The Calculation of Multidimensional Hermite Polynomials and Gram-Charlier Coefficients*

By S. Berkowitz and F. J. Garner

Abstract. The paper documents derivations of:
   (a) a recurrence relation for calculating values of multidimensional Hermite polynomials,
   (b) a recurrence relation for calculating an approximation to the Gram-Charlier co-efficients of the probability density distribution associated with a random process, based on (a),
   (c) an efficient algorithm to utilize the formulae in (a) and (b).

I. **Introduction.** The paper documents derivations of: (a) a recurrence relation for calculating values of multidimensional Hermite polynomials; (b) a recurrence relation for calculating an approximation to the Gram-Charlier coefficients of the probability density distribution associated with a random process, based on (a); (c) an efficient algorithm to utilize the formulae in (a) and (b).

The relations for producing Hermite polynomials are generalizations of formulae derived in Appell and Kampé de Fériet [1]. The recurrence relation mentioned in (a) above appears without derivation in Vilenkin [5], which further references a dissertation by Sirazhdinov [4].

Since Hermite polynomials form complete, biorthogonal systems with respect to the Gaussian probability density, one basic use of the polynomials is to expand a near-Gaussian probability density distribution in terms of the polynomials in a so-called Gram-Charlier series. As we shall demonstrate below, the coefficients of the series can be calculated directly from the time series generated by a random process.

Finally, we present and prove an algorithm which computes a Hermite polynomial or Gram-Charlier coefficient of vector order m by means of the above recurrence relations. The algorithm requires the smallest number of suborder polynomials and/or coefficients possible.

The derivation of the recurrence relations is the responsibility of S. Berkowitz, and the algorithm represents the work of both authors. The algorithm has been implemented in FORTRAN subroutines by Mr. Garner.

II. Multidimensional Hermite Polynomials.

A. *Definitions.*

1. $\mathbf{m} = (m_1, \cdots, m_n)$ is an order vector of degrees m_i. The polynomial or argument x and of index m is said to be of degree m_i in the argument x_i ($i = 1, \cdots, n$) and of total degree $m = \sum_{i=1}^{n} m_i$.

2. $\mathbf{x} = (x_i, \cdots, x_n)$ is a *vector of arguments* x_i.

3. $R \equiv Q^{-1}$ is a positive definite matrix. One calls R a covariance matrix for the particular application of expanding joint probability density distributions in a Gram-Charlier series. $R \equiv [r_{ij}]$; $Q \equiv [q_{ij}]$.

4. The complete biorthogonal systems of Hermite polynomials $\{H_m(\mathbf{x})\}$ and $\{G_m(\mathbf{x})\}$ are defined by the generating functions:

(1) $$\exp[\mathbf{a}^T Q \mathbf{x} - \tfrac{1}{2}\mathbf{a}^T Q \mathbf{a}] = \sum_{m=0}^{\infty} c_m(\mathbf{a}) H_m(\mathbf{x}),$$

(2) $$\exp[\mathbf{a}^T \mathbf{x} - \tfrac{1}{2}\mathbf{a}^T R \mathbf{a}] = \sum_{m=0}^{\infty} c_m(\mathbf{a}) G_m(\mathbf{x}),$$

where

   (i) $c_m(\mathbf{a}) = \prod_{i=1}^{n} a_i^{m_i}/m_i!$,

   (ii) the a_i are arbitrary.

The sets $\{H_m(\mathbf{x})\}$ and $\{G_m(\mathbf{x})\}$ are biorthogonal with respect to $(\det Q)^{1/2}(2\pi)^{-n/2}$ $\cdot \exp[-\tfrac{1}{2}\mathbf{x}^T Q \mathbf{x}]$ over the entire real n-space. In the case of unidimensional polynomials ($n = 1$), both $\{H_m(\mathbf{x})\}$ and $\{G_m(\mathbf{x})\}$ reduce to a single orthogonal, complete set $\{H_m(x)\}$, which is generated by the following relation (cf. Erdélyi [2]):

(3) $$\exp[ax - \tfrac{1}{2}a^2] = \sum_{m=0}^{\infty} \frac{a^m}{m!} H_m(x).$$

The polynomials $\{H_m(x)\}$ are orthogonal with respect to $\exp[-\tfrac{1}{2}x^2]$ over the entire real line.

B. *Recurrence Relations for* $H_m(\mathbf{x})$ *and* $G_m(\mathbf{x})$. Differentiating the generating functions (1) with respect to a_k produces

(4) $$\frac{\partial}{\partial a_k}\{\exp[\mathbf{a}^T Q \mathbf{x} - \mathbf{a}^T Q \mathbf{a}]\} = \sum_{j=1}^{n} q_{kj}(x_j - a_j) \sum_{m=0}^{\infty} c_m(\mathbf{a}) H_m(\mathbf{x})$$

$$= \sum_{m=0}^{\infty} \left(\prod_{i \neq k} \frac{a_i^{m_i}}{m_i!}\right) \frac{a_k^{m_k-1}}{(m_k - 1)!} H_m(\mathbf{x}).$$

By matching coefficients of $a_k^{m_k-1} \prod_{i \neq k} a_i^{m_i}$ in the right-hand equation of (4) one finds

(5) $$H_m(\mathbf{x}) = \left(\sum_{j=1}^{n} q_{kj} x_j\right) H_{m-e_k}(\mathbf{x}) - \sum_{j \neq k} q_{kj} m_j H_{m-e_k-e_j}(\mathbf{x}) - q_{kk}(m_k - 1) H_{m-2e_k}(\mathbf{x}),$$

$$k = 1, \cdots, n,$$

where e_k is a vector with a "1" as the kth component, and zeroes elsewhere. A similar operation on the generating function (2) produces:

(6) $$G_m(\mathbf{x}) = x_k G_{m-e_k}(\mathbf{x}) - \sum_{j \neq k} r_{kj} m_j G_{m-e_k-e_j}(\mathbf{x}) - (m_k - 1) r_{kk} G_{m-2e_k}(\mathbf{x}),$$

$$k = 1, \cdots, n.$$

For any particular $H_m(\mathbf{x})$ or $G_m(\mathbf{x})$, one has a choice of n recurrence relations. The value of any term in (5) or (6) with a negative index is zero and the jth relation must not be used if $m_j = 0$ as one would falsely conclude $H_m(\mathbf{x}) = 0$. If one chooses the pth recurrence relation to compute $H_m(\mathbf{x})$ or $G_m(\mathbf{x})$, then m_p is called the *pivot order*, and p is called the *pivot*.

III. Calculation of Gram-Charlier Series Coefficients. Consider the n-dimensional vector time series $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ... generated from a random process whose joint probability density is $p(\mathbf{x})$.

The components of the kth vector of the time series are:

$$(7) \qquad \mathbf{x}^{(k)} = x_1^{(k)}, \cdots, x_n^{(k)}.$$

Denote the Gaussian probability density by the formula:

$$p_0(\mathbf{x}) = (\det Q)^{1/2}(2\pi)^{-n/2} \exp\,[-\tfrac{1}{2}(\mathbf{x} - \mathbf{\mu})^T Q(\mathbf{x} - \mathbf{\mu})],$$

where (a) $\mathbf{\mu} = \mu_1, \cdots, \mu_n$ is usually taken as the vector of ensemble averages of the variates $x_1, \cdots, x_n$, respectively.

(b) Q is a positive definite n by n matrix, usually taken as the inverse covariance matrix of $\mathbf{x}$.

One can expand $p(\mathbf{x})$ in a Gram-Charlier series as follows:

$$p(\mathbf{x}) = p_0(\mathbf{x}) \sum_{m=0}^{\infty} A_m H_m(\mathbf{y}),$$

where (a) $\mathbf{y} = y_1, \cdots, y_n$ is a normalization of $\mathbf{x}$ such that $y_i = (x_i - \mu_i)/\sigma_i$, ($i = 1, \cdots, n$), and

(b) $\sigma_i = E[(x_i - \mu_i)^2]$ is the second moment of x_i about its expectation.

The biorthogonality property of the $H_m(\mathbf{x})$ and $G_m(\mathbf{x})$ is:

$$(8) \qquad \int_S p_0(\mathbf{x})G_m(\mathbf{x})H_p(\mathbf{x})\,d\mathbf{x} = b_m^{-1} \prod_{i=1}^{n} \delta_{m_i p_i},$$

where: (a) S indicates the entire real n-space,

(b) $b_m = \prod_{i=1}^{n} (m_i!)^{-1}$,

(c) δ_{mp} is the Kronecker delta.

By means of the biorthogonality property (8) and the Gaussian weighting function $p_0(\mathbf{x})$, one can formulate the Gram-Charlier coefficients as follows:

$$A_m = b_m \int_S G_m(\mathbf{y})p(\mathbf{x})\,d\mathbf{x},$$

or, simply,

$$(9) \qquad A_m = b_m E[G_m(\mathbf{y})].$$

From (9) and the recurrence relation (6), one deduces:

$$(10) \quad A_m = b_m \left[E[y_k G_{m-e_k}(\mathbf{y})] - \sum_{j=1}^{n} r_{kj} m_j E[G_{m-e_k-e_j}(\mathbf{y})] + r_{kk} E[G_{m-2e_k}(\mathbf{y})] \right].$$

For a sample set of M vectors of the type indicated by (7), the first term on the right-hand side of (10) may be written as

$$(11) \qquad E[y_k G_{m-e_k}(\mathbf{y})] \approx \frac{1}{M} \sum_{i=1}^{M} y_k^{(i)} G_{m-e_k}(\mathbf{y}^{(i)}).$$

The expectations μ_k and σ_k have approximations

$$(12) \qquad\qquad \mu_k \approx \frac{1}{M} \sum_{i=1}^{M} x_k^{(i)},$$

$$(13) \qquad\qquad \sigma_k \approx \frac{1}{M} \sum_{i=1}^{M} (x_k^{(i)} - \mu_k)^2.$$

Thus, by substituting (11) and (9) into (10) one has a recurrence relation for $A_{\mathbf{m}}$ (which, in turn, employs the recurrence relation (6) for $G_{\mathbf{m}}(\mathbf{x})$):

$$(14) \quad A_{\mathbf{m}} = \frac{1}{M} \left[b_{\mathbf{m}} \sum_{i=1}^{M} y_k^{(i)} G_{\mathbf{m}-\mathbf{e}_k}(\mathbf{y}^{(i)}) - \sum_{j=1}^{n} r_{kj} m_k^{-1} A_{\mathbf{m}-\mathbf{e}_k-\mathbf{e}_j} \right], \quad k = 1, \cdots, n.$$

Similarly, for the expansion

$$p(\mathbf{x}) = p_0(\mathbf{x}) \sum_{m=0}^{\infty} B_{\mathbf{m}} G_{\mathbf{m}}(\mathbf{y}),$$

one can derive the recurrence relation:

$$(15) \qquad B_{\mathbf{m}} = \frac{1}{M} \left[b_{\mathbf{m}} \sum_{i=1}^{M} \sum_{j=1}^{n} q_{kj} y_j^{(i)} H_{\mathbf{m}-\mathbf{e}_k}(\mathbf{y}^{(i)}) - \sum_{j=1}^{n} q_{kj} m_k^{-1} B_{\mathbf{m}-\mathbf{e}_k-\mathbf{e}_j} \right],$$

$$k = 1, \cdots, n.$$

As in the recurrence relations for Hermite polynomials, any term in (14), or (15) with a negative index is to be regarded as zero. In fact, since the indexing in (14) and (15) corresponds (with a bit of manipulation) to the indexing in (5) and (6), the problems of sequencing the recursive generation of Gram-Charlier coefficients are the same as those for sequencing Hermite polynomials, with the exception that only the vectors $\mathbf{m} - \mathbf{e}_k - \mathbf{e}_j$, $j = 1, \cdots, n$, need be generated for the Gram-Charlier coefficients. Therefore, in the remainder of the paper, we restrict ourselves to a discussion of Hermite polynomials and make only parenthetical restrictions where necessary for Gram-Charlier coefficients.

IV. An Algorithm for Generating Values of Hermite Polynomials (or Gram-Charlier Coefficients).

In recursively generating the value of $H_{\mathbf{m}}(\mathbf{x})$ or $G_{\mathbf{m}}(\mathbf{x})$ from (5) or (6) one could conceivably generate all $H_{\mathbf{v}}(\mathbf{x})$ or $G_{\mathbf{v}}(\mathbf{x})$ such that $v_i \leqq m_i$ ($i = 1, \cdots, n$). However, there is a test that permits one to determine, purely on the basis of the order vector $\mathbf{v}$, whether or not a polynomial value need be calculated from the recurrence relation. The test is embodied in an algorithm which we present below and prove in the next section. The algorithm requires a prior decision on the generating sequence as follows:

The order vector $\mathbf{m}$ is permuted according to an arbitrary but fixed pivot sequence vector

$$(16) \qquad\qquad \mathbf{\delta} = \sigma(1), \cdots, \sigma(n),$$

where the set $\{\sigma(j)\}$ is a permutation of $1, \cdots, n$. The classes V_p, defined by

$$V_p = \{\mathbf{v} = v_1, \cdots, v_n \mid 0 \leqq v_{\sigma(j)} \leqq m_{\sigma(j)}, 1 \leqq j \leqq p - 1;$$

$$0 < v_{\sigma(p)} \leqq m_{\sigma(p)}; v_{\sigma(j)} = 0, p + 1 \leqq j \leqq n\},$$

$$p = 1, \cdots, n,$$

are each generated according to an ordering S_p which is defined by

$$S_p = \left\{ \mathbf{v}, \mathbf{v}' \in V_p \mid \mathbf{v} = (v_{\sigma(1)}, \cdots, v_{\sigma(p)}, 0, \cdots, 0), \right.$$

$$\mathbf{v}' = (v'_{\sigma(1)}, \cdots, v'_{\sigma(p)}, 0, \cdots, 0), \text{ and}$$

$$\left. \mathbf{v} < \mathbf{v}' \text{ iff } \sum_{j=1}^{p} (v_{\sigma(j)} - v'_{\sigma(j)}) b^{p-j} < 0, \text{ where } b > \max_i (m_i) \right\}.$$

The class V_p stipulates that the $\sigma(p)$th recurrence relation is to be used for calculating $H_\mathbf{v}(x)$ if $\mathbf{v} \in V_p$. The sequence S_p is merely an ordering of the real numbers $\{(v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(p)} b\}$ in a base b sufficiently large so that no $v_{\sigma(j)}$ exceeds $b - 1$, thus insuring the availability of lower-ordered polynomial values for the calculation of a given polynomial value in the recurrence relation.

The algorithm now follows:

1. Set $H_0(x) = 1$ (or $G_0(x) \equiv A_0 \equiv B_0 \equiv 1$).
2. Initialize the pivot sequence at $p = 1$.
3. Initialize the order vector sequence in S_p.
4. For the ordering S_p over V_p, generate the sth order vector $\mathbf{v}$.
5. (a) If

$$\sum_{j=p+1}^{n} m_{\sigma(j)} + (m_{\sigma(p)} - v_{\sigma(p)}) \geqq \sum_{j=1}^{p-1} (m_{\sigma(j)} - v_{\sigma(j)}),$$

compute $H_\mathbf{v}(x)$ (or $G_\mathbf{v}(x)$) from the $\sigma(p)$th recurrence relation and store. (For the computation of a Gram-Charlier coefficient, we require also that $\sum_{i=1}^{n} (m_i - v_i)$ be even.)

(b) Otherwise, proceed directly to 6.

6. (a) If V_p is not exhausted, update s by one and return to 4 to increment V_p according to S_p.

(b) If V_p is exhausted, and $p < n$, update p by one and return to 3 to initialize S_p.

(c) If $p = n$, exit.

The number of polynomial values required to compute the value of a given $H_m(x)$ or $G_m(x)$ depends heavily upon the manner in which one chooses the order sequence. The order sequence (16) which minimizes the number of required polynomial values is

(17) $$j \leqq k \quad \text{iff} \quad m_{\sigma(j)} \geqq m_{\sigma(k)}.$$

We will justify this assertion after presenting the proof for the algorithm.

V. Proof of the Algorithm.

A. *Definitions.* The reverse sequence of indices in the following definitions facilitates the description of index increments at the top of the partial ordering of order vectors required by the recurrence relation.

1. *Order Vectors.*

(a) *Target Order Vector* $\mathbf{h} \equiv h_n, \cdots, h_1, \equiv m_{\sigma(1)}, \cdots, m_{\sigma(n)}$.

(b) *Test Order Vector* $\mathbf{w} \equiv w_n, \cdots, w_1, \equiv v_{\sigma(1)}, \cdots, v_{\sigma(n)}$.

(c) *Test Difference* $d_j \equiv h_j - w_j$, $j = 1, \cdots, n$.

(d) *Masking Vector* $\mathbf{e}'_j \equiv \mathbf{e}_{n-j+1} \equiv (\delta_{nj}, \cdots, \delta_{1j})$, $j = 1, \cdots, n$.

2. *Pivot Classes.*

(a) $C_p \equiv V_{n-p+1}$, $p = 1, \cdots, n$. (Note that the C_p form a partition of the set of suborder vectors to be considered in computing $\mathbf{h}$.)

(b) $T_p = \{\mathbf{w}|\mathbf{w} \in C_p \text{ and } \sum_{j=1}^{p} d_j \geqq \sum_{j=p+1}^{n} d_j, p = 1, \cdots, n\}$.

(c) $R = \{\mathbf{w}|\mathbf{w} \text{ is needed in a recurrence relation to compute } H_m(\mathbf{x}) \text{ or } G_m(\mathbf{x})\}$.

(d) The pivot associated with C_p or T_p is $\sigma(n - p + 1)$; that is, if $\mathbf{w} \in T_p \cap R$, then the $\sigma(n - p + 1)$th recurrence relation is used to compute $H_v(\mathbf{x})$ or $G_v(\mathbf{x})$.

3. *Increment Classes.*

(a) According to the recurrence relation (5) or (6), there are three types of increments:

Type I—The pivot order w_p is incremented by one.

Type II—The pivot order w_p is incremented by two. (This type is equivalent to two Type I increments, and will not be used below.)

Type III—The pivot order w_p is incremented by one, and some other order w_j ($j > p$) is also incremented by one.

(b) A vector $\mathbf{a} = a_n, \cdots, a_1$ is in $L_p(t, \mathbf{w})$ if and only if $\mathbf{a}$ is the result of zero or more increments from $\mathbf{w}$ with pivot $\sigma(n - p + 1)$ and increment Type(s) $t = $ I, II, or III.

B. *The Proof.* The following two notes labelled (N1) and (N2) will be employed in the proof.

> When a pivot remains constant throughout incrementing from $\mathbf{w}$ to $\mathbf{a}$,
(N1) the recurrence relations demand that $a_p - w_p \geqq \sum_{i \neq p} (a_i - w_i)$,
> where the pivot is $\sigma(n - p + 1)$.

(N2) If any chain of increments can be constructed from $\mathbf{w}$ to $\mathbf{h}$, then $\mathbf{w} \in R$.

We seek to prove that $R = \bigcup_k T_k$. However, since the C_k form a partition of the order vectors under consideration, $R = \bigcup_k (R \cap C_k)$, it suffices to prove that

$$R \cap C_k = T_k, \qquad k = 1, \cdots, n,$$

by induction on k.

1. For $k = 1$.

(a) Assume $\mathbf{w} \in R \cap C_1$. The pivot is $\sigma(n)$, since $\mathbf{w} \in C_1$. Since, by (N1), $d_1 \geqq \sum_{j=2}^{n} d_j$, it is true that $\mathbf{w} \in T_1$ and thus $T_1 \subset R \cap C_1$.

(b) Assume $\mathbf{w} \in T_1$. Thus, $d_1 \geqq \sum_{j=2}^{n} d_j$.

(i) If the right-hand side of the inequality is zero, then $\mathbf{h} \in L_1(\text{I}, \mathbf{w})$ and, by (N2), $\mathbf{w} \in R$. Since, $T_1 \subset C_1$, $\mathbf{w} \in R \cap C_1$.

(ii) If the right-hand side of the inequality is positive, then $\exists w_1 = \mathbf{h} - (d_1 - \sum_{j=2}^{n} d_j)\mathbf{e}'_1 \in L_1(\text{III}, \mathbf{w})$. But $\mathbf{h} \in L_1(\text{I}, \mathbf{w}_1)$ and, by (N2), $\mathbf{w} \in R$. As in (i) then, $T_1 \subset R \cap C_1$.

$$\therefore T_1 = R \cap C_1.$$

2. Suppose $T_k = R \cap C_k$, $1 \leqq k \leqq m - 1 < n$. (H1)

We show now that $T_m = R \cap C_m$.

(a) Assume $\mathbf{w} \in R \cap C_m$. There are three cases:

(i) $\mathbf{w} = \mathbf{h}$;

(ii) $\exists\, \mathbf{a} = (a_n, \cdots, a_{p+1}, w_p + 1, 0, \cdots, 0) \in L_p(\text{I or III}, \mathbf{w})$, for $p < m$. By definition, $\mathbf{a} \in C_p$. Moreover, if $\mathbf{a}$ *is also in* R, then, by (H1), $\mathbf{a} \in T_p$. If one defines $d'_j = h_j - a_j$, $j = 1, \cdots, n$, then one has

$$\sum_{j=1}^{m} d_j \geq \sum_{j=1}^{p} d_j = \sum_{j=1}^{p} d'_j + 1$$

$$\geq \sum_{j=p+1}^{n} d'_j + \sum_{j=p+1}^{n} (a_j - w_j) \qquad \text{(by (H1) and (N1))}$$

$$\geq \sum_{j=m+1}^{n} d_j.$$

Thus, $\mathbf{w} \in T_m$ if any $\mathbf{a}$ of the above type is in R;

(iii) $\exists\, \mathbf{b} = (b_n, \cdots, b_{m+1}, h_m, 0, \cdots, 0) \in L_m(\text{I and/or II and/or III}, \mathbf{w})$. By definition, $\mathbf{b} \in C_m$. Consider any $\mathbf{c} = (c_n, \cdots, c_{p+1}, 1, 0, \cdots, 0) \in L_p(\text{I or III}, \mathbf{b})$ for $p < m$. If $\mathbf{c} \in R$, then, as in (ii) above, $\mathbf{b} \in T_m$. Let $d'_j = h_j - b_j$, $j = 1, \cdots, n$. The following inequalities show that $\mathbf{w} \in T_m$ (if $\mathbf{c} \in R$):

$$\sum_{j=1}^{m} d_j \geq \sum_{j=1}^{m-1} d'_j + \sum_{j=m+1}^{n} (b_j - w_j) \qquad \text{(by (N1))}$$

$$\geq \sum_{m+1}^{n} d'_j + \sum_{j=m+1}^{n} (b_j - w_j) \qquad \text{(since } \mathbf{b} \in T_m)$$

$$= \sum_{j=m+1}^{n} d_j.$$

But increments of $\mathbf{b}$ for pivot $\sigma(n - p + 1)$ with $p < m$ constitute *all* the increments of $\mathbf{b}$. Moreover, any increment chain upward from $\mathbf{w}$ produces at least one of the vectors represented by $\mathbf{h}$, $\mathbf{a}$, or $\mathbf{b}$. Hence, at least one of the vectors must be in R. Therefore, $\mathbf{w} \in T_m$ and $R \cap C_m \subset T_m$.

(b) Assume $\mathbf{w} \in T_m$. We show that $\mathbf{w} \in R \cap C_m$ by induction on d_m.

(i) $d_m = 0$, so that

(18)
$$\sum_{i=1}^{m-1} d_i \geq \sum_{j=m+1}^{n} d_j.$$

Case One. If the right-hand side of (18) is zero, then either $\mathbf{w} = \mathbf{h}$ (in which case $\mathbf{w} \in R \cap C_1$), or $\exists\, p = \max\{j \mid h_j > 0, 1 \leq j < m\}$, and $\exists\, \mathbf{c} = \mathbf{w} + \mathbf{e}'_p \in L_p(\text{I}, \mathbf{w})$. But $\mathbf{c} \in T_p$, since $\sum_{j=1}^{p} h_j \geq 1$. Hence, by (H1), $\mathbf{c} \in R \cap C_p$, and, by (N2), $\mathbf{w} \in R$. Since $T_m \subset C_m$, $\mathbf{w} \in R \cap C_m$.

Case Two. On the other hand, if the right-hand side of the inequality (18) is positive, $\mathbf{w} \neq \mathbf{h}$ and $\exists\, k$, $m + 1 \leq k \leq n$, such that $d_k > 0$. Thus, for p chosen as in Case One—p exists, since otherwise $\mathbf{w} = \mathbf{h}$—there is a vector $\mathbf{g} = g_n, \cdots, g_1 =$

$\mathbf{w} + \mathbf{e}'_k + \mathbf{e}'_p \in L_p(\mathrm{III}, \mathbf{w})$. If one defines $d'_j \equiv g_j - w_j$, $1 \leq j \leq n$, then

$$\sum_{j=1}^{p} d'_j = \sum_{j=1}^{m-1} d_j - 1 \geq \sum_{j=m}^{n} d_j - 1 = \sum_{j=p+1}^{n} d'_j.$$

Thus, as in Case One, $\mathbf{w} \in R \cap C_m$. Therefore, for $d_m = 0$, $T_m \subset R \cap C_m$.

(ii) Suppose that $T_m \subset R \cap C_m$ for $0 \leq d_m < t$.   (H2)

We show now that if $\mathbf{w} \in T_m$ and if $d_m = t$, so that

(19)
$$\sum_{j=1}^{m-1} d_j + t \geq \sum_{j=m+1}^{n} d_j,$$

then $\mathbf{w} \in R \cap C_m$.

Case One. If the inequality in (19) is strict, then $\exists\, \mathbf{y} = y_n, \cdots, y_1 = \mathbf{w} + \mathbf{e}'_m \in L_m(\mathrm{I}, \mathbf{w})$. If $d'_j \equiv h_j - y_j$, $1 \leq j \leq n$, then

$$\sum_{j=1}^{m} d'_j = \sum_{j=1}^{m-1} d_j + (t - 1) \geq \sum_{j=m+1}^{n} d_j = \sum_{j=m+1}^{n} d'_j.$$

Thus $\mathbf{y} \in T_m$, and, since $d'_m = t - 1$, $\mathbf{y} \in R \cap C_m$ by (H2). As in Case One of (i) *supra*, we conclude that $\mathbf{w} \in R \cap C_m$.

Case Two. If, in (19), the inequality is in fact an equality, then $\exists\, k$, $m + 1 \leq k \leq n$, such that $d_k > 0$, and $\exists\, \mathbf{z} = z_n, \cdots, z_1 = \mathbf{w} + \mathbf{e}'_m + \mathbf{e}'_k \in L_m(\mathrm{III}, \mathbf{w})$. If $d'_j \equiv h_j - z_j$, $1 \leq j \leq n$, then

$$\sum_{j=1}^{m} d'_j = \sum_{j=1}^{m-1} d_j + (t - 1) = \sum_{j=m+1}^{n} d_j - 1 = \sum_{j=m+1}^{n} d'_j.$$

Thus, $\mathbf{z} \in T_m$, and, exactly as in Case One of (ii) $\mathbf{w} \in R \cap C_m$.

Therefore, $T_m \subset R \cap C_m$, and, indeed, $T_m = R \cap C_m$.

In applying the algorithm to Gram-Charlier coefficients, it was necessary that $\sum_{j=1}^{n} (m_j - v_j)$ be even. Let $\mathbf{u}$ be a type t *decrement* of $\mathbf{w}$ if, and only if, $\mathbf{w}$ is a type t increment of $\mathbf{u}$. Then, in the recurrence relation (14) or (15), decrements of types II or III were needed to designate required order vectors for the A_m or B_m. Hence for any required order vector, $\sum_{j=1}^{n} d_j$ must be even. Q.E.D.

Let us now justify the assertion that the order sequence (16) which minimizes the number of order vectors in R is

$$j \leq k \quad \text{iff } m_{\sigma(j)} \geq m_{\sigma(k)} \quad \text{(cf. (17))}.$$

According to the algorithm, the vectors $\mathbf{w} \in R \cap C_p$ are those specified by

(20)
$$w_p - \sum_{j=p+1}^{n} w_j \leq k_p, \qquad p = 1, \cdots, n,$$

where

$$k_p = \sum_{j=1}^{p} h_j - \sum_{j=p+1}^{n} h_j, \qquad h_j \geq w_j \geq 0, \qquad j = 1, \cdots, n.$$

Consider a Euclidean $(n - p + 1)$-space E_p with orthogonal axes $w_p, \cdots, w_n$. If one replaces the inequality in (20) by an equality, one has a hyperplane R_p of (positive) unity slope with w_p-intersect at $w_p = k_p$. The points in $R \cap C_p$ form a closed subset

of E_p, bounded by the hyperplanes R_p, $w_i = 0$, and $w_i = h_i$ $(i = p, \cdots, n)$, where for any i, $w_j \geqq 0$ if $j \neq i$. The number of points in $R \cap C_p$ is minimum for least w_p-intersect k_p, and thus for the minimum $\sum_{i=1}^{p} h_i$, a fact that justifies the assertion (17).

Naval Ship Research and Development Center
Washington, D. C. 20034

1. P. Appell & J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques*: *Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.
2. A. Erdélyi, et al., *Higher Transcendental Functions*, McGraw-Hill, New York, 1953. MR 15, 419.
3. J. Kampé de Fériet, *The Gram-Charlier Approximation of the Normal Law and the Statistical Description of a Homogeneous Turbulent Flow near Statistical Equilibrium*, Applied Mathematics Laboratory, NSRDC Report No. 2013, March, 1966.
4. S. Kh. Sirazhdinov, *Contribution to the Theory of Hermite Multidimensional Polynomials*, Candidate Dissertation, Tashkent State University, USSR, 1949. (Russian)
5. S. Ya. Vilenkin, "Determination of the maximum time (critical path time) distribution," *Avtomat. i Telemeh.*, v. 1965, no. 7, pp. 1247–1252 = *Automat. Remote Control*, v. 1965, no. 7, pp. 1333–1337.