \|D^m f\|_{L^2[a,b]}, \tag{3.12}$$

which is the required result for this special case.

Otherwise, since $m \leqq z$, applying Rolle's Theorem to $D^{2m-2-z}(f - \mathcal{S}_m f) \in C^{z-m+1}[a, b]$, which vanishes at every mesh point, we have that for each $0 \leqq j \leqq z - m + 1$, there exist points $\{\xi_l^{(j)}\}_{l=0}^{M+1-j}$ in $[a, b]$ such that

$$
\begin{aligned}
D^{2m-2-z+j}(f - \mathcal{S}_m f)(\xi_l^{(j)}) = 0, \qquad & 0 \leqq j \leqq m - 1 - (2m - 2 - z),\\
= z - m + 1, \qquad & 0 \leqq l \leqq M + 1 - j,
\end{aligned} \tag{3.13}
$$

$$a = \xi_0^{(j)} < \xi_1^{(j)} < \cdots < \xi_{M+1-j}^{(j)} = b, \qquad 0 \leqq j \leqq z - m + 1, \tag{3.14}$$

$$\xi_l^{(j)} \leqq \xi_l^{(j+1)} < \xi_{l+1}^{(j)}, \quad \text{for all } 0 \leqq l \leqq M + 1 - j,\ 0 \leqq j \leqq z - m + 1 \tag{3.15}$$

and

$$|\xi_{l+1}^{(j)} - \xi_l^{(j)}| \leqq (j + 1)\bar{\Delta}, \qquad 0 \leqq l \leqq M - j,\ 0 \leqq j \leqq z - m + 1, \tag{3.16}$$

i.e., choose $\xi_l^{(0)} = x_l$, $0 \leqq l \leqq M + 1$.

Thus, applying the Rayleigh-Ritz inequality, we have

$$\int_{\xi_l(j)}^{\xi_{l+1}(j)} (D^{2m-2-z+j}(f - \mathcal{S}_m f)(x))^2\,dx \leq \left[\frac{(j+1)\bar{\Delta}}{\pi}\right]^2 \int_{\xi_l(j)}^{\xi_{(l+1)}(j)} (D^{2m-2-z+(j+1)}(f - \mathcal{S}_m f))^2\,dx \tag{3.17}$$

for all $0 \leqq l \leqq M - j$, $0 \leqq j \leqq z - m + 1$. Summing (3.17) with respect to l from 0 to $M - j$, we have

$$||D^{2m-2-z+j}(f - \mathcal{S}_m f)||_{L^2[a,b]} \leqq \frac{(j+1)\bar{\Delta}}{\pi} ||D^{2m-2-z+(j+1)}(f - \mathcal{S}_m f)||_{L^2[a,b]}, \tag{3.18}$$

$0 \leqq j \leqq z - m + 1$. Using (3.18) repeatedly along with (3.2) we have

$$\begin{aligned} ||D^{2m-1-z}(f - \mathcal{S}_m f)||_{L^2[a,b]} &\leqq \frac{(z+2-m)!}{\pi^{z-m+1}} (\bar{\Delta})^{z-m+1} ||D^m(f - \mathcal{S}_m f)||_{L^2[a,b]} \\ &\leqq \frac{(z+2-m)!}{\pi^{z-m+1}} (\bar{\Delta})^{z-m+1} ||D^m f||_{L^2[a,b]}. \end{aligned} \tag{3.19}$$

Combining (3.11) with (3.19), we have that

$$||D^j(f - \mathcal{S}_m f)||_{L^2[a,b]} \leqq \frac{(z+2-m)!}{\pi^{m-j}} (\bar{\Delta})^{m-j} ||D^m f||_{L^2[a,b]}, \tag{3.20}$$

if $0 \leqq j \leqq 2m - 2 - z$. Otherwise, it follows from (3.18) that

$$||D^j(f - \mathcal{S}_m f)||_{L^2[a,b]} \leqq \frac{(z+2-m)!}{j!\,\pi^{m-j}} ||D^m f||_{L^2[a,b]}. \tag{3.21}$$

Finally, we prove the left-hand inequality of (3.6). This inequality follows directly from a fundamental result of Kolmogorov, cf. [4, p. 146], which states that

$$\lambda_{t+1}^{-1/2}(m - j) \leqq \Lambda(m, m, z, j), \tag{3.22}$$

where $t \equiv$ dimension $D^j(S(2m - 1, \Delta, z))$, for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a, b)$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq m$. But the space $D^j(S(2m - 1, \Delta, z))$ has dimension $t \equiv (2m-j)(M+1) - (z+1-j)M = (M+1)(2m-z+1) + z-j+1$. Q.E.D.

We remark that in this case it is easy to verify that there exists a positive constant, K, such that

$$\begin{aligned} \lambda_d^{-1/2} &\geqq \left(\frac{b-a}{\pi}\right)^{m-j} \frac{1}{(M+1)^{m-j}} \frac{1}{s^{m-j}} \frac{1}{1 + Ks^{-1}(M+1)^{-1}} \\ &\geqq \frac{1}{\pi^{m-j}} \frac{1}{s^{m-j}} \frac{1}{1 + Ks^{-1}(M+1)^{-1}} (\underline{\Delta})^{m-j}, \end{aligned}$$

where $s \equiv (2m - z + 1 + (z - j + 2)/(M + 1))$, and thus that splines are "quasi-optimal".

The next result generalizes Theorem 9 of [5].

THEOREM 3.5.

$$\lambda_d^{-1/2}(2m - j) \leqq \Lambda(m, 2m, z, j) \leqq K_{m,2m,z,j}(\bar{\Delta})^{2m-j} \tag{3.23}$$

where

$$d \equiv (M+1)(2m-z+1)+z-j+2 \tag{3.24}$$

and

$$K_{m,2m,z,j} \equiv (K_{m,m,z,j})(K_{m,m,z,0}), \quad \textit{for all } 1 \leqq m,\ 0 \leqq M,\ \Delta \in \mathcal{P}_M(a,b),\ m-1 \leqq z \leqq 2m-2, \textit{ and } 0 \leqq j \leqq m. \tag{3.25}$$

Proof. Applying the Cauchy-Schwarz inequality to the Second Integral Relation yields the inequality

$$\|D^m(f-\mathcal{S}_m f)\|^2_{L^2[a,b]} \leqq \|D^{2m}f\|_{L^2[a,b]}\|f-\mathcal{S}_m f\|_{L^2[a,b]}. \tag{3.26}$$

Applying the proof of Theorem 3.4, we have

$$\|D^j(f-\mathcal{S}_m f)\|_{L^2[a,b]} \leqq K_{m,m,z,j}\|D^m(f-\mathcal{S}_m f)\|_{L^2[a,b]}(\bar{\Delta})^{m-j}. \tag{3.27}$$

Using (3.27) for the special case of $j = 0$ in (3.26) yields

$$\|D^m(f-\mathcal{S}_m f)\|_{L^2[a,b]} \leqq \|D^{2m}f\|_{L^2[a,b]}K_{m,m,z,0}(\bar{\Delta})^m. \tag{3.28}$$

Using (3.28) to bound the right-hand side of (3.27) gives us the right-hand inequality of (3.23). The left-hand inequality of (3.23) follows as in Theorem 3.4. Q.E.D.

We now recall a fundamental inequality of E. Schmidt which will be used several times in the remainder of this paper.

LEMMA 3.1. *If $p_N(x)$ is a polynomial of degree N,*

$$\|Dp_N\|_{L^2[a,b]} \leqq \frac{E_N}{b-a}\|p_N\|_{L^2[a,b]}, \tag{3.29}$$

where $E_N \equiv (N+1)^2\sqrt{2}$.

Proof. Cf. [2]. Q.E.D.

THEOREM 3.6.

$$\lambda_d^{-1/2}(p-j) \leqq \Lambda(m,p,z,j) \leqq K_{m,p,z,j}(\bar{\Delta})^{p-j}, \tag{3.30}$$

where

$$d \equiv (M+1)(2m-z+1)+z-j+2 \tag{3.31}$$

and

$$K_{m,p,z,j} \equiv \left\{K_{p,p,2m-1,j} + K_{m,2m,z,j}\cdot 2^{(1/2)(2m-p)}\left[\frac{p!}{(2p-2m)!}\right]^2(\bar{\Delta}/\underline{\Delta})^{2m-p}\right\} \tag{3.32}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a,b)$, $m < p < 2m$, $4m-2p-1 \leqq z \leqq 2m-2$, *and* $0 \leqq j \leqq m$.

Proof. Consider $S(2p-1, \Delta, 2m-1) \subset K^{2m}[a,b]$. This space is well defined since $2p-2 \geqq 2(m+1)-2 = 2m$. Moreover, if $\mathcal{S}_m$ denotes the interpolation mapping of $C^{m-1}[a,b]$ into $S(2m-1,\Delta,z)$ and $\mathcal{S}_p$ denotes the interpolation mapping of $C^{p-1}[a,b]$ into $S(2p-1,\Delta,2m-1)$, then $\mathcal{S}_m(\mathcal{S}_p f) = \mathcal{S}_m f$ for all $f \in C^{p-1}[a,b]$. In fact, $D^k\mathcal{S}_p f$ interpolates $D^k f$ at x_i, $1 \leqq i \leqq M$, for all $0 \leqq k \leqq 2p-(2m-1)-2 = 2p-2m-1$, while $D^k\mathcal{S}_m f$ interpolates $D^k f$ at x_i, $1 \leqq i \leqq M$, for all $0 \leqq k \leqq 2m-z-2 \leqq 2m-(4m-2p-1)-2 = 2p-2m-1$.

Thus,

$$\|D^j(f - \mathcal{S}_m f)\|_{L^2[a,b]} \leqq \|D^j(f - \mathcal{S}_p f)\|_{L^2[a,b]} + \|D^j(\mathcal{S}_p f - \mathcal{S}_m(\mathcal{S}_p f))\|_{L^2[a,b]}, \qquad 0 \leqq j \leqq m. \tag{3.33}$$

By Theorem 3.4,

$$\|D^j(f - \mathcal{S}_p f)\|_{L^2[a,b]} \leqq K_{p,p,2m-1,j}(\bar{\Delta})^{p-j}\|D^p f\|_{L^2[a,b]}, \tag{3.34}$$

and by Theorem 3.5

$$\|D^j(\mathcal{S}_p f - \mathcal{S}_m(\mathcal{S}_p f))\|_{L^2[a,b]} \leqq K_{m,2m,z,j}(\bar{\Delta})^{2m-j}\|D^{2m}\mathcal{S}_p f\|_{L^2[a,b]}. \tag{3.35}$$

But by Schmidt's inequality and the First Integral Relation, since $\mathcal{S}_p f$ is a piecewise polynomial of degree $2p - 1$ with $p > m$, we have

$$\begin{aligned} \|D^{2m}\mathcal{S}_p f\|_{L^2[a,b]} &\leqq \frac{\left(\prod_{i=1}^{2m-p} E_{2p-2m-1+i}\right)\|D^p f\|_{L^2[a,b]}}{(\underline{\Delta})^{2m-p}} \\ &\leqq 2^{(2m-p)/2}\left[\frac{p!}{(2p+2m)!}\right]^2 \frac{\|D^p f\|_{L^2[a,b]}}{(\underline{\Delta})^{2m-p}}. \end{aligned} \tag{3.36}$$

The required result now follows from (3.33), (3.34), (3.35), and (3.36). Q.E.D.

4. L^2-Error Bounds for Higher Order Derivatives. In this section we give *explicit upper* bounds for the quantities $\Lambda(m, p, z, j)$ in the special cases of $m < p \leqq 2m$ and $m < j \leqq p$. Since $\mathcal{S}_m f$ is not necessarily in $K^j[a, b]$ if $z + 1 < j \leqq p$, it is necessary to modify the definition of $\Lambda(m, p, z, j)$ given in (3.1). The new definition is given by

$$\Lambda(m, p, z, j) \equiv \operatorname{Sup}\left\{\left(\sum_{i=0}^{M}\|D^j(f - \mathcal{S}_m f)\|^2_{L^2[x_i,x_{i+1}]}\right)^{1/2} \Big/ \|D^p f\|_{L^2[a,b]} \;\middle|\; f \in K^p[a, b], \|D^p f\|_{L^2[a,b]} \neq 0\right\}. \tag{4.1}$$

The main result of this section is

THEOREM 4.1.

$$\Lambda(m, p, z, j) \leqq K_{m,p,z,j}(\bar{\Delta})^{p-j}, \tag{4.2}$$

where

$$K_{m,p,z,j} \equiv \left[K_{p,p,p,j} + (K_{m,p,z,m} + K_{p,p,p,m})2^{(j-m)/2}\left[\frac{(2p+m)!}{(2p-j)!}\right]^2\left(\frac{\bar{\Delta}}{\underline{\Delta}}\right)^{j-m}\right], \tag{4.3}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a, b)$, $m < p \leqq 2m$, $4m - 2p - 1 \leqq z \leqq 2m - 2$, *and* $m < j \leqq p$.

Proof. By Theorem 3.6,

$$\|D^m(f - \mathcal{S}_m f)\|_{L^2[a,b]} \leqq K_{m,p,z,m}(\bar{\Delta})^{p-m}, \tag{4.4}$$

and by Theorem 3.4,

$$\|D^k(f - \mathcal{S}_p f)\|_{L^2[a,b]} \leqq K_{p,p,p,k}(\bar{\Delta})^{p-k}, \qquad 0 \leqq k \leqq p. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$(4.6) \qquad \|D^m(\mathcal{S}_m f - \mathcal{S}_p f)\|_{L^2[a,b]} \leqq (K_{m,p,z,m} + K_{p,p,p,m})(\bar{\Delta})^{p-m}.$$

Using the Schmidt inequality in (4.6), we obtain

$$(4.7) \qquad \begin{aligned} \|D^j(\mathcal{S}_m f - \mathcal{S}_p f)\|_{L^2[a,b]} &\leqq \frac{\left(\prod_{i=1}^{j-m} E_{(2p-1)-j+i}\right)}{(\underline{\Delta})^{j-m}} \|D^m(\mathcal{S}_m f - \mathcal{S}_p f)\|_{L^2[a,b]} \\ &\leqq (K_{m,p,z,m} + K_{p,p,p,m})\left(\prod_{i=1}^{j-m} E_{2p-1-j+i}\right)(\bar{\Delta})^{p-j}(\bar{\Delta}/\underline{\Delta})^{j-m}. \end{aligned}$$

The required result follows from (4.5), (4.7), and

$$(4.8) \qquad \|D^j(f - \mathcal{S}_m f)\|_{L^2[a,b]} \leqq \|D^j(f - \mathcal{S}_p f)\|_{L^2[a,b]} + \|D^j(\mathcal{S}_p f - \mathcal{S}_m f)\|_{L^2[a,b]}.$$

Q.E.D.

We remark that in those cases in which $\mathcal{S}_m f \in K^j[a, b]$, lower bounds of the form introduced in Section 3 can be given for $\Lambda(m, p, z, j)$.

5. L^∞**-Error Bounds.** In this section, we give *explicit upper* bounds for the quantities $\Lambda^\infty(m, p, z, j)$, $1 \leqq m$, $m \leqq p \leqq 2m$, $m - 1 \leqq z \leqq 2m - 2$, and $0 \leqq j \leqq p$, defined by

$$(5.1) \qquad \Lambda^\infty(m, p, z, j) \equiv \operatorname{Sup}\left\{ \max_{0 \leqq i \leqq M} (\|D^j(f - \mathcal{S}_m f)\|_{L^\infty[x_i, x_{i+1}]}) / \|D^p f\|_{L^2[a,b]} \; \middle| \; f \in K^p[a, b], \|D^p f\|_{L^2[a,b]} \neq 0 \right\}.$$

We obtain the following results as corollaries of the results of Section 3 and Section 4. As an improvement of Theorem 6 of [5], we have

THEOREM 5.1.

$$(5.2) \qquad \Lambda^\infty(m, m, z, j) \leqq K^\infty_{m,m,z,j}(\bar{\Delta})^{m-j-1/2},$$

where

$$(5.3) \qquad \begin{aligned} K^\infty_{m,m,z,j} &\equiv K_{m,m,z,j+1}, \quad \textit{if } m - 1 = z,\ 0 \leqq j \leqq m - 1, \\ &\equiv K_{m,m,z,j+1}, \quad \textit{if } m - 1 < z \leqq 2m - 2,\ 0 \leqq j \leqq 2m - 2 - z, \\ &\equiv (j - 2m + 3 + z)^{1/2} K_{m,m,z,j+1}, \quad \textit{if } m - 1 < z \leqq 2m - 2, \\ &\qquad 2m - 2 - z < j \leqq m - 1, \end{aligned}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a, b)$, $m - 1 \leqq z \leqq 2m - 2$, *and* $0 \leqq j \leqq m - 1$

Proof. We give the proof in the special case of $m - 1 = z$, $0 \leqq j \leqq m - 1$, as the proof in the other cases is analogous. Given any $x \in [a, b]$, there exists a point $y \in [a, b]$ such that $D^j(f - \mathcal{S}_m f)(y) = 0$ and $|x - y| \leqq \bar{\Delta}$. Hence, $D^j(f - \mathcal{S}_m f)(x) = \int_y^x D^{j+1}(f - \mathcal{S}_m f)(t)\, dt$ and

$$\|D^j(f - \mathcal{S}_m f)\|_{L^\infty[a,b]} \leqq (\bar{\Delta})^{1/2} \|D^{j+1}(f - \mathcal{S}_m f)\|_{L^2[a,b]}.$$

The result now follows from applying Theorem 3.4 to the right-hand side of the preceding inequality. Q.E.D.

As in Theorem 5.1, we have as an improvement of Theorem 8 of [5].

THEOREM 5.2.

$$\Lambda^\infty(m, 2m, z, j) \leqq K^\infty_{m,2m,z,j}(\bar{\Delta})^{2m-j-1/2}, \tag{5.4}$$

where

$$\begin{aligned} K^\infty_{m,2m,z,j+1} &\equiv K_{m,2m,z,j+1}, \quad \text{if } m-1 = z,\ 0 < j \leqq m-1, \\ &\equiv K_{m,2m,z,j+1}, \quad \text{if } m-1 < z \leqq 2m-2,\ 0 \leqq j \leqq 2m-2-z, \\ &\equiv (j-2m+3+z)^{1/2} K_{m,2m,z,j+1}, \quad \text{if } m-1 < z \leqq 2m-2, \\ &\qquad\qquad 2m-2-z < j \leqq m-1, \end{aligned} \tag{5.5}$$

for all $1 < m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a, b)$, $m-1 \leqq z \leqq 2m-2$, *and* $0 \leqq j \leqq m-1$.

As in Theorem 3.6, we have

THEOREM 5.3.

$$\Lambda^\infty(m, p, z, j) \leqq K^\infty_{m,p,z,j}(\bar{\Delta})^{p-j-1/2}, \tag{5.6}$$

where

$$K^\infty_{m,p,z,j} \equiv \left\{ K^\infty_{p,p,2m-1,j} + K^\infty_{m,2m,z,j} \cdot 2^{(2m-p)/2} \left[\frac{p!}{(2p-2m)!}\right]^2 \left(\frac{\bar{\Delta}}{\underline{\Delta}}\right)^{2m-p} \right\}, \tag{5.7}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a, b)$, $m < p < 2m$, $4m-2p-1 \leqq z \leqq 2m-2$, *and* $0 \leqq j \leqq m-1$.

Finally, to give a result analogous to Theorem 4.1, we need an inequality due to A. A. Markov.

LEMMA 5.1. *If $p_N(x)$ is a polynomial of degree N, then*

$$\|DP_N\|_{L^\infty[a,b]} \leqq \frac{M_N}{b-a} \|p_N\|_{L^\infty[a,b]}, \tag{5.8}$$

where $M_N \equiv 2N^2$.

Proof. Cf. [6]. Q.E.D.

As an extension of Theorem 10 of [5], we prove

THEOREM 5.4.

$$\Lambda^\infty(m, p, z, j) \leqq K^\infty_{m,p,z,j}(\bar{\Delta})^{p-j-1/2}, \tag{5.9}$$

where

$$K^\infty_{m,p,z,j} \equiv \left\{ K^\infty_{p,p,p,j} + (K^\infty_{m,p,z,j} + K^\infty_{p,p,p,j}) 2^{j-m+1} \left(\frac{(2p-m)!}{(2p-j-1)!}\right)^2 \left(\frac{\bar{\Delta}}{\underline{\Delta}}\right)^{j-m+1} \right\} \tag{5.10}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathcal{P}_M(a, b)$, $m < p \leqq 2m$, $4m-2p-1 \leqq z \leqq 2m-2$ *and* $m \leqq j \leqq p-1$.

Proof. From Theorem 5.1, we have that

$$\|D^k(f - \mathcal{S}_p f)\|_{L^\infty[a,b]} \leqq K^\infty_{p,p,p,k}(\bar{\Delta})^{p-k-1/2} \|D^p f\|_{L^2[a,b]}, \quad 0 \leqq k \leqq p-1, \tag{5.11}$$

and from Theorem 5.3

$$\|D^{m-1}(f - \mathcal{S}_m f)\|_{L^\infty[a,b]} \leqq K^\infty_{m,p,z,m-1}(\bar{\Delta})^{p-m+1/2} \|D^p f\|_{L^2[a,b]}. \tag{5.12}$$

Combining (5.11) and (5.12), we have

$$\|D^{m-1}(\mathcal{S}_m f - \mathcal{S}_p f)\|_{L^\infty[a,b]} \leqq (K^\infty_{m,p,2,m-1} + K^\infty_{p,p,p,k})(\bar{\Delta})^{p-m+1/2}\|D^p f\|_{L^2[a,b]}. \tag{5.13}$$

But,

$$\begin{aligned} \|D^j(\mathcal{S}_m f - \mathcal{S}_p f)\|_{L^\infty_\Delta[a,b]} &\leqq \frac{\left(\prod_{i=1}^{j-m+1} M_{2p-1-j+i}\right)}{(\underline{\Delta})^{j-m+1}} \|D^{m-1}(\mathcal{S}_m f - \mathcal{S}_p f)\|_{L^\infty_\Delta[a,b]} \\ &\leqq 2^{j-m+1}\left(\frac{(2p-m)!}{(2p-j-1)!}\right)^2 \frac{1}{(\underline{\Delta})^{j-m+1}} \\ &\qquad \cdot \|D^{m-1}(\mathcal{S}_m f - \mathcal{S}_p f)\|_{L^\infty_\Delta[a,b]}, \end{aligned} \tag{5.14}$$

where

$$\|\cdot\|_{L^\infty_\Delta[a,b]} \equiv \max_{0 \leqq i \leqq m} \|\cdot\|_{L^\infty[x_i, x_{i+1}]}.$$

The required result follows directly from (5.11), (5.13), (5.14), and the observation that

$$\|D^j(f - \mathcal{S}_m f)\|_{L^\infty_\Delta[a,b]} \leqq \|D^j(f - \mathcal{S}_p f)\|_{L^\infty_\Delta[a,b]} + \|D^j(\mathcal{S}_p f - \mathcal{S}_m f)\|_{L^\infty_\Delta[a,b]}.$$

Q.E.D.

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