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On Hadamard Matrices Constructible by Circulant Submatrices

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Abstract. Let V_{2n} be an H-matrix of order $2n$ constructible by using circulant $n \times n$ submatrices. A recursive method has been found to construct V_{4n} by using circulant $2n \times 2n$ submatrices which are derived from $n \times n$ submatrices of a given V_{2n}. A similar method can be applied to a given W_{4n}, an H-matrix of Williamson type with odd n, to construct W_{8n}. All V_{2n} constructible by the standard type, for $1 \leqq n \leqq 16$, and some V_{2n}, for $n \geqq 20$, are listed and classified by this method.

Let H_n be an $n \times n$ Hadamard matrix. Although it is conjectured that no circulant H_{4n}-matrix exists for $n > 1$ (see [3]), it is known that many H_{4n}-matrices can be constructed by using circulant submatrices of order n or $2n$. (For H-matrices of Williamson type, see [1], [2], [4].)

Let V_{2n} be an H_{2n}-matrix constructible by using circulant $n \times n$ submatrices. Then V_{2n} can be constructed by the following standard type:

$$(*) \qquad M_{2n} = \begin{bmatrix} A & B \\ -B^T & A^T \end{bmatrix}, \qquad \text{where } A, B \text{ are } n \times n \text{ circulant matrices}$$

and C^T means the transposed matrix of C.

A recursive method has been found to construct V_{4n} by circulant $2n \times 2n$ matrices which are derived by circulant $n \times n$ submatrices of a given V_{2n}. (See Theorem 1, below.) Likewise, let W_{4n} be an H_{4n}-matrix of Williamson type with odd n; W_{8n} can be constructed by using $2n \times 2n$ symmetric circulant matrices which are derived from $n \times n$ symmetric circulant submatrices of a given W_{4n}. (See Theorem 2.)

Let $S_n = ((e_i))$ be the $n \times n$ circulant matrix with the first row entries e_i, ($0 \leqq i \leqq n - 1$), all zero except for $e_1 = 1$. Then $n \times n$ circulant matrices A, B of (*) can be written as polynomials in S. (We shall omit the suffix n of S_n and others when there is no confusion.)

$$A = A_n(S) = \sum_{i=0}^{n-1} a_i S^i, \qquad B = B_n(S) = \sum_{i=0}^{n-1} b_i S^i,$$

with coefficients $a_i, b_i = 1$ or -1; where $S^0 = I_n =$ the $n \times n$ identity matrix.

A sufficient condition for the matrix M_{2n} of type (*) being an H-matrix is that $M_{2n}M_{2n}^T = 2nI_{2n}$ which is equivalent to

$$(1) \qquad AA^T + BB^T = 2nI_n.$$

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Let $P = P_n(S)$, $Q = Q_n(S)$ be matrices obtained by replacing -1 by 0 in A, B respectively. Then the condition (1) is equivalent to

$$PP^T + QQ^T = (p_n + q_n - r_n)I + r_n J, \tag{2}$$

where $J = J_n = \sum_{i=0}^{n-1} S^i$ and p_n, q_n are, respectively, the numbers of 1's in each row of P, Q. Here, p_n, q_n and r_n must be solutions of the following necessary conditions for existence of V_{2n}.

$$(n - 2p_n)^2 + (n - 2q_n)^2 = 2n, \tag{3}$$

$$p_n + q_n - r_n = \tfrac{1}{2}n. \tag{4}$$

Similarly, by taking $Q' = J - Q$, instead of Q in (2), (3), and (4), which is possible since whenever A and B satisfy the condition (1), so do A and $-B$, we obtain the corresponding conditions:

$$PP^T + Q'Q'^T = (p_n + q'_n - r'_n)I + r'_n J, \tag{5}$$

$$(n - 2p_n)^2 + (n - 2q'_n)^2 = 2n, \tag{6}$$

$$p_n + q'_n - r'_n = \tfrac{1}{2}n. \tag{7}$$

Since $q'_n = n - q_n$, we also obtain from (7) and (4),

$$r'_n = 2p_n - r_n. \tag{8}$$

THEOREM 1. *Let M_{2m} be a given V_{2m}-matrix of type (*) satisfying the conditions (2), (3), and (4). Then M_{4m}, a V_{4m}-matrix of type (*), can be found as follows:*

$$P_{2m}(s) = P_m(s^2) + s^k Q_m(s^2), \qquad Q_{2m}(s) = P_m(s^2) + s^k Q'_m(s^2), \tag{**}$$

where $s = S_{2m}$, $Q'_m = J_m - Q_m$, and k is any odd integer.

Proof. Since $p_{2m} = p_m + q_m$, $q_{2m} = p_m + (m - q_m)$, $r_{2m} = 2p_m$ are solutions of the conditions (3) and (4) for $n = 2m$ whenever p_m, q_m, r_m are solutions of (3) and (4) for $n = m$, it is sufficient to show that P_{2m} and Q_{2m} satisfy the condition (2), i.e.

$$P_{2m}P_{2m}^T + Q_{2m}Q_{2m}^T = mI_{2m} + 2p_m J_{2m}. \tag{5}$$

From (**), the left side of (5) equals, (since $P^T(s) = P(s^{-1})$),

$$\begin{aligned}
&(P(s^2)P(s^{-2}) + Q(s^2)Q(s^{-2})) + (P(s^2)P(s^{-2}) + Q'(s^2)Q'(s^{-2})) \\
&\quad + [s^k P(s^{-2}) + s^{-k}P(s^2)]\, J_m(s^2), \qquad [\text{since } Q(s^2) + Q'(s^2) = J_m(s^2) = J_m(s^{-2})] \\
&= \frac{1}{2}\, mI + r_m \sum_{i=0}^{m-1} s^{2i} + \frac{1}{2}\, mI + (2p_m - r_m) \sum_{i=0}^{m-1} s^{2i} + 2p_m \sum_{i=0}^{m-1} s^{2i+1} \\
&= mI + 2p_m J.
\end{aligned}$$

Let N_{4n} be a $4n \times 4n$ matrix such that

$$N_{4n} = \begin{bmatrix} A, & B, & C, & D \\ -B, & A, & -D, & C \\ -C, & D, & A, & -B \\ -D, & -C, & B, & A \end{bmatrix}$$

where A, B, C, D are $n \times n$ symmetric circulant $(+1, -1)$-matrices. Then a sufficient condition for N_{4n} being a W_{4n}-matrix is that

$$N_{4n} N_{4n}^T = 4n I_{4n}.$$

Let P, Q, K, and G be matrices obtained by replacing -1 by 0 in A, B, C, and D, respectively. Then, corresponding to the conditions (2)–(4), we obtain

$$P^2 + Q^2 + K^2 + G^2 = (t_n - r_n)I + r_n J, \tag{2'}$$

where $t_n = p + q + k + g$; p, q, k, and g are the numbers of 1's in each row of A, B, C, and D, respectively.

$$(n - 2p)^2 + (n - 2q)^2 + (n - 2k)^2 + (n - 2g)^2 = 4n. \tag{3'}$$

$$t_n - r_n = n. \tag{4'}$$

Similarly, corresponding to the conditions (5)–(8), we obtain

$$P^2 + Q'^2 + K^2 + G'^2 = (t_n' - r_n')I + r_n' J, \tag{5'}$$

where $Q' = J - Q$, $G' = J - G$, and $t_n' = p + q' + k + g'$; q' and g' are, respectively, the numbers of 1's in each row of Q' and G'.

$$(n - 2p)^2 + (n - 2q')^2 + (n - 2k)^2 + (n - 2g')^2 = 4n. \tag{6'}$$

$$t_n' - r_n' = n. \tag{7'}$$

$$r_n' = 2(p + k) - r_n. \tag{8'}$$

THEOREM 2. *Let N_{4m} be a given W_{4m}-matrix with odd m satisfying the conditions* (2′), (3′) *and* (4′). *Then N_{8m}, a W_{8m}-matrix, can be found as follows*:

$$P_{2m}(s) = P(s^2) + s^m Q(s^2), \qquad Q_{2m}(s) = P(s^2) + s^m Q'(s^2),$$

$$K_{2m}(s) = K(s^2) + s^m G(s^2), \qquad G_{2m}(s) = K(s^2) + s^m G'(s^2);$$

where $s = S_{2m}$, $Q' = J_m - Q$, and $G' = J_m - G$.

Proof. We know that P_{2m}, Q_{2m}, K_{2m}, and G_{2m} are also symmetric circulant and, as in the proof of Theorem 1, that $p_{2m} = p + q$, $q_{2m} = p + (n - q)$, $k_{2m} = k + g$, and $g_{2m} = k + (n - g)$; $r_{2m} = 2(p + k)$ are solutions of (3′) and (4′) for $n = 2m$ whenever p, q, k, g, and r_m are solutions of (3′) and (4′) for $n = m$. Therefore, it is sufficient to prove that the condition (2′) is also satisfied, i.e.

$$P_{2m}^2 + Q_{2m}^2 + K_{2m}^2 + G_{2m}^2 = 2mI + 2(p + k)J. \tag{2''}$$

The condition (2″) can be checked easily since the process of proof is exactly similar to that of Theorem 1.

Let $\{u_i\}$ and $\{v_i\}$ be two finite sequences respectively of

$$PP^T = \sum_{i=0}^{n-1} u_i S^i \qquad \text{and} \qquad QQ^T = \sum_{i=0}^{n-1} v_i S^i,$$

where P, Q are $n \times n$ circulant (0, 1)-matrices; in this case, we also obtain $w_{n-i} = w_i$ for $w = u$ or v.

The following Table I, of all constructible V_{2n} ($1 \leqq n \leqq 16$) of type (*) with the restriction $p_n \leqq q_n \leqq \frac{1}{2}n$, is obtained by matching two finite sequences $\{u_i\}$ and

$\{v_i\}$, respectively of PP^T and QQ^T, such that $u_i + v_i = r_n$ for $1 \leqq i \leqq \frac{1}{2}n$. Here, Theorem 1 serves as a tool of classifying these finite sequences.

Note. 1. $s = S_n^k$, where k is any integer relatively prime to n.

2. When $q_n = \frac{1}{2}n$, $Q_n(s)$ and $Q_n'(s)$ produce the same finite sequence.

3. * indicates the class of $P_n(s)$ and $Q_n(s)$ unobtainable by Theorem 1.

It should also be noted that for a given $n \times n$ circulant matrix $K(S)$, all matrices $M(i, j) = S^i K(S^j)$, for any integers i and j with $(n, j) = 1$, produce the same finite sequence corresponding to $M(i, j)M^T(i, j)$. Among all $M(i, j)$ regarded as polynomials in S, there is a polynomial, say R, of least nonnegative degree; we list R, as the representative of all matrices $M(i, j)$ producing the same finite sequence, as $R_n(s)$ in the Table I.

In Table I, Classes I and II of $n = 16$ are respectively derived from the corresponding classes of $n = 8$. Although P_8 and Q_8 of Class II cannot be derived from P_4 and Q_4, they produce P_{16} and Q_{16} of Class II, by Theorem 1. In this case, P_{16} and Q_{16} are interchangeable since $p = q = 6$, and we have

TABLE I

n	$P_n(s)$	$Q_n(s)$
1	0	0
2	0	I
4	I	I
8-I	$I + s$	$I + s + s^3 + s^5$
II*	$I + s^2$	$I + s + s^3 + s^4$
10	$I + s + s^3$	$I + s + s^4 + s^6$
16-I	$I + s + s^2 + s^3 + s^6 + s^{10}$ or $I + s + s^2 + s^4 + s^7 + s^8$	$I + s + s^3 + s^6 + s^8 + s^{12}$ or $I + s + s^4 + s^6 + s^8 + s^{11}$
II	$I + s + s^2 + s^4 + s^5 + s^{10}$ or $I + s + s^2 + s^5 + s^6 + s^8$	$I + s + s^3 + s^7 + s^9 + s^{12}$ or $I + s + s^4 + s^7 + s^9 + s^{11}$
III*	$I + s + s^2 + s^4 + s^6 + s^9$ or $I + s^2 + s^3 + s^4 + s^6 + s^{11}$ or $I + s + s^3 + s^5 + s^7 + s^8$	$I + s + s^5 + s^7 + s^8 + s^{11}$ or $I + s + s^2 + s^6 + s^9 + s^{12}$ or $I + s + s^4 + s^6 + s^9 + s^{10}$

$$P(s, k) = P_8(s^2) + s^k Q_8(s^2) = I + s^4 + s^k(I + s^2 + s^6 + s^8),$$

$$Q(s, k) = P_8(s^2) + s^k Q_8'(s^2) = I + s^4 + s^k(s^4 + s^{10} + s^{12} + s^{14}).$$

We obtain

$$P_{16}(s) = I + s + s^2 + s^4 + s^5 + s^{10} = s\,Q(s, 5)$$

or

$$= I + s + s^2 + s^5 + s^6 + s^8 = s\,P(s, -1),$$

since these two polynomials are of distinct type (in the sense of [5]) and of least positive degree in $s = S$ producing the same finite sequence among all $P(s, k)$ and $Q(s, k)$ for this case.

When $n = 20$, we obtain two subclasses of matrices P and Q by Theorem 1. We have the following cases:

Subclass-1:

$$P(s, k) = P_{10}(s^2) + s^{-k} Q_{10}(s^2) = I + s^2 + s^6 + s^{-k}(I + s^2 + s^8 + s^{12})$$

and

$$Q(s, k) = P_{10}(s^2) + s^{-k} Q_{10}'(s^2)$$
$$= I + s^2 + s^6 + s^{-k}(s^4 + s^6 + s^{10} + s^{14} + s^{16} + s^{18});$$

Subclass-2:

$$P(s, k) = P_{10}(s^2) + s^{-k} Q_{10}(s^{-2})$$
$$= I + s^2 + s^6 + s^{-k}(I + s^{-2} + s^{-8} + s^{-12})$$

and

$$Q(s, k) = P_{10}(s^2) + s^{-k} Q_{10}'(s^2)$$
$$= I + s^2 + s^6 + s^{-k}(s^4 + s^6 + s^{10} + s^{14} + s^{16} + s^{18});$$

Each one of the subclasses produces five distinct designs corresponding to $k = 1, 3, 5, 7$, and 9. For example, the finite sequence $\{u_{2i+1}\}$ of odd components (since the even components $u_{2i} = r = 2$ for all i, it is sufficient to consider only odd components of $\{u_i\}$) corresponding to $P(S, k)$ are: $(u_1, u_3, u_5, u_7, u_9) = (4, 1, 3, 2, 2), (2, 4, 2, 2, 2), (2, 3, 3, 2, 2), (3, 1, 3, 3, 2)$, and $(2, 3, 1, 3, 3)$ for Subclass-1 respectively of $k = 1, 3, 5, 7$, and 9; and $(2, 2, 3, 2, 3), (1, 3, 3, 2, 3), (2, 2, 2, 4, 2), (3, 1, 3, 3, 2), (2, 4, 1, 2, 3)$ for Subclass-2.

The following Table II is obtained by taking $s = S^k$ with k, an integer relatively prime to $n = 20$ for $P_{20} = P(s, 9)$ of Subclass-2, i.e. $P_{20}(S^k) = I + S^{2k} + S^{3k} + S^{6k} + S^{9k} + S^{11k} + S^{19k}$.

Starting from $P = Q = I$ for $n = 4$, and repeating applications of Theorem 1, we obtain, for example, the following P_n, Q_n for $n = 32$ and 64:

$$P_{32} = \sum_\alpha s^\alpha, \quad \text{where } \alpha \in \{0, 1, 2, 3, 4, 8, 9, 13, 14, 16, 17, 23\}$$

and

$$Q_{32} = \sum_\beta s^\beta, \quad \text{where } \beta \in \{0, 2, 4, 5, 7, 8, 11, 14, 15, 16, 19, 21, 25, 27, 29, 31\};$$

$$P_{64} = \sum_\alpha s^\alpha, \qquad Q_{64} = \sum_\beta s^\beta,$$

TABLE II

k	*(+1, −1)-matrix A corresponding to* P_{20}	$\{u_{2i+1}\}$
1	+ − + + − − + − − + − + − − − − − − − +	2, 4, 1, 2, 3
3	+ − − − − − + + − + − − − + − − − + + −	2, 2, 1, 3, 4
7	+ + + + − − − − − − − − − + + − − + − −	4, 3, 1, 2, 2
9	+ + − − − − − + − − − + − − + − − − + +	3, 2, 1, 4, 2

where $\alpha \in \{0, 1, 2, 4, 5, 6, 8, 9, 11, 15, 16, 17, 18, 23, 26, 28, 29, 31, 32, 33, 34, 39, 43, 46, 51, 55, 59, 63\}$ and $\beta \in \{0, 2, 3, 4, 6, 7, 8, 13, 16, 18, 19, 21, 25, 26, 27, 28, 32, 34, 35, 37, 41, 45, 46, 47, 49, 53, 57, 61\}$.

It should be noted that Theorem 3 of Williamson [4] produces Williamson type matrices of the same order, but of different construction, as given by Theorem 2 of this paper. When $n = 29$, we obtain a W_{4n}-matrix (see [7]) with submatrices

$$P_{29} = \sum_{\alpha} t_{\alpha}, \qquad Q_{29} = \sum_{\beta} t_{\beta}, \qquad K_{29} = \sum_{\gamma} t_{\gamma}, \qquad G_{29} = \sum_{\delta} t_{\delta},$$

where $t_k = S^k + S^{29-k}$; $\alpha \in \{2, 3, 5, 6, 8, 12\}$, $\beta \in \{4, 7, 9, 10, 11\}$, $\gamma \in \{3, 4, 5, 8, 9, 11, 13, 14\}$, and $\delta \in \{1, 3, 4, 5, 8, 9, 11\}$. By applying Theorem 2, we obtain W_{8n}-matrix with submatrices

$$P_{58} = \sum_{\alpha} t_{\alpha}, \qquad Q_{58} = \sum_{\beta} t_{\beta}, \qquad K_{58} = \sum_{\gamma} t_{\gamma} \quad \text{and} \quad G_{58} = \sum_{\delta} t_{\delta},$$

where $t_k = s^k + s^{58-k}$ for $k \neq 29$ and $t_{29} = s^{29}$; and $\alpha \in \{4, 6, 7, 9, 10, 11, 12, 15, 16, 21, 24\}$, $\beta \in \{1, 3, 4, 5, 6, 10, 12, 13, 16, 17, 19, 23, 24, 25, 27, 29\}$, $\gamma \in \{6, 7, 8$ $10, 11, 13, 16, 18, 19, 21, 22, 23, 26, 27, 28\}$, and $\delta \in \{1, 3, 5, 6, 8, 9, 10, 15, 16, 17, 18, 22, 25, 26, 28, 29\}$.

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