MATHEMATICS OF COMPUTATION, VOLUME 24, NUMBER 112, OCTOBER 1970

A Note on Solid Partitions

By Donald E. Knuth

Abstract. The problem of enumerating partitions which satisfy a given partial order relation is reduced to the problem of enumerating permutations satisfying that relation. This theorem is applied to the enumeration of solid partitions; existing tables of solid partitions are extended.

A *plane partition* of n is an arrangement of nonnegative integers

$$\begin{matrix} n_{00} & n_{01} & n_{02} & \cdots \\ n_{10} & n_{11} & n_{12} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

which sum to n, and which are nonincreasing in both rows and columns:

$$n_{ij} \geqq n_{i(j+1)}, \qquad n_{ij} \geqq n_{(i+1)j}.$$

For example,

$$\begin{matrix} 5 & 4 & 2 & 1 & 1 \\ 3 & 2 & & & \\ 2 & 2 & & & \end{matrix} \tag{1}$$

is a plane partition of 22. (Blank entries are zero.)

Let $b(n)$ denote the total number of plane partitions of n. In 1912, Major Percy A. MacMahon triumphantly announced [5] a proof of the remarkable formula

$$1 + b(1)z + b(2)z^2 + b(3)z^3 + \cdots = 1/(1 - z)(1 - z^2)^2(1 - z^3)^3 \cdots$$

which he had previously verified by numerous empirical calculations, but which he had been unable to prove six months earlier [4]. In his enthusiasm he concluded his paper by saying, "We have evidently, potentially, the complete solution of the problem of three-dimensional partition, and it remains to work it out and bring it to the same completeness as has been secured in this Part for the problem in two dimensions. This will form the subject of Part VII of this Memoir." The problem of enumerating three-dimensional ("solid") partitions has never been resolved, however, and Part VII of MacMahon's classic Memoir never appeared. No constructive proof of MacMahon's formula for the two-dimensional case was known until 1969 [2].

MacMahon at one time conjectured that, if $c(n)$ is the number of solid partitions of n, the formula

$$1 + c(1)z + c(2)z^2 + c(3)z^3 + \cdots = 1/(1 - z)(1 - z^2)^3(1 - z^3)^6(1 - z^4)^{10} \cdots$$

Received November 5, 1969, revised February 18, 1970.

AMS 1969 *subject classifications.* Primary 1048, 0510; Secondary 0620.

Key words and phrases. Plane partitions, solid partitions, partially-ordered partitions, partially-ordered permutations, index of permutation, backtracking.

TABLE 1

n	$c(n)$	$c'(n)$	$d(n)$	$d'(n)$	$e(n)$	$e'(n)$
0	1	1	1	1	0	0
1	1	1	0	0	1	1
2	4	4	2	2	3	3
3	10	9	5	4	6	5
4	26	22	12	9	10	7
5	59	46	24	16	15	9
6	140	102	56	35	20	10
7	307	206	113	63	26	12
8	684	427	248	129	34	16
9	1464	841	503	234	46	21
10	3122	1658	1043	445	68	29
11	6500	3173	2080	798	97	32
12	13426	6038	4169	1458	120	22
13	27248	11251	8145	2568	112	−2
14	54804	20807	15897	4561	23	−39
15	108802	37907	30545	7924	−186	−67
16	214071	68493	58402	13770	−496	−48
17	416849	122338	110461	23584	−735	64
18	805124	216819	207802	40301	−531	277
19	1541637	380637	387561	68097	779	576
20	2930329	663417	718875	114646	3894	848
21	5528733	1147033	1324038	191336	9323	981
22	10362312	1969961	2425473	317893	16472	771
23	19295226	3359677	4416193	524396	23056	40
24	35713454	5694592	7999516	861054	23850	−1498
25	65715094	9592063	14411507	1405130	10116	−4276
26	120256653	16065593	25837198	2282651	−31613	−8745
27	218893580	26756430	46092306	3688254	−120720	−15062
28	396418699	44328414	81851250	5933463	−283202	−21702

might be valid. (See [6, pp. 175–176].) But recently-computed tables [1] show that this formula gives the wrong answer for $n = 6$. In order to find out the true nature of $c(n)$, the only known approach is to prepare tables, by brute force, and to examine these tables with the hope of finding some pattern.

The purpose of this note is to describe a slightly sophisticated computational method for extending the existing tables of $c(n)$, by showing how the coefficients $d(n)$ in the formula

$$\begin{aligned} &1 + c(1)x + c(2)x^2 + \cdots \\ &\quad = (1 + d(1)x + d(2)x^2 + \cdots)/(1 - x)(1 - x^2)(1 - x^3) \cdots \end{aligned}$$

can be computed for small n. The computational technique we will discuss was essentially used by MacMahon [4] in his examination of the two-dimensional case, and later also in his preliminary study of solid partitions [5, pp. 360–373]. Since the ideas apply under fairly general circumstances, we will reformulate MacMahon's method for the case of arbitrary partially-ordered sets.

Theory. Let P be a set of elements that are *partially-ordered* by the relation $\prec$ and *well-ordered* by the total order relation $<$. We will assume that the partial order is embedded in the total order, in the sense that

$$x \prec y \quad \text{implies} \quad x < y. \tag{2}$$

By a *labelling* of P we mean a function $n(x)$ taking the elements of P into the set N of nonnegative integers, satisfying the two conditions

(L1) $x \prec y$ implies $n(x) \geqq n(y)$.

(L2) Only finitely many x have $n(x) > 0$.

We wish to count the number of labellings of P satisfying certain restrictions.

It is not difficult to show that there is a one-to-one correspondence between labellings of P and pairs of sequences

$$\begin{gathered} n_1 \geqq n_2 \geqq \cdots \geqq n_m, \\ x_1, x_2, \cdots, x_m, \end{gathered} \tag{3}$$

where $m \geqq 0$, the n_i are positive integers, and the x_i are distinct elements of P, subject to the following two conditions:

(S1) For $1 \leqq j \leqq m$ and $x \in P$, $x \prec x_j$ implies $x = x_i$ for some $i < j$.

(S2) $x_i > x_{i+1}$ implies $n_i > n_{i+1}$, for $1 \leqq i < m$.

To construct such a correspondence, we may proceed as follows. Given a labelling, let $n_1, n_2, \cdots, n_m$ be the nonzero labels in nonincreasing order, and let x_i be such that $n(x_i) = n_i$; the x's are arranged so that we put x before y when $n(x) = n(y)$ and $x < y$. Then (S1) is satisfied, since $x \prec x_j$ implies that $n(x) \geqq n(x_j)$ and $x < x_j$; and (S2) is satisfied, since $n_i = n_{i+1}$ implies that $x_i < x_{i+1}$. Conversely, given sequences (3) satisfying (S1) and (S2) we define a labelling by setting $n(x_i) = n_i$ for $1 \leqq i \leqq m$ and $n(x) = 0$ for all other x. Clearly (L2) is satisfied, and so is (L1), for if $x \prec y$ we have either $n(x) \geqq 0 = n(y)$ or $y = x_j$, $x = x_i$, $i < j$, $n(x) = n_i \geqq n_j = n(y)$. It is easy to verify that these two constructions are inverses of each other, since (S2) and the relation $n(x_i) = n_i$ uniquely defines the sequence of x's.

For example, let P be the set of points $\{(i, j) \mid i, j \in N\}$ of the plane, subject to the partial order

$$(i, j) \preceq (i', j'), \quad \text{if and only if} \quad i \leqq i' \quad \text{and} \quad j \leqq j',$$

and the well-order

$$(i, j) < (i', j'), \quad \text{if and only if} \quad i < i' \quad \text{or} \quad (i = i' \quad \text{and} \quad j < j').$$

A labelling of this set P is essentially a plane partition; for example, (1) has $n((0, 0)) = 5$, $n((0, 1)) = 4, \cdots, n((0, 5)) = 0$, etc. The sequences (3) corresponding to (1) are

$$\begin{array}{l} n_1, \cdots, n_m = \quad 5 \;\;,\;\; 4 \;\;,\;\; 3 \;\;,\;\; 2 \;\;,\;\; 2 \;\;,\;\; 2 \;\;,\;\; 2 \;\;,\;\; 1 \;\;,\;\; 1 \;\;; \\ x_1, \cdots, x_m = (0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (2, 1), (0, 3), (0, 4). \end{array} \tag{4}$$

We are interested primarily in cases where P is countably infinite, and when the labels sum to a given number n:

$$\sum_{x \in P} n(x) = n. \tag{5}$$

Let us say that such a labelling is a *P-partition* of n. For this case we can refine the above correspondence in order to minimize the dependence of the n's on the x's. Every P-partition of n corresponds uniquely to a pair of sequences

$$\begin{gathered} n_1 \geqq n_2 \geqq n_3 \geqq \cdots \geqq 0, \qquad \sum n_i = n; \\ x_1, x_2, \cdots, x_m; \end{gathered} \tag{6}$$

where $m \geqq 0$, the n_i form an infinite sequence of nonnegative integers, the x_i are distinct elements of P, and conditions (S1), (S2), (S3), and (S4) hold, where

(S3) Either $m = 0$ or $n_m > n_{m+1}$;

(S4) If $m > 0$, there exists $x \in P$ such that $x < x_m$ and $x \neq x_i$ for $1 \leqq i \leqq m$.

Such pairs of sequences correspond uniquely to pairs of sequences (3) of the former type, when P is infinite, if we extend the n's by adding infinitely many zeroes, and if we contract the x's, if necessary, by removing x_m if it is the least element of $P - \{x_1, \cdots, x_{m-1}\}$. For example, (4) corresponds to

$$\begin{gathered} 5\ ,\ 4\ ,\ 3\ ,\ 2\ ,\ 2\ ,\ 2\ ,\ 2\ ,\ 1,\ 1,\ 0,\ 0,\ 0,\ \cdots; \\ (0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (2, 1). \end{gathered} \tag{7}$$

It is not difficult to verify that this process defines a one-to-one correspondence.

Let us say that a sequence of x's satisfying conditions (S1) and (S4) is a *topological sequence*, since such sequences arise in connection with "topological sorting" (see [3, pp. 258–268]). The *index* of a topological sequence is defined to be

$$\sum \{j \mid 1 \leqq j < m \quad \text{and} \quad x_j > x_{j+1}\} + m. \tag{8}$$

Given a topological sequence, every sequence $n_1 \geqq n_2 \geqq \cdots$ of nonnegative integers satisfying (S2) and (S3) corresponds uniquely to a sequence $p_1 \geqq p_2 \geqq \cdots$ obtained by subtracting unity from $n_1, \cdots, n_j$, in turn, for each j such that $x_j > x_{j+1}$ or $j = m$.

For example, the topological sequence

$$x_1, \cdots, x_7 = (0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (2, 1) \tag{9}$$

has index $3 + 7$; and the sequence p_1, p_2, p_3; corresponding to (7) is obtained by subtracting 1 from n_1, n_2, n_3, then subtracting 1 from each of $n_1, n_2, \cdots, n_7$:

$$p_1, p_2, p_3 = 3, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, \cdots. \tag{10}$$

Note that the sum of the p_i is the sum of the n_i diminished by the index; this is the only interdependence between the x's, the p's, and n.

In summary, our observations have the following consequence:

THEOREM. *Let P be an infinite partially-ordered set. There is a one-to-one correspondence between P-partitions of n, and ordered pairs* $(\mathbf{x}, \mathbf{p})$ *where* $\mathbf{x} = x_1, \cdots, x_m$ *is a topological sequence and* $\mathbf{p} = p_1, p_2, \cdots$ *is a partition* (*in the ordinary sense*) *of* $n - k$, k *the index of* $\mathbf{x}$.

Since the generating function for ordinary partitions is $1/(1 - z)(1 - z^2) \cdots$, we have

COROLLARY. *Let P be an infinite partially-ordered set; let $s(n)$ be the number of P-partitions of n, and let $t(k)$ be the number of topological sequences of P having index k. Then,*

$$1 + s(1)z + s(2)z^2 + \cdots$$
$$= (1 + t(1)z + t(2)z^2 + \cdots)/(1-z)(1-z^2)(1-z^3)\cdots.$$

Consequently, we can enumerate P-partitions by enumerating only the topological sequences, and the latter are easier to enumerate. Note that the definition of topological sequence involves an assumed well ordering $<$; but the corollary shows that *the number of topological sequences of given index is independent of the well ordering*. Therefore we can choose any convenient well-ordering relation (e.g., the lexicographic order in our examples) when doing the enumeration.

An Application. To find the number $c(n)$ of solid partitions of n, let P be the set of three-dimensional lattice points

$$\{(i, j, k) \mid i, j, k \in N\},$$

with the partial ordering

$$(i, j, k) \preceq (i', j', k'),$$

if and only if

$$i \leqq i', \qquad j \leqq j', \qquad k \leqq k';$$

and with the lexicographic well-ordering

$$(i, j, k) < (i', j', k'),$$

if and only if

$$i < i' \quad \text{or} \quad (i = i' \text{ and } j < j') \quad \text{or} \quad (i = i', j = j', \text{ and } k < k').$$

If $d(n)$ is the corresponding number of topological sequences having index n, we have the formula

$$1 + c(1)z + c(2)z^2 + \cdots$$
$$= (1 + d(1)z + d(2)z^2 + \cdots)/(1-z)(1-z^2)(1-z^3)\cdots.$$

The following ALGOL program enumerates $d(0), d(1), \cdots, d(n)$, using a fairly standard backtracking method:

```
procedure count (n, d); integer n; integer array d;
begin comment d[i] is set to the number of topological sequences of index i, for
0 ≦ i ≦ n;
   integer array cc[0 : n, 0 : n];   comment number of columns in given plane, row;
   integer array rr[0 : n];          comment number of rows in given plane;
   integer pp;                       comment number of planes;
   integer k;                        comment recursion depth;
   integer array row[0 : n];         comment row of given move;
   integer array plane[0 : n];       comment plane of given move;
```

```
integer array index[0 : n];      comment partial sum for index;
integer p, r, c;                 comment current plane, row, column;
integer t;                       comment temporary storage;
for p := 0 step 1 until n do
begin rr[p] := 0; for r := 0 step i until n do cc[p, r] := 0 end;
   for k := 1 step 1 until n do d[k] := 0;
   pp := k := index[0] := plane[0] := row[0] := 0; d[0] := 1;
   up : k := k + 1; p := pp; r := 0;
   try : c := cc[p, r];
   if p > 0 then if cc[p - 1, r] ≦ c then go to again;
   if r > 0 then if cc[p, r - 1] ≤ c then go to again;
   if p < plane[k - 1] then go to less;
   if p = plane[k - 1] then if r < row[k - 1] then go to less;
   t := index[k - 1]; go to move;
   less: t := index[k - 1] + k - 1;
   if k + t > n then go to nope;
   move: begin comment We have now decided to choose the point (p, r, c) as
      the kth element of the topological sequence; end;
   index[k] := t;
   if p + r > 0 then d[k + t] := d[k + t] + 1;
   if t + k ≧ n then go to again;
   if r + c = 0 then pp := pp + 1;
   if c = 0 then rr[p] := rr[p] + 1;
   cc[p, r] := c + 1;
   plane[k] := p; row[k] := r; go to up;
   again: if r > 0 then begin r := r - 1; go to try end;
   if p > 0 then begin p := p - 1; r := rr[p]; go to try end;
   nope: k := k - 1;
   if k > 0 then begin p := plane[k]; r := row[k];
                     c := cc[p, r] - 1; cc[p, r] := c;
              if c = 0 then rr[p] := rr[p] - 1;
              if r + c = 0 then pp := pp - 1;
              go to again end;
end count.
```

Simple modifications to this program make it possible to count $c'(n)$, the number of solid partitions restricted to at most two planes. Table 1 shows the values of $c(n)$, $c'(n)$, $d(n)$, $d'(n)$, $e(n)$ and $e'(n)$, for $n \leqq 28$, where the exponents $e(n)$ are defined by the relation

$$1 + c(1)z + c(2)z^2 + \cdots = 1/(1-z)^{e(1)}(1-z^2)^{e(2)}(1-z^3)^{e(3)} \cdots .$$

The numbers $d'(n)$, $e'(n)$ are defined similarly. Unfortunately no pattern is evident in these numbers, so this table should suffice to disprove most simple conjectures about solid partitions.

Institute for Defense Analyses
Princeton, New Jersey 08540

1. A. O. L. Atkin, P. Bratley, I. G. MacDonald & J. K. S. McKay, "Some computations for m-dimensional partitions," *Proc. Cambridge Philos. Soc.*, v. 63, 1967, pp. 1097–1100. MR **36** #124.
2. Edward A. Bender & Donald E. Knuth, "Enumeration of plane partitions," *J. Combinatorial Theory*. (To appear.)
3. Donald E. Knuth, *The Art of Computer Programming*. Vol. 1, Addison-Wesley, Reading, Mass., 1968.
4. Major P. A. MacMahon, "Memoir on the theory of the partitions of numbers. V: Partitions in two-dimensional space," *Philos. Trans. Roy. Soc. London Ser.* A, v. 211, 1912, pp. 75–110.
5. Major P. A. MacMahon, "Memoir on the theory of the partitions of numbers. VI: Partitions in two-dimensional space, to which is added an adumbration of the theory of the partitions in three-dimensional space," *Philos. Trans. Roy. Soc. London Ser.* A, v. 211, 1912, pp. 345–373.
6. Major P. A. MacMahon, *Combinatory Analysis*. Vol. 2, Cambridge Univ. Press, Cambridge, 1916; reprint, Chelsea, New York, 1960.