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The Generalized Serial Test Applied to Expansions of Some Irrational Square Roots in Various Bases*

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Abstract. A brief summary is given of the application of the generalized serial test for randomness to the digits of irrational $\sqrt{n}$ in bases t where $2 \leqq n, t \leqq 15$. The results are consistent, except for a few aberrations, with the hypothesis of randomness of the digits.

This is a brief report on the application of the generalized serial test (see [1]) for randomness to the expansions of some irrational $\sqrt{n}$ in various bases. It can be considered as an extension of the work Good and Gover [1] have done with 10,000 binary digits of $\sqrt{2}$. I. J. Good (private communication) has remarked that some of the tests used in [4] are special cases of the generalized serial test.

The conclusion resulting from this study is that the results, except for a few aberrations given in Table 3, are consistent with the hypothesis of randomness of the digits of the square roots investigated. Since a total of about 2400 tests were made, perhaps the aberrations are not surprising, if one notes that $2400^{-1} = .0004$.

For completeness, a recapitulation is given (from [1]) of the generalized serial test for randomness as used here. Let a sequence of N digits in base t ($t = 2, 3, \cdots$) be given and let the sequence be circularized; i.e., the last digit is considered as being followed by the first digit. Let n_I be the number of occurrences of the ν-plet $I = (i_1, i_2, \cdots, i_\nu)$ in the circularized sequence. Define

$$\psi_0^2 = \psi_{-1}^2 = 0, \qquad \psi_\nu^2 = \frac{t^\nu}{N} \sum_I \left(n_I - \frac{N}{t^\nu}\right)^2,$$

and

$$\nabla^2 \psi_\nu^2 = \psi_\nu^2 - 2\psi_{\nu-1}^2 + \psi_{\nu-2}^2$$

for $\nu \geqq 1$. The distributions $\nabla^2 \psi_\nu^2$ are asymptotically chi-square with the number of degrees of freedom equal to $t^\nu - 2t^{\nu-1} + t^{\nu-2}$.

The authors [3], [4] have computed N digits of the fractional part of $\sqrt{n}$, base t, for $n = 2, 3, 5, 6, 7, 10, 11, 13, 14$, and 15 in accordance with Table 1.

This was accomplished by computing $88064 = 43 \cdot 2^{11}$ binary digits of the fractional part of $\sqrt{n}$ and then changing base. In the conversion from binary to base t, it can be shown that $N = 2^{11}[43 \log 2/\log t]$ digits are accurate, where [ ] denotes largest integer. (Note that for $t = 6$, $N = 2^{15}$.) Because of some minor technical difficulties, the last 1 or 2 digits may not be accurate. For this reason 88062 replaces 88064 in the title of [3].

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t	N	t	N
2	88046	11	24576
3	55296	12	22528
5	36864	13	22528
6	32768	14	22528
7	30720	15	20480
10	24576		

TABLE 1. *Number of computed digits of $\sqrt{n}$ in base t.*

The quantity $\nabla^2\psi_\nu^2$ is computed for each whole block of length $x\cdot 1000$ of $(\sqrt{n})_t$; i.e., for the blocks with digits: $1 \to x\cdot 1000$, $(x\cdot 1000 + 1) \to 2x\cdot 1000$, etc., and also for the entire sequence of N digits of $(\sqrt{n})_t$. The values of ν and x selected are listed in Table 2.

This paper will list only the aberrations observed; if the chi-square level (tail area) is less than .007, it is listed in Table 3. The remainder of the data has a χ^2 level greater than .007.

It should be noted that with the exception of $((13)^{1/2})_{12}$, the aberrations occur in the intermediate digits and disappear when a larger sample is examined.

Remark. Since the work of Good and Gover [1] is referred to several times, it should also be mentioned that their suggestion for calculating $\sqrt{2}$ (or $\sqrt{m}$ in [2]) is generalized in [5].

Acknowledgment. The computations were made on the MANIAC II computer in our laboratory.

Appendix. Certain errors are noted in the report of the values of $\nabla^2\psi_\nu^2$ $(1 \leqq \nu \leqq 10)$ for 10,000 binary digits as given in Table 1 of Good and Gover [1]. These errors apparently arose from two sources: rounding error in the value of $\nabla^2\psi_\nu^2$ itself and the fact that their 10,000th binary digit was obtained as 0, whereas in fact it is a 1. For $\nu = 8$, block 5 should be 37.2 and block 8 should be 78.8. The entries in block 10 should be replaced by the sequence (starting with $\nu = 1$): 1.3, 0.0, 0.7, 10.3,

t	ν	x
2	1, 2, $\cdots$, 10	10
3, 5	1, 2, 3, 4	10
6, 7, 10	1, 2, 3	5
11, 12, 13, 14, 15	1, 2	5

TABLE 2. *Values of ν and x for the various bases t.*

n	t	ν	Block	Level
2	2	2	10001 → 20000	.0058
2	14	1	20001 → 25000	.0053
3	3	1	10001 → 20000	.00025
3	7	1	20001 → 25000	.0015
5	15	1	total	.004
7	7	1	25001 → 30000	.006
10	5	2	30001 → 40000	.005
11	5	2	10001 → 20000	.0066
13	3	2	40001 → 50000	.00027
13	12	1	total	.0015
14	6	2	15001 → 20000	.0023

TABLE 3. *Aberrations.*

6.1, 16.6, 22.1, 67.2, 130, and 236. The entries in the block marked "whole" should read: **.4**, **.5**, 3.4, **2.6**, 9.1, 19.8, 51.5, 58.9, **128**, 242. The four which are boldface were correct in their original table.

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