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"Best" Interpolation, Differentiation, and Integration Approximations on the Hardy Space H^2

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Abstract. A general formula is developed which gives the "best" approximation for any linear functional on the Hardy space H^2. Some "best" approximations are given for interpolation, differentiation, and integration and are compared to polynomial approximations.

I. Introduction. Several recent papers [3]–[6] have investigated the numerical differentiation and interpolation of analytic functions by using the Cauchy integral formula to express the derivative or function value in terms of a contour integral and then using various numerical methods to approximate the contour integral. Without loss of generality we can assume all functions are analytic in the unit disk and approximations are made at $z = 0$. Then the resulting approximations are of the form $f^{(n)}(0) = \sum_{k=1}^{N} a_k f(re^{2\pi ik/N})$, where $0 < r < 1$, and error estimates are in terms of the Hardy H^2 norm

$$\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{f(e^{i\theta})}\, d\theta. \tag{1.1}$$

Now any linear approximation of the form $\sum_{k=1}^{N} a_k f(r\xi_k)$, where $\xi_k = \exp(2\pi ik/N)$, is a linear functional on f, and in particular the error introduced by such an approximation, $f^{(n)}(0) - \sum_{k=1}^{N} a_k f(r\xi_k)$, is itself a linear functional on f. Sard [7, Chapter 2] and Davis [1, Chapter 14] present methods of generating "best approximation" formulas by the use of such error functionals. This paper uses techniques similar to those of Davis to develop a class of "best approximations" as follows: Given a value of r $(0 < r < 1)$ and an integer $N \geqq 1$, we are concerned with the best approximation having the form

$$Lf = \sum_{k=1}^{N} a_k f(r\xi_k), \tag{1.2}$$

where

$$\xi_k = e^{2\pi ik/N}. \tag{1.3}$$

Some examples are given to illustrate the resulting "best approximations." In Section III, it is shown that the best approximation does not differ significantly from polynomial interpolation and differentiation using the same interpolating points, whereas in Section IV it is shown that the "best" integration method is significantly

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better than polynomial approximations to the integral using the same interpolating points.

II. Error Functionals on the Hardy Space H^2. The Hardy space H^2 consists of those functions which (a) are analytic in the unit disk and (b) have a finite Hardy H^2 norm. (See Hoffman [2] for a development of Hardy spaces.) An inner product is defined on H^2 by

$$(f(z), g(z)) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta.$$

This inner product is compatible with the Hardy H^2 norm in the sense that

$$\|f\|^2 = (f(z), f(z)) \tag{2.1}$$

and with this inner product, H^2 is a separable Hilbert space. Hence, by the Riesz representation theorem, for every bounded linear functional L defined on H^2, there exists a function $h(z)$ in H^2, called the representer of L, such that $Lf(z) = (f(z), h(z))$ for all $f(z)$ in H^2. For example, using the method given in [1, Theorem 12.6.6], it is easy to show that the representer of the bounded linear functional which maps the function f into $f(w)$ for a fixed w in the unit disk is $(1 - \bar{w}z)^{-1}$.

Now, the magnitude of the error functional E for the N term approximation of the linear functional L is

$$|Ef| = \left|Lf - \sum_{k=1}^{N} a_k f(r\xi_k)\right| = \left(f(z),\, h(z) - \sum_{k=1}^{N} \bar{a}_k(1 - r\bar{\xi}_k z)^{-1}\right), \tag{2.2}$$

where $h(z)$ is the representer of L and $(1 - r\bar{\xi}_k z)^{-1}$ is the representer of the functional which evaluates a function at $r\xi_k$ where $0 < r < 1$. Applying the Schwarz inequality to (2.2) gives

$$\begin{aligned} |Ef| &\leqq \|f(z)\|\cdot\left\|h(z) - \sum_{k=1}^{N} \bar{a}_k(1 - r\bar{\xi}_k z)^{-1}\right\| \\ &\leqq C\,\|f(z)\|, \end{aligned}$$

where

$$C = \left\|h(z) - \sum_{k=1}^{N} \bar{a}_k(1 - r\bar{\xi}_k z)^{-1}\right\| \tag{2.3}$$

is independent of the function f.

The "best approximation" is defined as the one which minimizes C for a given r and N. According to Davis [1, Theorem 8.6.3] the coefficients $\bar{a}_k$ of the best approximation $h(z) - \sum_{k=1}^{N} \bar{a}_k(1 - r\bar{\xi}_k z)^{-1}$ are given by the solution of N equations in N unknowns $\mathbf{RA} = \mathbf{H}$ where

$$\mathbf{R}_{jk} = (1 - r^2\xi_j\bar{\xi}_k)^{-1}, \qquad \mathbf{A}_k = \bar{a}_k, \quad \text{and} \quad \mathbf{H}_k = h(r\xi_k).$$

The solution to this system of equations can be obtained by a slight variation of a method given by Wilf [8]. The $\bar{a}_j$ are

$$\bar{a}_j = \frac{(1 - r^{2N})^2}{N^2 r^{2N-2}} \sum_{k=1}^{N} h(r\xi_k)\xi_{k-j}\,(1 - r^2\xi_{k-j})^{-1}, \qquad j = 1, \cdots, N. \tag{2.4}$$

Thus the approximation to Lf which minimizes the error estimate $|Ef|$, the "best approximation", is $\sum_{k=1}^{N} a_k f(r\xi_k)$ where $\bar{a}_k$ is given by (2.4). This error estimate is sharp in the sense that it is an equality for $f(z) = h(z) - \sum_{k=1}^{N} \bar{a}_k(1 - r\bar{\xi}_k z)^{-1}$ where $\bar{a}_k$ is determined by (2.4).

III. Interpolation and Differentiation. For $n = 0, 1, \cdots$, the representer for $f^{(n)}(0)$ is $n!z^n$. So we can consider interpolation and differentiation at zero simultaneously. Substituting $n!z^n$ into (2.4) and simplifying gives

$$a_k = (1 - r^{2N})(n!)r^{2mN-n}\xi_k^{-n}/N \quad \text{for} \quad k = 1, 2, \cdots, N, \tag{3.1}$$

where m is the smallest nonnegative integer such that $mN \leqq n < (m + 1)N$. This gives the "best approximation"

$$f^{(n)}(0) \simeq [(1 - r^{2N})(n!)r^{2mN-n}/N] \sum_{k=1}^{N} \xi_k^{-n} f(r\xi_k) \quad \text{for} \quad n = 0, 1, \cdots . \tag{3.2}$$

To evaluate the error constant C we substitute $h(z) = n!z^n$ and expression (3.1) for the a_k's into (2.3) and simplify to obtain

$$C^2 = (n!)^2[1 - (1 - r^{2N})r^{2mN}].$$

One interesting feature of this approximation is that it allows approximation with $N < n$, something that is impossible with polynomial approximation methods. Unfortunately for $N < n$ the error constant C is an increasing function of m. Its minimum value, which occurs at $m = 1$ and $r^{2N} = \frac{1}{2}$, is $0.87n!$. Furthermore the limit, as $m \to \infty$, of C is $n!$, so there is a nonzero lower bound for the error estimate which increases with n. Practically then, such approximation schemes have little to recommend them.

When $N > n$ (and so $m = 0$) the above formulas take the simpler form

$$C = (n!)r^N \quad \text{and} \quad |Ef| \leqq (n!)r^N \, ||f||.$$

Many approximations are based upon making the approximation exact for all polynomials up to some degree, whereas the ones presented here minimize the error over all functions in H^2 at the cost of polynomial approximations. Consider approximation (3.2) with $N > n$ acting on z^k. The error is 0 except when $k - n$ is a multiple of N. Since every function f in H^2 has a Taylor series expansion about zero, we see that the error Ef is introduced by the terms z^n, z^{n+N}, z^{n+2N}, $\cdots$.

Using the same interpolating points, Lyness [5] has developed a polynomial approximation which is exact for all polynomials of degree equal to or less than $N + n - 1$. (Interestingly, the polynomial coefficients differ from the "best" ones only by a multiplicative constant.) The error constant C for the polynomial approximation is $(n!)r^N/(1 - r^{2N})^{1/2}$ and the error Ef is introduced by the terms $z^{n+N}, z^{n+2N}, \cdots$ in the Taylor expansion of f. Thus the "best approximation" reduces the polynomial error constant by the factor $(1 - r^{2N})^{1/2}$ at the cost of introducing the z^n term in Ef. The "best approximation" is significantly better than the polynomial approximation only for small values of N and values of r close to 1.

IV. Integration. The representer for integration along the real axis from $-b$ to b is

$$h(z) = \frac{1}{z} \ln |(1 + bz)/(1 - bz)|. \tag{4.1}$$

This leads to the quadrature rule

$$If \simeq \sum_{k=1}^{N} a_k f(r\xi_k), \tag{4.2}$$

where

$$\bar{a}_k = \frac{(1 - r^{2N})^2}{N^2 r^{2N-1}} \sum_{j=1}^{N} \frac{1}{(\xi_k - r^2 \xi_j)} \ln \left| \frac{1 + br\xi_j}{1 - br\xi_j} \right| \tag{4.3}$$

and as before

$$\xi_j = e^{2i\pi j/N}. \tag{4.4}$$

To evaluate the error constant C, the a_k's were computed from (4.3) and these were substituted in (2.3) to compute C. This was done numerically for various values of b and r. The computations were ill-conditioned in the sense that even using 16-significant-digit arithmetic, it was difficult to compute C to 3 significant digits when $N > 15$ or $C < 10^{-8}$. The condition of the computations is also dependent upon r. For values of $r < b/2$ the a_k's tend to be large and vary in sign and the computations are ill-conditioned. Whereas for $r \geqq b$, the values of the a_k's are positive and of the same order of magnitude and the computations are well conditioned.

Since no function in H^2 has singularities inside the unit disk and every point outside the unit disk is a singularity for at least one function in H^2, we expect the magnitude of C to decrease as b decreases and the interval of integration moves away from the region containing singularities. This is indeed the case.

For $b = .125$ all singularities are some distance from $[-b, b]$ and to 3 significant digits (for $N \leqq 15$) the values of C fit the curve $C^2 = .0622r^{2N}$. (This curve for C and the one given in the next paragraph were obtained from the numerically computed values of C.)

As $[-b, b]$ approaches possible singularities, the behavior of C becomes more erratic. For $b = 0.5$ and $r = 0.75$, $C^2 \simeq 1.01r^{2N}$. But for $r = 0.5$, C^2 is below this curve for odd values of N and above it for even values of N with the fit improving as N increases. For $r = 0.625$, C^2 is also above and below the same curve with the fit improving until for $N \geqq 11$ the fit is accurate to 3 significant digits.

By the time b increases to 0.875, the behavior of C is so erratic that for $N \leqq 15$ it never comes close to fitting a simple curve.

Lyness [6] has developed some polynomial approximations to the integral using interpolation at the same points used here. The error constant for the polynomial approximations is $C \sim 2br^N/(1 - r^2)^{1/2}$ for even values of N and $C \sim 2br^{N-1}/(1 - r^2)^{1/2}$ for odd values of N. For b close to zero, as expected, the polynomial approximation is close to the "best approximation". However, as the region of integration approaches possible singularities, the "best approximation" becomes significantly better than polynomial approximation.

Remark. For large values of N, Lyness's polynomial approximations are much better, in the sense of having a smaller constant C, than polynomial approximations using equally spaced points on the real axis. For example, for $b = 0.5$, Hämmerlin [9] has shown that the trapezoid method, Simpson's method, and Boole's method have an error constant of the form $C \leqq C_i(N - 1)^{-2i}$ where i is the degree of the polynomial approximation and C_i is a constant dependent upon the method.

Conclusion. We have developed a general formula for directly computing the "best" approximation to any linear functional on H^2. This formula was then used to develop some "best" interpolation, differentiation, and integration approximations. It is interesting to note, that for interpolation and differentiation, polynomial approximations using the same interpolating points are very close to the "best" approximations. However, this is not true for approximate integration.

The methods discussed in this paper rely on function values at equally spaced points on the circle with radius r less than one and center at the origin. The error estimates for interpolation, differentiation, and integration using these points are of the form $C \sim O(r^N)$ whereas the error estimates using equally spaced points on the real line, at least for integration, are of the form $C \sim O(N^{-i})$ where i is a constant dependent upon the method.

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