advanced students. Nevertheless, it should serve equally well as a lucid introduction to this subject in other school systems, such as that in this country.

Volume 1 provides in the space of nine chapters a very readable introduction to such topics as the use of hand-calculating machines; rounding errors; flow charts; curve tracing and the graphical solution of equations; iterative methods for the solution of equations in one or more variables; differences of a polynomial and their application in locating and correcting tabular errors; solution of linear simultaneous equations by the methods of elimination, triangular decomposition, and Gauss-Seidel iteration; numerical solution of polynomial equations; linear interpolation; and numerical integration by the trapezoidal, mid-ordinate, and Simpson rules.

Volume 2 treats equally clearly and concisely in eight chapters such topics as the interpolation formulas of Gregory-Newton, Bessel, and Everett (including throwback); inverse interpolation; Lagrange interpolation (including Aitken's method); numerical integration using differences; numerical differentiation; numerical solution of ordinary differential equations of the first and second orders; curve fitting by least squares; and the summation of slowly convergent series by Euler's method and the Euler-Maclaurin formula.

Each volume is well supplied with illustrative examples as well as with exercises (and answers) for the student. Also included are short bibliographies of material for further reading and study.

J. W. W.

66[2.10].—F. G. Lether & G. L. Wise, *Ralston Quadrature Constants*, Tables appearing in the microfiche section of this issue.

An n-point quadrature rule of the form

$$\int_{-1}^{1} f(x)\,dx \simeq \sum_{j=2}^{n-1} a_j f(x_j) + a_1(f(-1) - f(1))$$

which is of polynomial degree $2n - 4$ is termed a Ralston Quadrature Rule. A list of weights and abscissas for $n = 3(1)9$ is given, together with coefficients e_1 and e_2 which may be used to bound the approximation error in terms of bounds on the first or second derivatives of $f(x)$.

Rules of this type may be used in cytolic integration. Because $a_1 = -a_n$ and $x_1 = -x_n = -1$, if the integration interval is divided into N equal panels and the n point rule used in each, only $N(n - 2) + 2$ distinct function values are required for a result of polynomial degree $2n - 4$. This may be compared with $N(n - 2)$ distinct function values using a Gauss Legendre formula to obtain a result of polynomial degree $2n - 5$.

The weights and abscissas are given to between nine and eleven significant figures. The authors also list the coefficients in the polynomials whose roots are the abscissas. This information may be useful both to users and to theoreticians, and I am happy to see its inclusion with the tables.

J. N. L.

ON [illegible] QUADRATURE CONSTANTS

Frank G. Lether and Donald L. [illegible]

INTRODUCTION. In 1852 Lobatto [8] developed a class of quadrature rules of the form

$$(1.1)\qquad \int_{-1}^{1} f(x)dx = w_1[f(-1)+f(1)] + \sum_{j=2}^{n-1} w_j f(x_j)$$

which have precision (2n-3) where $n \geq 3$. As noted by Lobatto, a composite quadrature rule based on (1.1) has the property that the contributions at the endpoints of each subinterval coalesce. We also have the advantage of a rule of one higher degree on each subinterval at the actual cost of only two more functional evaluations over the whole interval of integration. A composite rule based on the n-point rule (1.1) requires only two more functional evaluations than a composite rule based on a (n-2)-point Gauss-Legendre rule. The former Lobatto rule will have precision (2n-3) while the latter Legendre rule will have precision (2n-5).

The cases n = 3 and n = 4 are discussed in detail by Lobatto. Lobatto shows that the error term in (1.1) has a derivative of two higher order than the (n-2)-point Gauss-Legendre analogue. For this reason Lobatto remarks that composite rules based on (1.1)

The previous objection can be circumvented. In this paper we give the abscissas and weights in (1.1) for n = [illegible]. For each of these rules we calculate the L_1 norm of the Peano kernel. These norms can be used to determine an error bound which depends on the first or second derivative of the integrand [2, p. 81, p. 84]. The norms of the Peano kernels for the n-point Lobatto rule (1.1) are less than or equal to those of the (n−1)-point Gauss-Legendre analogue.

We note that (1.1) can also be used to generate composite cross product rules over a hypercube with even greater profit. When the hypercube is subdivided into congruent hypercubic subdivisions, the contributions of the points on the faces of the interior hypercubes cancel. For n = [illegible] the resulting cross-product rule is similar to the cubature rule considered by Thacher [4]. Error bounds for the Lobatto cross-product rules can be obtained from the constants we give in Table 2 by using a technique due to Stroud [3]. The details can be found in [2, p. 73].

2. **Computation.** For each n, [illegible] ≤ n ≤ [illegible], the algebraic method outlined by Hildebrand [1, p. [illegible]] was used to compute the abscissas and weights in (1.1). The a_j in (1.1), [illegible] ≤ n ≤ [illegible], are the zeros of the polynomials listed in Appendix A of this paper. All of the numerical calculations were carried out on the Univac 1108 computer using double precision floating point arithmetic (approximately 18

$$[illegible]$$

where [illegible] and compared these summations with the true values. Based on these comparisons we feel that the maximum end-figure error in the abscissas and weights given in Table I is less than [illegible] of 1 unit.

3. **Error Bounds.** Suppose we wish to integrate over the interval [a, b] by using a composite rule based on the fixed n-point quadrature rule

$$(3.1)\qquad \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} a_i f(x_i)$$

Let N be a positive integer and define

$$x_j = a + jh, \quad j = 0, 1, \dots, N$$

where

$$h = (b-a)/N.$$

If [illegible] then

$$\int_a^b f(x)\,dx = [illegible]$$

where

$$[illegible]$$

and c_n is a constant which does not depend on f(x). The error coefficient c_n is the [illegible] of the [illegible] error [illegible] the rule [illegible].

It should be noted that if we take the (n-1)-point Gauss-Legendre rule for (2.1), then we have precision (2n-3) and (n-1) evaluations of f(x) are needed.

4. **The Quadrature Constants.** In Table 1 we list the abscissas and weights for the n-point Lobatto rule (1.1). In Table 2 we list the corresponding error coefficients c_1 and c_2.

For the purpose of comparison we have included in Table 3 the values c_1 and c_2 for the (n-1)-point Gauss-Legendre rule, $3 \leq n \leq 8$. These values were previously computed by Stroud and Secrest [5, p. 100]. From Tables 2 and 3 we see that the error coefficients c_1 and c_2 for the n-point Lobatto rule are less than or equal to those of the (n-1)-point Gauss-Legendre analogue.

The entries in Tables 1 and 2 are accurate to at least 4 significant digits.

TABLE I

[illegible]

[illegible]	[illegible]
[illegible]	
[illegible]	[illegible]
[illegible]	
[illegible]	
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	
[illegible]	
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	
[illegible]	
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

TABLE 1 (continued)

n = 7

[illegible]	
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

n = 8

[illegible]	
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

n = 9

[illegible]	
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

TABLE [illegible]

(n-point [illegible] rule)

n	e_1	e_2
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]

TABLE [illegible]

(n-point Gauss-Legendre rule)

n	e_1	e_2
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]

Appendix A

The $(n-2)$ interior dimensions a_j $(j=2$ [illegible] in [1] are the zeros of the following polynomials

$n = 3$ a [illegible]

$n = 4$ a^2 [illegible]

$n = 5$ a^3 [illegible]

$n = 6$ a^4 [illegible]

$n = 7$ a^5 [illegible]

$n = 8$ a^6 [illegible]

$n = 9$ a^7 [illegible]

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The editorial committee would welcome readers' comments about this microfiche feature. Please send comments to Professor Eugene Isaacson, MATHEMATICS OF COMPUTATION, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012.

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