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The Initial-Value Problem for the Equation $(tu_t)_t = u_{xx}$

By Alan Solomon and Faiza Solomon

Abstract. It is shown that the initial-value problem of the equation $(tu_t)_t = u_{xx}$ with the value of u prescribed for $t = 0$ has a unique solution satisfying a maximum principle. In addition, several numerical schemes for its solution are proposed.

Introduction. Consider the initial-value problem

Problem A. *Find a function* $u(x, t)$ *satisfying the conditions*

$$u_{xx} = (tu_t)_t \qquad (= tu_{tt} + u_t), \qquad t > 0, \quad -\infty < x < \infty, \tag{1}$$

$$u(x, 0) = f(x), \qquad -\infty < x < \infty, \tag{2}$$

for a twice continuously differentiable function $f(x)$.

This problem has arisen from the consideration of heat conduction problems with delay, related to the equivalent equation

$$u_t(x, t) = \frac{1}{t}\int_0^t u_{xx}(x, \tau)\, d\tau. \tag{3}$$

Equation (1) governs the motion of a homogeneous rope with one free end when the variables t, x are interchanged (see [1, p. 390]). It is clearly hyperbolic for $t > 0$ and reduces to the heat equation

$$u_t = u_{xx} \qquad (= f''(x)), \tag{4}$$

for $t = 0$, whence Problem A is a Goursat type problem in which u_t is to be found initially from (4).

In this paper we study both the theoretical and numerical aspects of solving Problem A. In Section 1 we briefly sketch the proofs of existence, of uniqueness and of a strong maximum principle, based on well-known concepts; in Section 2 we examine an explicit difference scheme with variable time steps for the numerical solution, while in Section 3 other numerical schemes are described.

1. Solution of the Problem. The characteristics of Eq. (1) are parabolas

$$t = \tfrac{1}{4}(x - c)^2, \qquad c = \text{const}, \tag{5}$$

two of which pass through every point of the upper half-plane. In order to prove the

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unique existence of a solution to Problem A and to derive a maximum principle we first obtain the following:

THEOREM 1. *Let $P = (x_0, t_0)$ be any point of the upper half-plane. Let C_-, C_+ be the portions of the characteristics*

$$(6)\qquad \begin{aligned} t &= t_1(x) = \tfrac{1}{4}[x - (x_0 - 2t_0^{1/2})]^2, \\ t &= t_2(x) = \tfrac{1}{4}[x - (x_0 + 2t_0^{1/2})]^2, \end{aligned}$$

joining P to the points

$$A = (x_0 - 2t_0^{1/2}, 0), \qquad B = (x_0 + 2t_0^{1/2}, 0).$$

Let L be the interval

$$[x_0 - 2t_0^{1/2}, x_0 + 2t_0^{1/2}]$$

of the initial line $t = 0$.

If $u(x, t)$ is a solution to Eq. (1), *then*

$$(7)\qquad u(x_0, t_0) = \frac{1}{4t_0^{1/2}} \int_{x_0 - 2t_0^{1/2}}^{x_0} u(s, t_1(s))\, ds + \frac{1}{4t_0^{1/2}} \int_{x_0}^{x_0 + 2t_0^{1/2}} u(s, t_2(s))\, ds.$$

Theorem 1 is proved by integrating Eq. (1) over the region D bounded by C_-, C_+, L and by using Green's theorem.

Let $M = \max f(x)$, $m = \min f(x)$ over L. Then we assert the

COROLLARY 1 (MAXIMUM PRINCIPLE).

$$(8)\qquad m \leqq u(x_0, t_0) \leqq M.$$

It suffices for us to prove the right-hand inequality.

By Eq. (7)

$$(9)\qquad \min_{C_-, C_+} u(x, t) \leqq u(x_0, t_0) \leqq \max_{C_-, C_+} u(x, t)$$

with strict inequality holding, unless u is constant on C_-, C_+. If u is not constant on C_-, C_+, then by inequality (9) there is a point $Q = (x_1, t_1)$ on these characteristics for which $u(x_1, t_1) > u(x_0, t_0)$. However, inequality (9) can itself be applied to $u(x_1, t_1)$ with respect to the characteristics passing through Q, a process which can then be applied to other points on these characteristics. The same argument can be applied to the left-hand inequality of (8); thus, inequality (8) is proved.

COROLLARY 2 (STRONG MAXIMUM PRINCIPLE). *If $f'(x)$ is not identically zero on L, then*

$$(10)\qquad m < u(x_0, t_0) < M.$$

For, let a point $(x^*, 0)$ of the base L exist at which $f'(x^*) \neq 0$ and

$$(11)\qquad m < f(x^*) < M.$$

Then, on the characteristic parabola emanating from $(x^*, 0)$

$$t = \tfrac{1}{4}(s - x^*)^2, \qquad x = s,$$

we have

$$\frac{d}{dt} u(s, t) \bigg|_{s=x^*} = u_x \frac{dx}{ds} + u_t \frac{dt}{ds} \bigg|_{s=x^*} = f'(x^*) \neq 0,$$

whence u is not constant on the characteristic. Let R be the point of intersection of this characteristic with C_+. Clearly, inequality (8) implies $m < u(R) < M$, and hence, by the inequality (9), $u(x_0, t_0)$ cannot attain either of the values m, M; Corollary 2 is thus proved.

From this result and the linearity of Eq. (1) we have

COROLLARY 3. *If a solution to Problem* A *exists, then it is unique.*

Representation (7) can be used in order to prove the existence of a solution to Problem A, based on Picard iteration (see, i.e., [2, pp. 466–471]). Due to the similarity of this proof to that for the ordinary wave equation, we omit it, and summarize our results as:

THEOREM 2. *There exists a unique solution to Problem* A *obeying a strong maximum principle.*

2. An Explicit Numerical Scheme with Variable Time Steps for Problem A. Numerical schemes for the solution of Problem A can be formulated in many different ways. In this section we examine a scheme with fixed space mesh length and variable time mesh lengths, which was most effective among all schemes tested.

Let us choose some $\Delta x > 0$ and a sequence of numbers $\Delta t_1, \Delta t_2, \Delta t_3, \cdots$, and let us define

$$x_j = j\Delta x, \qquad j = 0, \pm 1, \cdots,$$

$$t_n = \sum_{j=1}^{n} \Delta t_j, \qquad t_0 = \Delta t_0 = 0, \qquad n = 1, 2, \cdots.$$

For any indices j, n, Eq. (3) implies

$$(t_n + \tfrac{1}{2}\Delta t_{n+1})u_t(x_j, t_n + \tfrac{1}{2}\Delta t_{n+1}) - (t_n - \tfrac{1}{2}\Delta t_n)u_t(x_j, t_n - \tfrac{1}{2}\Delta t_n) = \int_{t_n-(1/2)\Delta t_n}^{t_n+(1/2)\Delta t_{n+1}} u_{xx}(x, \tau)\, d\tau. \tag{12}$$

Replacing the derivatives u_t on the left-hand side of (12) by centered differences, and the integral on the right-hand side by a centered difference at (x_j, t_n), we are led to the explicit difference scheme

$$\mathcal{L}_\Delta U_j^n = \frac{(t_n/\Delta t_{n+1} + \frac{1}{2})(U_j^{n+1} - U_j^n) - (t_n/\Delta t_n - \frac{1}{2})(U_j^n - U_j^{n-1})}{\frac{1}{2}(\Delta t_n + \Delta t_{n+1})} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0, \tag{13}$$

where U_j^n represents the approximation to $u(x_j, t_n)$. The value U_j^1 is to be found by the heat equation analog

$$\frac{U_j^1 - U_j^0}{\Delta t_1} = \frac{U_{j+1}^0 - 2U_j^0 + U_{j-1}^0}{\Delta x^2} \tag{13a}$$

with $U_j^0 = f(x_j)$.

In the case of equal mesh lengths $\Delta t_n \equiv \Delta t$, Eq. (13) reduces to the standard explicit scheme

$$(14) \qquad (t_n + \tfrac{1}{2}\Delta t)(U_j^{n+1} - U_j^n) - (t_n - \tfrac{1}{2}\Delta t)(U_j^n - U_j^{n-1}) = \lambda^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

with $\lambda = \Delta t/\Delta x$.

By a simple calculation we find that, for $\mathcal{L}u = (tu_t)_t - u_{xx}$, we have, for any smooth function u, $\mathcal{L}_\Delta u = \mathcal{L}u + E$ with

$$E = (\Delta t_{n+1} - \Delta t_n)(t_n u_{3t}/3 + u_{2t}/2) + \left(\frac{\Delta t_{n+1}^3 + \Delta t_n^3}{\Delta t_{n+1} + \Delta t_n}\right)(t_n u_{4t}/12 + u_{3t}/6)$$
$$+ \cdots + 2\left(\frac{\Delta t_{n+1}^\nu + (-1)^{\nu+1}\Delta t_n^\nu}{\Delta t_{n+1} + \Delta t_n}\right)(t_n u_{(\nu+1)t}/(\nu+1)! + u_{\nu t}/2\nu!)$$
$$- x^2 u_{4x}/12 - \cdots - 2\Delta x^{2(\mu-1)} u_{2\mu x}/(2\mu)! - \cdots,$$

where derivatives of u are evaluated at (x_j, t_n).

Although the first term in E vanishes for $\Delta t_{n+1} = \Delta t_n$, the best numerical results have been obtained for $\Delta t_n = O(t_n^{1/2}\,\Delta x)$, a reflection of the fact that the succeeding term in E is of order $O(\Delta t_n^2 + t_n \Delta x^2)$.

Using a modification of the usual method of von Neumann, we have been led to a stability criterion for Eq. (13) of the form

$$(15a) \qquad \lambda_n \lambda_{n+1} \leqq t_n,$$

$$(15b) \qquad \Delta t_{n+1} \leqq t_n \Delta t_n / t_{n-1}, \qquad \lambda_n = \Delta t_n/\Delta x.$$

We may justify these conditions by means of the following argument: Let us seek a solution to Eq. (13) in the form $U_j^n = g_n \exp(ij\alpha\Delta x)$, with g_0, g_1 known values. Introducing this expression into Eq. (13) we obtain the difference equation

$$(16) \qquad \begin{aligned} &(t_{n+1}/\Delta t_{n+1} - \tfrac{1}{2})(g_{n+1} - g_n) - (t_n/\Delta t_n - \tfrac{1}{2})(g_n - g_{n-1}) \\ &\quad + 2\,\frac{(\lambda_n + \lambda_{n+1})}{\Delta x}\, g_n \sin^2(\alpha\Delta x/2) = 0. \end{aligned}$$

We wish to find conditions under which the terms g_n will remain bounded as n tends to infinity. If we define the vector

$$\vec{h}_n = \begin{bmatrix} g_{n+1} \\ g_n \end{bmatrix}, \qquad n = 0, 1, \cdots,$$

then Eq. (16) can be rewritten in the form

$$(17) \qquad \vec{h}_{n+1} = C_n \vec{h}_n,$$

where

$$C_n = \begin{bmatrix} 1 + (\beta - 2\gamma)/\delta & -\beta/\delta \\ 1 & 0 \end{bmatrix},$$

$$\delta = t_{n+1}/\Delta t_{n+1} - 1/2, \qquad \beta = t_n/\Delta t_n - 1/2,$$

$$\gamma = \frac{\lambda_n + \lambda_{n+1}}{\Delta x} \sin^2(\alpha\Delta x/2).$$

It is easily seen that the conditions (15a) and (15b) guarantee that the spectral radius $\rho(C_n)$ of C_n is not greater than 1. By Eq. (17) $\overrightarrow{h_n} = \prod_{j=1}^{n} C_j \overrightarrow{h_0}$, whence stability of the scheme (13), or equivalently boundedness of the vectors $\overrightarrow{h_n}$ uniformly in n, will be guaranteed, if

$$\prod_{j=0}^{n} ||C_j|| \leqq M, \tag{18}$$

for some constant M independent of n, where $||\cdot||$ is the maximum matrix norm. However, since for any C_n, $\rho(C_n) \leqq ||C_n||$, we find from (18) that $\prod_{j=0}^{n} \rho(C_j) \leqq M$, for which conditions (15a, b) are sufficient.

The results of numerical experiments justify the use of (15a, b) as stability conditions, as well as the use of the scheme (13) with time steps varying in a manner dictated by this condition. Thus, in particular, the violation of either (15a) or (15b) has led to instabilities in our numerical experiments.

In Table I are listed the values of the exact solution

$$u(x, t) = J_0(2t^{1/2}) \cos x$$

to Problem A, with the conditions

$$u(0, t) = u(2\pi, t) = J_0(2t^{1/2}) \tag{19}$$

on the interval $[0, 2\pi]$ for the times $t = 1.99958, 100{,}028$ and points $x_j = j\Delta x$, $\Delta x = 2\pi/100$, $j = 0, \cdots, 50$. Here J_0 denotes the "zeroth" Bessel function.

Among possible mesh lengths Δt_n satisfying (15a), (15b) we have used the sequence defined by

$$\lambda_n \lambda_{n-1} = t_{n-1} - \alpha, \qquad \alpha > 0,$$

or, equivalently,

$$\begin{aligned} \Delta t_n &= n(\Delta x^2/2), && n = \text{odd}, \\ &= n(\Delta x^2/2) - 2\alpha, && n = \text{even}, \end{aligned} \tag{20}$$

for α a small parameter of order $O(\Delta x^2)$. Thus,

$$\begin{aligned} t_n &= \tfrac{1}{4}n(n+1)\Delta x^2 - (n-1)\alpha, && n = \text{odd}, \\ &= \tfrac{1}{4}n(n+1)\Delta x^2 - n\alpha, && n = \text{even}. \end{aligned}$$

A second choice of mesh lengths Δt_n is that of fixed $\Delta t_n = \Delta t$. In this case the initially parabolic behavior of (1) dictates the choice

$$\Delta t = \tfrac{1}{2}\Delta x^2. \tag{21}$$

In Table II we show the relative error

$$R = \frac{\text{computed value} - \text{real value}}{\text{real value}}$$

for results obtained by (20) and (21) for $t = 1.99958$, $x = 2\pi/100$, $\alpha = 10^{-4}$. Due to symmetry, only the values for $0 \leqq x \leqq \pi$ are shown. We see that the scheme (20) yields consistently more accurate results although it was obtained in 46 time steps, versus 1014 time steps in the case of (21). We note too that only the last time step in using

TABLE I. *Exact Solution*

x	$u(x, 1.99958)$	$u(x, 100.028)$
0	-1.96430×10^{-1}	1.66835×10^{-1}
6.28319×10^{-2}	-1.96042×10^{-1}	1.66506×10^{-1}
1.25664×10^{-1}	-1.94881×10^{-1}	1.65519×10^{-1}
1.88496×10^{-1}	-1.92950×10^{-1}	1.63880×10^{-1}
2.51327×10^{-1}	-1.90259×10^{-1}	1.62593×10^{-1}
3.14159×10^{-1}	-1.86816×10^{-1}	1.58669×10^{-1}
3.76991×10^{-1}	-1.82636×10^{-1}	1.55119×10^{-1}
4.39823×10^{-1}	-1.77735×10^{-1}	1.50957×10^{-1}
5.02655×10^{-1}	-1.73133×10^{-1}	1.46498×10^{-1}
5.65487×10^{-1}	-1.65851×10^{-1}	1.40863×10^{-1}
6.28319×10^{-1}	-1.58915×10^{-1}	1.34972×10^{-1}
6.91150×10^{-1}	-1.51352×10^{-1}	1.28548×10^{-1}
7.53982×10^{-1}	-1.43191×10^{-1}	1.21617×10^{-1}
8.16814×10^{-1}	-1.34465×10^{-1}	1.14206×10^{-1}
8.79646×10^{-1}	-1.25209×10^{-1}	1.06345×10^{-1}
9.42478×10^{-1}	-1.15459×10^{-1}	9.80630×10^{-2}
1.00531	-1.05252×10^{-1}	8.93946×10^{-2}
1.06814	-9.46308×10^{-2}	8.03733×10^{-2}
1.13097	-8.36357×10^{-2}	7.10348×10^{-2}
1.19381	-7.23106×10^{-2}	6.14160×10^{-2}
1.25664	-6.07001×10^{-2}	5.15548×10^{-2}
1.31947	-4.88501×10^{-2}	4.14901×10^{-2}
1.38230	-3.68073×10^{-2}	3.12617×10^{-2}
1.44513	-2.46192×10^{-2}	2.09099×10^{-2}
1.50796	-1.23339×10^{-2}	1.04756×10^{-2}
1.57080	-5.03517×10^{-13}	4.27655×10^{-13}
1.63363	1.23339×10^{-2}	-1.04756×10^{-2}
1.69646	2.46192×10^{-2}	-2.09099×10^{-2}
1.75929	3.68073×10^{-2}	-3.12617×10^{-2}
1.82212	4.88501×10^{-2}	-4.14901×10^{-2}
1.88496	6.07001×10^{-2}	-5.15548×10^{-2}
1.94779	7.23106×10^{-2}	-6.14160×10^{-2}
2.01062	8.36357×10^{-2}	-7.10348×10^{-2}
2.07345	9.46308×10^{-2}	-8.03733×10^{-2}
2.13628	1.05252×10^{-1}	-8.93946×10^{-2}
2.19911	1.15459×10^{-1}	-9.80630×10^{-2}
2.26195	1.25209×10^{-1}	-1.06345×10^{-1}
2.32478	1.34465×10^{-1}	-1.14206×10^{-1}
2.38761	1.43191×10^{-1}	-1.21617×10^{-1}
2.45044	1.51352×10^{-1}	-1.28548×10^{-1}
2.51327	1.58915×10^{-1}	-1.34972×10^{-1}
2.57611	1.65851×10^{-1}	-1.40863×10^{-1}
2.63894	1.72133×10^{-1}	-1.46198×10^{-1}
2.70177	1.77735×10^{-1}	-1.50957×10^{-1}
2.76460	1.82636×10^{-1}	-1.55119×10^{-1}
2.82743	1.86816×10^{-1}	-1.58669×10^{-1}
2.89027	1.90259×10^{-1}	-1.61593×10^{-1}
2.95310	1.92950×10^{-1}	-1.63880×10^{-1}
3.01593	1.94881×10^{-1}	-1.65519×10^{-1}
3.07876	1.96042×10^{-1}	-1.66506×10^{-1}
3.14159	1.96430×10^{-1}	-1.66835×10^{-1}

TABLE II. *Relative Errors for* $t = 1.99958$

x	*Scheme* (20) ($\alpha = 10^{-4}$) $R \times 10^2$	*Scheme* (21) $R \times 10^2$
0	0	0
6.28319×10^{-2}	-3.74205×10^{-4}	1.56451×10^{-3}
1.25664×10^{-1}	1.05931×10^{-3}	3.02897×10^{-3}
1.88496×10^{-1}	2.33711×10^{-3}	4.42017×10^{-3}
2.51327×10^{-1}	3.48479×10^{-3}	5.76665×10^{-3}
3.14159×10^{-1}	4.47864×10^{-3}	7.09625×10^{-3}
3.76991×10^{-1}	5.34131×10^{-3}	8.44004×10^{-3}
4.39823×10^{-1}	6.04008×10^{-3}	9.82903×10^{-3}
5.02655×10^{-1}	6.59408×10^{-3}	1.12993×10^{-2}
5.65487×10^{-1}	6.95875×10^{-3}	1.28979×10^{-2}
6.28319×10^{-1}	7.14686×10^{-3}	1.46707×10^{-2}
6.91150×10^{-1}	7.09633×10^{-3}	1.66889×10^{-2}
7.53982×10^{-1}	6.80748×10^{-3}	1.90225×10^{-2}
8.16814×10^{-1}	6.18842×10^{-3}	2.17873×10^{-2}
8.79646×10^{-1}	5.21147×10^{-3}	2.51170×10^{-2}
9.42478×10^{-1}	3.72714×10^{-3}	2.92034×10^{-2}
1.00531	1.63963×10^{-3}	3.43399×10^{-2}
1.06814	-1.33291×10^{-3}	4.09176×10^{-2}
1.13097	-5.47894×10^{-3}	4.96085×10^{-2}
1.19381	-1.14628×10^{-2}	6.14974×10^{-2}
1.25664	-2.02631×10^{-2}	7.84993×10^{-2}
1.31947	-3.41116×10^{-2}	1.04556×10^{-1}
1.38230	-5.78567×10^{-2}	1.48721×10^{-1}
1.44513	-1.06585×10^{-1}	2.38161×10^{-1}
1.50796	-2.54662×10^{-1}	5.09119×10^{-1}
1.57080	$-7.31755 \times 10^{+9}$	1.33469×10^{10}
1.63363	3.44220×10^{-1}	-5.82442×10^{-1}
1.69646	1.96102×10^{-1}	-3.11666×10^{-1}
1.75929	1.47313×10^{-1}	-2.22408×10^{-1}
1.82212	1.23475×10^{-1}	-1.78469×10^{-1}
1.88496	1.09513×10^{-1}	-1.52719×10^{-1}
1.94779	1.00566×10^{-1}	-1.36119×10^{-1}
2.01062	9.44130×10^{-2}	-1.24714×10^{-1}
2.07345	9.00611×10^{-2}	-1.16532×10^{-1}
2.13628	8.68571×10^{-2}	-1.10538×10^{-1}
2.19911	8.44986×10^{-2}	-1.06111×10^{-1}
2.26195	8.27134×10^{-2}	-1.02805×10^{-1}
2.32478	8.13898×10^{-2}	-1.00303×10^{-1}
2.38761	8.03890×10^{-2}	-9.84168×10^{-2}
2.45044	7.96650×10^{-2}	-9.70460×10^{-2}
2.51327	7.91361×10^{-2}	-9.61149×10^{-2}
2.57611	7.87805×10^{-2}	-9.55339×10^{-2}
2.63894	7.85471×10^{-2}	-9.52043×10^{-2}
2.70177	7.84207×10^{-2}	-9.50344×10^{-2}
2.76460	7.83694×10^{-2}	-9.49552×10^{-2}
2.82743	7.83575×10^{-2}	-9.49212×10^{-2}
2.89027	7.83575×10^{-2}	-9.49076×10^{-2}
2.95310	7.83575×10^{-2}	-9.49027×10^{-2}
3.01593	7.83575×10^{-2}	-9.49008×10^{-2}
3.07876	7.83575×10^{-2}	-9.49003×10^{-2}
3.14159	7.83575×10^{-2}	-9.49001×10^{-2}

TABLE III. *Relative Errors for* $t = 100.028$

x	*Scheme* (20) ($\alpha = 10^{-4}$) $R \times 10^2$	*Scheme* (23) $R \times 10^2$
0	-2.18056×10^{-9}	0
6.28319×10^{-2}	5.12180×10^{-4}	2.19268×10^{-1}
1.25664×10^{-1}	7.49287×10^{-4}	4.38328×10^{-1}
1.88496×10^{-1}	9.91638×10^{-4}	6.58115×10^{-1}
2.51327×10^{-1}	1.24254×10^{-3}	8.79622×10^{-1}
3.14159×10^{-1}	1.50529×10^{-3}	1.10394
3.76991×10^{-1}	1.78542×10^{-3}	1.33230
4.39823×10^{-1}	2.08626×10^{-3}	1.56616
5.02655×10^{-1}	2.41603×10^{-3}	1.80726
5.65487×10^{-1}	2.77926×10^{-3}	2.05770
6.28319×10^{-1}	3.18920×10^{-3}	2.32017
6.91150×10^{-1}	3.65357×10^{-3}	2.59806
7.53982×10^{-1}	4.19272×10^{-3}	2.89578
8.16814×10^{-1}	4.82270×10^{-3}	3.21924
8.79646×10^{-1}	5.57887×10^{-3}	3.57646
9.42478×10^{-1}	6.49597×10^{-3}	3.97868
1.00531	7.64118×10^{-3}	4.44215
1.06814	9.09941×10^{-3}	4.99122
1.13097	1.10286×10^{-2}	5.66421
1.19381	1.36939×10^{-2}	6.52478
1.25664	1.75476×10^{-2}	7.68850
1.31947	2.34817×10^{-2}	9.38478
1.38230	3.35593×10^{-2}	$1.21504 \times 10^{+1}$
1.44513	5.39752×10^{-2}	$1.75954 \times 10^{+1}$
1.50796	1.15745×10^{-1}	$3.37697 \times 10^{+1}$
1.57080	$3.03904 \times 10^{+9}$	7.89280×10^{11}
1.63363	-1.32713×10^{-1}	$-3.04941 \times 10^{+1}$
1.69646	-7.09079×10^{-2}	$-1.43319 \times 10^{+1}$
1.75929	-5.04643×10^{-2}	-8.90710
1.82212	-4.03224×10^{-2}	-6.16978
1.88496	-3.43518×10^{-2}	-4.50992
1.94779	-3.04027×10^{-2}	-3.39087
2.01062	-2.76406×10^{-2}	-2.58265
2.07345	-2.55945×10^{-2}	-1.97029
2.13628	-2.40531×10^{-2}	-1.48993
2.19911	-2.28367×10^{-2}	-1.10327
2.26195	-2.18836×10^{-2}	-7.85955×10^{-1}
2.32478	-2.11012×10^{-2}	-5.21784×10^{-1}
2.38761	-2.04763×10^{-2}	-2.99554×10^{-1}
2.45044	-1.99492×10^{-2}	-1.11292×10^{-1}
2.51327	-1.95262×10^{-2}	4.88156×10^{-2}
2.57611	-1.91629×10^{-2}	1.85095×10^{-1}
2.63894	-1.88742×10^{-2}	3.00814×10^{-1}
2.70177	-1.86236×10^{-2}	3.98471×10^{-1}
2.76460	-1.84307×10^{-2}	4.79988×10^{-1}
2.82743	-1.82631×10^{-2}	5.46847×10^{-1}
2.89027	-1.81431×10^{-2}	6.00184×10^{-1}
2.95310	-1.80412×10^{-2}	6.40854×10^{-1}
3.01593	-1.79813×10^{-2}	6.69483×10^{-1}
3.07876	-1.79353×10^{-2}	6.86496×10^{-1}
3.14159	-1.79290×10^{-2}	6.92140×10^{-1}

(20) was done for very small $\Delta t_{46} = 2 - t_{45}$, which violates (15b) and introduces a matrix C_{45} in (17) whose spectral radius is greater than 1.

In the next section similar results are obtained by comparing (20) with a simple implicit scheme.

3. Other Numerical Schemes.

A. *A DuFort-Frankel Scheme.* The presence of the coefficient t in the Eq. (1) obviates the necessity of choosing extremely small Δt in order to meet the stability requirement (15a) of the explicit scheme examined above. Nevertheless, due to the initially parabolic nature of Eq. (1), we are led to seek an explicit, unconditionally stable difference scheme based on the method of DuFort and Frankel (see [3]). This is obtained by replacement of both sides of Eq. (1) by centered differences for Δt fixed, and replacement of the value of the solution at the point about which these differences are formed by the average of the values at the following and the previous time steps, and results in the scheme

$$U_i^{n+1}(\lambda + \tfrac{1}{2}) = U_i^{n-1}(\tfrac{1}{2} - \lambda) + \lambda(U_{i+1}^n + U_{i-1}^n), \tag{22}$$

for $\lambda = \Delta t/\Delta x$. By a simple calculation we find that this scheme is consistent with the equation $\lambda^2 u_{tt} + u_t - u_{xx} = 0$, whence consistency of Eq. (22) with Eq. (1) would be possible only if $t \equiv \lambda^2$, which is impossible.

B. *An Implicit Scheme.* The use of implicit schemes for the solution of hyperbolic equations is not as urgent as in the case of parabolic equations because stability requirements do not ordinarily demand that Δt be of order $O(\Delta x^2)$. In addition, care must be taken that such schemes are not "overstable" and do not artificially damp the solution (see [4]). Nevertheless it is of interest to examine a simple implicit scheme obtained by replacement of the term $(tu_t)_t$ by a centered difference and u_{xx} by a centered difference at the latest time step; this results in a scheme of the form

$$U_i^{n+1}(2\lambda^2 + (n + \tfrac{1}{2})\,\Delta t) - \lambda^2(U_{i+1}^{n+1} + U_{i-1}^{n+1}) = U_i^n(2n\Delta t) - (n - \tfrac{1}{2})\,\Delta t U_i^{n-1}, \tag{23}$$

with $\lambda = \Delta t/\Delta x$, for which the discretization error is

$$E = -\frac{\Delta t}{t}\int_0^t u_{4x}(x, \tau)\,d\tau + \Delta x^2(\lambda^2(u_{3t}/6 - u_{ttxx}/2) - u_{4x}/12) + O(\Delta x^3).$$

The leading term in E is of order $O(\Delta t)$ and dominates E. On the basis of arguments identical to those concerned with the explicit scheme we find that the scheme (23) is unconditionally stable for all values of λ. Nevertheless due to the presence of the leading term of E this scheme is not as accurate as the explicit scheme with variable time mesh, and yields results which are not satisfactory. In Table III we see the relative errors obtained by using the scheme (23) for fixed $\Delta t = \Delta x$ (in column 1) and the scheme (13) with mesh lengths given by (20) for the problem (19), at the time $t = 100$. The implicit scheme has now run for 1593 time steps, whereas the explicit scheme requires 319 time steps. We note that the phenomena of overstability noted by Zajac in [4] for the wave equation has not appeared in this or other experiments with implicit schemes for Eq. (1).

C. *A Method of Characteristics.* In addition to using difference schemes it is possible to solve (1) numerically by using the relation (7). To do this we choose some $\Delta x > 0$ and define a mesh of points (x_i, t_n) of intersection of the characteristics passing through the points $x_j = j\Delta x$, $j = 0, \pm 1, \ldots,$ of the initial line. By approximating the integral in (7) by a trapezoidal rule we have obtained results accurate to within an error of order $O(\Delta x^2)$.

Department of Mathematics
Tel Aviv University
Tel Aviv, Israel

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