

A Fourth-Order Finite-Difference Approximation for the Fixed Membrane Eigenproblem*

By J. R. Kuttler

Abstract. The fixed membrane problem $\Delta u + \lambda u = 0$ in Ω , $u = 0$ on $\partial\Omega$, for a bounded region Ω of the plane, is approximated by a finite-difference scheme whose matrix is monotone. By an extension of previous methods for schemes with matrices of positive type, $O(h^4)$ convergence is shown for the approximating eigenvalues and eigenfunctions, where h is the mesh width. An application to an approximation of the forced vibration problem $\Delta u + qu = f$ in Ω , $u = 0$ in $\partial\Omega$, is also given.

1. Introduction. Let Ω be a bounded region of the plane with smooth boundary $\partial\Omega$. We consider the fixed membrane problem

$$(1.1) \quad \Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega,$$

where Δ is the Laplacian. In [6], this problem was approximated by difference schemes which were of positive type in the interior of the region. Here, we consider a difference scheme for (1.1) which is only monotone. However, by appropriate modifications of the techniques of [6], we can prove that this scheme yields $O(h^4)$ approximations to the eigenvalues and eigenvectors of (1.1). The principal result is Theorem 8.1. An application to a forced vibration problem is also given in Section 9.

2. The Difference Scheme. Let $h > 0$ be given and define the mesh S_h by

$$\{(ih, jh) : i, j \text{ are integers}\}.$$

Points $x, y \in S_h$ will be called nearest neighbors if $|x - y| = h$, where we write

$$|x - y| = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}.$$

Let $\Omega_h^{(3)}$ be the set of points in $S_h \cap \Omega$ having at least one nearest neighbor not in Ω . One such point might be $x = (x_1, x_2)$ with $(x_1 - \alpha h, x_2), (x_1, x_2 - \beta h) \in \partial\Omega$ for $0 < \alpha, \beta \leq 2$. If $(x_1 + h, x_2), (x_1 + 2h, x_2), (x_1, x_2 + h), (x_1, x_2 + 2h) \in \Omega$, we define

Received September 11, 1969, revised September 9, 1970.

AMS 1970 subject classifications. Primary 35P15, 35J05, 65N10, 65N15, 65N25; Secondary 15A48, 15A42.

Key words and phrases. Finite-differences, membrane, fixed membrane, eigenvalues, elliptic partial differential equations, monotone matrices, forced vibration problem, discrete Green's function.

* This work was supported by the Department of the Navy, Bureau of Naval Weapons, under Contract No. w-62-0604-c.

$$\begin{aligned}
 h^2 l_h(x, y) &= \frac{3 - \alpha}{\alpha} + \frac{3 - \beta}{\beta}, & y = x, \\
 &= -\frac{2(2 - \alpha)}{1 + \alpha}, & y = (x_1 + h, x_2), \\
 &= -\frac{2(2 - \beta)}{1 + \beta}, & y = (x_1, x_2 + h), \\
 (2.1) \quad &= \frac{1 - \alpha}{2 + \alpha}, & y = (x_1 + 2h, x_2), \\
 &= \frac{1 - \beta}{2 + \beta}, & y = (x_1, x_2 + 2h), \\
 &= 0, & \text{otherwise.}
 \end{aligned}$$

Similar formulas apply at other points of $\Omega_h^{(3)}$. One special case may arise, as shown in Fig. 1, where $(x_1, x_2 + h), (x_1, x_2 + 2h)$ do not lie in Ω .

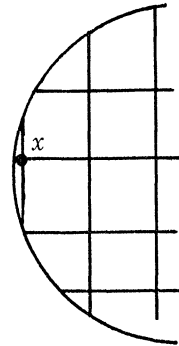


FIGURE 1

In such a case x would be excluded from the difference scheme altogether and the point $(x_1 + h, x_2)$ would be added to $\Omega_h^{(3)}$. For the new point, formula (2.1) would be used with $1 < \alpha \leq 2$. If $\partial\Omega$ has bounded curvature and h is sufficiently small, there will be no difficulty with the new point.

Next, let $\Omega_h^{(2)}$ be those points of $S_h \cap \Omega$, not in $\Omega_h^{(3)}$ or excluded, which have a nearest neighbor in $\Omega_h^{(3)}$. For $x \in \Omega_h^{(2)}$ define

$$\begin{aligned}
 h^2 l_h(x, y) &= 4, & y = x, \\
 (2.2) \quad &= -1, & |x - y| = h, \quad y \in S_h, \\
 &= 0, & \text{otherwise.}
 \end{aligned}$$

Finally, let Ω_h' be those points of $S_h \cap \Omega$ not in $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ or excluded. For $x \in \Omega_h'$ define

$$\begin{aligned}
 h^2 l_h(x, y) &= 5, & y = x, \\
 (2.3) \quad &= -\frac{4}{3}, & |x - y| = h, \quad y \in S_h, \\
 &= \frac{1}{2}, & |x - y| = 2h, \quad y \in S_h, \\
 &= 0, & \text{otherwise.}
 \end{aligned}$$

Let $\Omega_h = \Omega'_h \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$. We approximate the Laplacian of a function u vanishing on $\partial\Omega$ by

$$(2.4) \quad -\Delta_h u(x) = \sum_{y \in \Omega_h} l_h(x, y)u(y), \quad x \in \Omega_h.$$

Let us agree to use C as a generic constant, whose value may change at each usage, but which is always independent of h . Then, if also $u \in C^6(\bar{\Omega})$ (u has continuous sixth derivatives on the closure of Ω), it can be seen from Taylor series expansions that

$$(2.5) \quad \begin{aligned} |\Delta u(x) - \Delta_h u(x)| &\leq Ch^4, & x \in \Omega'_h, \\ &\leq Ch^2, & x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}. \end{aligned}$$

Bramble and Hubbard used Δ_h in [2] in approximating the Dirichlet problem for Poisson's equation.

Our difference scheme approximating (1.1) is

$$(2.6) \quad \Delta_h U_h(x) + \lambda_h U_h(x) = 0, \quad x \in \Omega_h.$$

Problem (2.6) is equivalent to finding the eigenvalues and eigenvectors of the matrix $[l_h(x, y)]_{x, y \in \Omega_h}$. In the next section, we develop some tools to use in studying this matrix which, however, have some independent interest.

3. Monotone Matrices. Let $A = (a_{ij})$ be an $n \times n$ matrix. We say $A \geq 0$ if each $a_{ij} \geq 0$ and $A \leq B$ if $B - A \geq 0$. The matrix A is *monotone* if $Ax \geq 0$ implies $x \geq 0$ for all x . Thus, A is monotone if and only if A^{-1} exists and $A^{-1} \geq 0$. An easily recognized type of monotone matrix is a matrix of *positive type*. The matrix A is of positive type if A is indecomposable, the diagonal of A is positive, the off-diagonal elements negative, and the row sums are nonnegative with at least one strictly positive. The following theorem is due to Price [8]:

THEOREM 3.1. *A is monotone if and only if there exists M monotone such that*

- (i) $M^{-1}(M - A) \geq 0$,
- (ii) $\rho(M^{-1}(M - A)) < 1$.

Here ρ denotes spectral radius, the maximum of the moduli of the eigenvalues. Here and in the corollaries, the "only if" part is trivial: take $M = A$. This theorem generalizes Theorem 2.7 of Bramble and Hubbard [2]. There are a number of important corollaries:

COROLLARY 3.2. *A is monotone if and only if there exists M monotone such that*

- (i) $M \geq A$,
- (ii) $\rho(M^{-1}(M - A)) < 1$.

COROLLARY 3.3. *A is monotone if and only if there exists M monotone and $x > 0$ such that*

- (i) $M \geq A$,
- (ii) $Ax > 0$.

Proof. By the Gerschgorin circle theorem (see [7, p. 152]),

$$\rho(M^{-1}(M - A)) \leq \max_i [M^{-1}(M - A)x]_i / x_i < 1,$$

since

$$0 \leq [M^{-1}(M - A)x]_i = x_i - [M^{-1}Ax]_i < x_i,$$

because $Ax > 0$, $M^{-1} \geq 0$ and no row of M^{-1} can be all zero.

COROLLARY 3.4. *A is monotone if and only if there exists M monotone and $x \geq 0$ such that*

- (i) $M \geq A$,
- (ii) $Ax > 0$.

Proof. Let $\delta = \min_i [Ax]_i > 0$ and let $\epsilon = \delta / (2 \max_i \sum_j |a_{ij}|)$. Then $x + \epsilon > 0$ and $A(x + \epsilon) > 0$, so the hypotheses of Corollary 3.4 are satisfied.

COROLLARY 3.5. *A is monotone if and only if there exist M_1, M_2 monotone such that*

$$M_1 \leq A \leq M_2.$$

Proof. Let x be such that M_1x is the vector with all components 1. Since M_1 is monotone, x exists and $x \geq 0$. Also, $Ax \geq M_1x > 0$, so the hypotheses of Corollary 3.4 are satisfied.

COROLLARY 3.6. *A is monotone if there is $\alpha > 0$ such that $A + \alpha I$ is monotone and every eigenvalue λ of A has positive real part.*

Proof. Apply Corollary 3.2. We need only show $\rho((A + \alpha I)^{-1}) < \alpha^{-1}$. But $\rho((A + \alpha I)^{-1}) = 1 / \min_\lambda |\alpha + \lambda|$, where λ runs over the eigenvalues of A.

At this time, we also note the following:

LEMMA 3.7. *If the partitioned matrix*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with A invertible has inverse

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

then $W - A^{-1} = -XCA^{-1}$. In particular, if $X \geq 0, A^{-1} \geq 0, C \leq 0$, then $A^{-1} \leq W$.

Proof. Since

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

we have $WA + XC = I$. Multiply on the right by A^{-1} .

4. Discrete Green's Functions. The main tools in our investigations will be discrete analogues of Green's function. These are inverses of matrices related to $[h^2 l_h(x, y)]_{x, y \in \Omega_h}$ and their nonnegativity is crucial to the investigation. This will be established, using results of the previous section.

We define then

$$(4.1) \quad -\Delta_{h,x} g_h(x, y) = h^{-2} \delta(x, y), \quad x \in \Omega'_h \cup \Omega_h^{(2)}, \quad g_h(x, y) = \delta(x, y), \quad x \in \Omega_h^{(3)},$$

for all $y \in \Omega_h$. This is the discrete Green's function considered by Bramble and Hubbard in [2, Eq. (4.5)]. From (4.1), we see that the matrix $[g_h(x, y)]_{x, y \in \Omega_h}$ is the inverse of the partitioned matrix

$$\mathfrak{M} \equiv \begin{bmatrix} A & B \\ 0 & I \end{bmatrix},$$

where $A = [h^2 l_h(x, y)]_{x, y \in \Omega_h' \cup \Omega_h^{(s)}}$, $B = [h^2 l_h(x, y)]_{x \in \Omega_h' \cup \Omega_h^{(s)}, y \in \Omega_h^{(s)}}$, and I is the identity on $\Omega_h^{(3)} \times \Omega_h^{(3)}$. It also follows from Lemma 3.7 that the matrix $[g_h(x, y)]_{x, y \in \Omega_h' \cup \Omega_h^{(s)}}$ is the inverse of A . In [2], it was shown that

$$(4.2) \quad g_h(x, y) \geq 0, \quad x, y \in \Omega_h,$$

i.e., \mathfrak{M} is monotone. Since g_h is the inverse, it follows that, for any function W defined on Ω_h , all $x \in \Omega_h$,

$$(4.3) \quad W(x) = h^2 \sum_{y \in \Omega_h' \cup \Omega_h^{(s)}} g_h(x, y) [-\Delta_h W(y)] + \sum_{y \in \Omega_h^{(s)}} g_h(x, y) W(y).$$

This is analogous to Poisson's formula. In [2], the following properties were proved of g_h :

$$(4.4) \quad \sum_{y \in \Omega_h^{(s)}} g_h(x, y) \leq 1,$$

$$(4.5) \quad \sum_{y \in \Omega_h^{(s)}} g_h(x, y) \leq C,$$

$$(4.6) \quad h^2 \sum_{y \in \Omega_h} g_h(x, y) \leq C,$$

for all $x \in \Omega_h$. Using these in (4.3), we have the inequality

$$(4.7) \quad \max_{\Omega_h} |W| \leq C \left[\max_{\Omega_h'} |\Delta_h W| + h^2 \max_{\Omega_h^{(s)}} |\Delta_h W| \right] + \max_{\Omega_h^{(s)}} |W|.$$

Now, on $\Omega_h^{(3)}$, we have

$$W(x) = \left[-h^2 \Delta_h W(x) - h^2 \sum_{y \in \Omega_h; y \neq x} l_h(x, y) W(y) \right] / h^2 l_h(x, x),$$

and from this and (2.1), we see that

$$(4.8) \quad \max_{\Omega_h^{(3)}} |W| \leq Ch^2 \max_{\Omega_h^{(s)}} |\Delta_h W| + \theta \max_{\Omega_h} |W|,$$

where

$$\theta = \max_{x \in \Omega_h^{(3)}} \sum_{y \in \Omega_h; y \neq x} |l_h(x, y)| / l_h(x, x) < 1.$$

Putting (4.8) into (4.7) and rearranging, we have

$$(4.9) \quad \max_{\Omega_h} |W| \leq C \left[\max_{\Omega_h'} |\Delta_h W| + h^2 \max_{\Omega_h^{(s)} \cup \Omega_h^{(3)}} |\Delta_h W| \right].$$

Let us now use (4.7) to estimate $W = \Phi_h - \varphi$ where φ is the torsion function defined by $\Delta \varphi = -1$ on Ω , $\varphi = 0$ on $\partial\Omega$ and $\Phi_h(x) = h^2 \sum_{y \in \Omega_h} g_h(x, y)$, which satisfies $\Delta_h \Phi_h = -1$ on $\Omega_h' \cup \Omega_h^{(2)}$. If $\partial\Omega$ is sufficiently smooth, φ satisfies (2.5) and we see from (4.7) that

$$\max_{\Omega_h} |\Phi_h - \varphi| \leq Ch^4 + \max_{\Omega_h^{(s)}} |\Phi_h - \varphi| \leq Ch^4 + \max_{\Omega_h^{(s)}} |\Phi_h| + \max_{\Omega_h^{(s)}} |\varphi|.$$

Now, $\varphi = 0$ on $\partial\Omega$, so $|\varphi(x)| \leq Ch$ for $|x - \partial\Omega| = \min_{y \in \partial\Omega} |x - y| \leq Ch$. Also, $\Phi_h = h^2$ on $\Omega_h^{(3)}$ by definition. Hence,

$$|\Phi_h(x)| \leq |\varphi(x)| + \max_{\Omega_h} |\Phi_h - \varphi| \leq Ch$$

for $|x - \partial\Omega| \leq Ch$, i.e.,

$$(4.10) \quad h^2 \sum_{y \in \Omega_h} g_h(x, y) \leq Ch.$$

Next, we consider the function

$$f_h(x, y) = C_1 - C_2 \log(|x - y|^2 + h^2).$$

It is easily verified that

$$\begin{aligned} -\Delta_{h,x} f_h(x, y) &\geq 0, & x \in \Omega'_h \cup \Omega_h^{(2)}, \quad y \neq x, \\ -\Delta_{h,x} f_h(x, y) &\geq h^{-2}, & x \in \Omega'_h \cup \Omega_h^{(2)}, \quad y = x, \end{aligned}$$

provided $C_2 \geq \frac{1}{4} \log 2$. If we choose

$$C_1 = C_2 \max_{x, y \in \Omega_h} \log(|x - y|^2 + h^2),$$

then $f_h(x, y) \geq 0$ for $x, y \in \Omega$. Thus, we see that

$$\mathfrak{M}(f_h - g_h) \geq 0,$$

and, since \mathfrak{M} is monotone,

$$(4.11) \quad 0 \leq g_h(x, y) \leq C_1 - C_2 \log(|x - y|^2 + h^2).$$

Analogous inequalities to (4.11) are proved by Bramble and Thomée in [3] for discrete Green's functions of positive-type operators. Here, we see monotonicity was sufficient.

An easy consequence of (4.11) is

$$(4.12) \quad h^2 \sum_{y \in \Omega_h} [g_h(x, y)]^2 \leq C.$$

5. More Inequalities for Green's Functions. This section will be devoted to derivations of some inequalities of more difficulty than those of the previous section.

Recall that $\mathfrak{G} = [g_h(x, y)]_{x, y \in \Omega_h}$ is the inverse of $[h^2 l_h(x, y)]_{x, y \in \Omega_h}$.

The inequality which we next wish to derive is

$$(5.1) \quad \sum_{y \in \Omega_h'} g_h(x, y) \leq C$$

for all $x \in \Omega_h$, where $\Omega_h' = \{x \in \Omega_h' : l_h(x, y) \neq 0 \text{ for some } y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}\}$. The method of proof is the matrix splitting technique employed by Bramble and Hubbard in [2]. The analysis which follows is regrettably detailed.

Let us write

$$(5.2) \quad \mathfrak{G} = [I - H_1 - H_2]^{-1} \tilde{D}^{-1},$$

where \tilde{D} is the diagonal matrix with

$$\begin{aligned} \tilde{d}_{xx}^{-1} &= 1, & x \in \Omega_h^{(3)}, \\ &= \frac{1}{4}, & x \in \Omega_h^{(2)}, \\ &= \frac{1}{5}, & x \in \Omega_h', \end{aligned}$$

and

$$\begin{aligned}
 [H_1]_{xy} &= \frac{2}{15}, & x \in \Omega'_h, & |x - y| = h, \\
 &= \frac{1}{8}, & x \in \Omega_h^{(2)}, & |x - y| = h, \\
 &= 0, & \text{otherwise,} & \\
 [H_2]_{xy} &= \frac{2}{15}, & x \in \Omega'_h, & |x - y| = h, \\
 &= -\frac{1}{80}, & x \in \Omega'_h, & |x - y| = 2h, \\
 &= \frac{1}{8}, & x \in \Omega_h^{(2)}, & |x - y| = h, \\
 &= 0, & \text{otherwise.} &
 \end{aligned}$$

Let us define the diagonal matrix D by

$$\begin{aligned}
 (d_{xx})^{-1} &= \sum_{y \in \Omega_h} (I - H_1)_{xy} = \frac{7}{15}, & x \in \Omega'_h, \\
 &= \frac{1}{2}, & x \in \Omega_h^{(2)}, \\
 &= 1, & x \in \Omega_h^{(3)},
 \end{aligned}$$

so that $D(I - H_1)$ has row sums one, i.e.,

$$(5.3) \quad \sum_{y \in \Omega_h} [D(I - H_1)]_{xy} = \sum_{y \in \Omega_h} [(I - H_1)^{-1} D^{-1}]_{xy} = 1.$$

We write $[I - H_1 - H_2] = [D^{-1}(I - H)] [D(I - H_1)]$, where $H = DH_2(I - H_1)^{-1} D^{-1}$. Thus, by (5.3),

$$\begin{aligned}
 \sum_{y \in \Omega_h} [D^{-1}(I - H)]_{xy} &= \sum_{y, z \in \Omega_h} [D^{-1}(I - H)]_{xy} [D(I - H_1)]_{yz} \\
 (5.4) \quad &= \sum_{z \in \Omega_h} [I - H_1 - H_2]_{xz} = 0, & x \in \Omega'_h \cup \Omega_h^{(2)}, \\
 &= 1, & x \in \Omega_h^{(3)}.
 \end{aligned}$$

Now, we consider the characteristic function of Ω'_h :

$$\begin{aligned}
 \chi(x) &= 1, & x \in \Omega'_h, \\
 &= 0, & x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.
 \end{aligned}$$

Then

$$\begin{aligned}
 1 \geq \chi(x) &= \{[(I - H)^{-1} D][D^{-1}(I - H)\chi]\}_x \\
 &= \sum_{y \in \Omega_h'} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)\chi]_y \\
 &\quad + \sum_{y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)\chi]_y \\
 &= \sum_{y \in \Omega_h'} [(I - H)^{-1} D]_{xy} \sum_{z \in \Omega_h} [D^{-1}(I - H)]_{yz} \\
 &\quad - \sum_{y \in \Omega_h'} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)(1 - \chi)]_y \\
 &\quad + \sum_{y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [(I - H)^{-1} D]_{xy} [D^{-1}(I - H)\chi]_y.
 \end{aligned}$$

By (5.4), the first term vanishes. Using the definitions of H and χ , this can be written as

$$(5.5) \quad \sum_{y \in \Omega_h'} [(I - H)^{-1} D]_{xy} \sum_{y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [H_2(I - H_1)^{-1} D^{-1}]_{yz} \\ - \sum_{y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [(I - H)^{-1} D]_{xy} \sum_{y \in \Omega_h'} [H_2(I - H_1)^{-1} D^{-1}]_{yz} \leq 1.$$

Now, we estimate the factors in each term. First, note that $(I - H)^{-1} \geq 0$. This is not obvious, but follows from $H \geq 0$ and $\rho(H) < 1$. That $H \geq 0$, is due to $0 \leq H_2(I - H_1)^{-1} = H_2 + H_2H_1 + \dots$, since the negative terms in H_2 are cancelled by positive terms in H_2H_1 as in [2]. That $\rho(H) < 1$ is due to $\rho(H) = \rho((I - H_1)^{-1}H_2) < 1$, since the row sums of

$$(I - (I - H_1)^{-1}H_2) = (I - H_1)^{-1}(I - H_1 - H_2) \\ = (I - H_1 - H_2) + H_1(I - H_1 - H_2) + \dots$$

are positive. Again negative row sums of $(I - H_1 - H_2)$ are cancelled by corresponding positive row sums of $H_1(I - H_1 - H_2)$.

Next, for $y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}$,

$$\sum_{z \in \Omega_h'} [H_2(I - H_1)^{-1} D^{-1}]_{yz} \\ \leq \sum_{z \in \Omega_h} [H_2(I - H_1)^{-1} D^{-1}]_{yz} = \sum_{z \in \Omega_h} [D^{-1} - D^{-1}(I - H)]_{yz} \\ \leq 1 - \sum_{z \in \Omega_h} [D^{-1}(I - H)]_{yz} = 1 - \sum_{z \in \Omega_h} [I - H_1 - H_2]_{yz} \leq 1.$$

Now, we consider, for $y \in \Omega_h''$, the term

$$(5.6) \quad \sum_{z \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [H_2(I - H_1)^{-1} D^{-1}]_{yz}.$$

Expanding the summand in a Neumann series, it becomes

$$[(H_2 + H_2H_1 + H_2H_1^2 + \dots) D^{-1}]_{yz}.$$

If $y \in \Omega_h''$, $z \in \Omega_h^{(2)} \cup \Omega_h^{(3)}$ is such that $|y - z| = 2h$, then $[H_2]_{yz} = -1/60$. However, let x be the point such that $|y - x| = |x - z| = h$. Then $[H_2H_1]_{yz}$ contains the term $[H_2]_{yx}[H_1]_{xz} = 4/225$. Similarly, each negative term in $H_2H_1^k$ is compensated for by a positive term in $H_2H_1^{k+1}$. Thus, for $y \in \Omega_h''$,

$$\sum_{z \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [H_2(I - H_1)^{-1} D^{-1}]_{yz} \geq \left[-\frac{1}{60} + \frac{4}{225} \right] \cdot \frac{1}{2} = \frac{1}{1800}.$$

It follows from (5.5) and the above that

$$(5.7) \quad \sum_{y \in \Omega_h''} [(I - H)^{-1} D]_{xy} \leq 1800 \left\{ 1 + \sum_{y \in \Omega_h^{(2)} \cup \Omega_h^{(3)}} [(I - H)^{-1} D]_{xy} \right\}.$$

By similar reasoning, using the function

$$\chi(x) = 1, \quad x \in \Omega_h' \cup \Omega_h^{(2)}, \\ = 0, \quad x \in \Omega_h^{(3)},$$

it can be shown that $\sum_{y \in \Omega_h^{(2)}} [(I - H)^{-1} D]_{xy} \leq C$. The argument is carried out in [2, Lemma 3.3]. Finally, we note from (5.4) that

$$(5.8) \quad 1 = \sum_{\nu \in \Omega_h} [(I - H)^{-1} D]_{x\nu} \sum_{z \in \Omega_h} [D^{-1}(I - H)]_{\nu z} = \sum_{\nu \in \Omega_h^{(s)}} [(I - H)^{-1} D]_{x\nu}.$$

Combining the above with (5.7), we see that

$$(5.9) \quad \sum_{\nu \in \Omega_h'} [(I - H)^{-1} D]_{x\nu} \leq C.$$

From (5.2) and (5.3), we finally have

$$\begin{aligned} \sum_{\nu \in \Omega_h''} g_h(x, y) &= \sum_{\nu \in \Omega_h''} [(I - H_1 - H_2)^{-1} \check{D}^{-1}]_{x\nu} = \frac{1}{5} \sum_{\nu \in \Omega_h''} [(I - H_1 - H_2)^{-1}]_{x\nu} \\ &= \frac{1}{5} \sum_{\nu \in \Omega_h''} \sum_{z \in \Omega_h' \cup \Omega_h^{(s)}} \{ [D(I - H_1)]^{-1} \}_{zz} [(I - H)^{-1} D]_{z\nu} \\ &\leq \frac{1}{5} \max_{z \in \Omega_h' \cup \Omega_h^{(s)}} \sum_{\nu \in \Omega_h''} [(I - H)^{-1} D]_{z\nu}, \end{aligned}$$

or, from (5.9),

$$(5.10) \quad \sum_{\nu \in \Omega_h''} g_h(x, y) \leq C,$$

the desired estimate.

We next define another Green's function G_h by

$$(5.11) \quad -\Delta_h G_h(x, y) = h^{-2} \delta(x, y), \quad x, y \in \Omega_h.$$

Although G_h may not be nonnegative, it is a perturbation of g_h . We have

THEOREM 5.1. For any mesh function S ,

$$(5.12) \quad \begin{aligned} \max_{x \in \Omega_h} \sum_{y \in \Omega_h} |[G_h(x, y) - g_h(x, y)]S(y)| \\ \leq C \left[\max_{\Omega_h^{(s)}} |S| + \max_{x \in \Omega_h' \cup \Omega_h^{(s)} \cup \Omega_h^{(s)}} \sum_{y \in \Omega_h} g_h(x, y) |S(y)| \right]. \end{aligned}$$

Proof. Let $x_0 \in \Omega$ be the point where $\sum_{y \in \Omega_h} |[G_h(x, y) - g_h(x, y)]S(y)|$ attains its maximum and let

$$W(x) = \sum_{y \in \Omega_h} [G_h(x, y) - g_h(x, y)]S^*(y),$$

where $S^*(y) = |S(y)| \operatorname{sgn} [G_h(x_0, y) - g_h(x_0, y)]$. Employing (4.9), we have

$$\begin{aligned} \max_{\Omega_h} |W| &\leq C \max_{\Omega_h^{(s)}} |h^2 \Delta_h W| \\ &\leq C \left[\max_{\Omega_h^{(s)}} |S| + \max_{x \in \Omega_h' \cup \Omega_h^{(s)} \cup \Omega_h^{(s)}} \left| \sum_{y \in \Omega_h} g_h(x, y) S^*(y) \right| \right], \end{aligned}$$

and (5.12) follows.

COROLLARY 5.2. For all $x, z \in \Omega_h$,

$$(5.13) \quad \sum_{\nu \in \Omega_h' \cup \Omega_h^{(s)} \cup \Omega_h^{(s)}} |G_h(x, \nu)| \leq C,$$

$$(5.14) \quad h^2 \sum_{\nu \in \Omega_h} |G_h(x, \nu)| \leq C,$$

$$(5.15) \quad |G_h(x, z)| \leq C |\log h|,$$

$$(5.16) \quad h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2 \leq C,$$

and for $|x - \partial\Omega| \leq Ch$,

$$(5.17) \quad h^2 \sum_{y \in \Omega_h} |G_h(x, y)| \leq Ch.$$

Proof. For (5.13), employ the characteristic function of $\Omega'_h \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$ as S in (5.13). Then apply the triangle inequality and (4.4), (4.5), and (5.10). For (5.14), let $S = h^2$ and use (4.6) and (4.10), respectively. For (5.15), let $S(y) = \delta(y, z)$ in (5.12), apply the triangle inequality and (4.11). For (5.16), let x_0 be the point where $\max_{x \in \Omega_h} h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2$ is attained, and let $S(y) = h^2 G_h(x_0, y)$ in (5.12), from which it follows that

$$h^2 \sum_{y \in \Omega_h} |G_h(x, y)|^2 \leq Ch^2 \max_{y \in \Omega_h^{(3)}} |G_h(x_0, y)| + \max_{z \in \Omega_h} h^2 \sum_{y \in \Omega_h} g_h(x, y) G_h(x_0, y).$$

Again, using (5.12) with $S(y) = h^2 g_h(x, y)$ for x fixed,

$$h^2 \sum_{y \in \Omega_h} G_h(x_0, y) g_h(x, y) \leq Ch^2 \max_{y \in \Omega_h^{(3)}} |g_h(x, y)| + \max_{z \in \Omega_h} h^2 \sum_{y \in \Omega_h} g_h(x_0, y) g_h(x, y).$$

By (4.11), this term can be seen to be bounded. Finally, letting $S(y) = h^2 \delta(y_0, y)$ in (5.12), we have, for any $y_0 \in \Omega_h$,

$$|h^2 G_h(x_0, y_0)| \leq C \left[h^2 + \max_{z \in \Omega_h} h^2 g_h(x, y_0) \right],$$

which indeed tends to zero as h does, by (4.11), and (5.16) follows. For (5.17) use $S = h^2$ and (4.10).

We require yet one more Green's function G'_h defined by

$$(5.18) \quad -\Delta_h G'_h(x, y) = h^{-2} \delta(x, y), \quad x \in \Omega'_h, \quad G'_h(x, y) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)},$$

for all $y \in \Omega_h$. Thus, the matrix $[G'_h(x, y)]_{x, y \in \Omega_h}$ is the inverse of the symmetric matrix $\mathfrak{L} = [h^2 I_h(x, y)]_{x, y \in \Omega_h}$. We show \mathfrak{L} is monotone by applying Corollary 3.6. First, we show $\mathfrak{L} + \frac{1}{3}I$ monotone from Corollary 3.5: we define M_1 by

$$\begin{aligned} [M_1]_{xy} &= \frac{16}{3}, & x = y, \\ &= -\frac{4}{3}, & |x - y| = h, \\ &= 0, & \text{otherwise,} \end{aligned}$$

for $x, y \in \Omega'_h$, and we define

$$\begin{aligned} [M_2]_{xy} &= \frac{8}{\sqrt{12}}, & x = y, \\ &= -\frac{1}{\sqrt{12}}, & |x - y| = h, \\ &= 0, & \text{otherwise.} \end{aligned}$$

Since M_1 and M_2 are of positive type, they are monotone, hence, so is M_2^2 , and it is easy to see that

$$M_1 \leq \mathfrak{L} + \frac{1}{3}I \leq M_2^2.$$

Thus, \mathfrak{E} is monotone if its eigenvalues, necessarily real by symmetry, are positive. But these are $h^2\mu_h^{(i)}$, where $\mu_h^{(i)}$ is the i th eigenvalue satisfying

$$(5.19) \quad \Delta_h V_h^{(i)}(x) + \mu_h^{(i)} V_h^{(i)}(x) = 0, \quad x \in \Omega'_h, \quad V_h^{(i)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

In the next section, we shall show that indeed $|\mu_h^{(i)} - \lambda^{(i)}| \rightarrow 0$ as $h \rightarrow 0$, for $\lambda^{(i)}$ the i th eigenvalue of (1.1), which is strictly positive. Thus, for h sufficiently small, \mathfrak{E} is monotone and G'_h nonnegative. Thus, as a consequence of Lemma 3.7,

$$(5.20) \quad 0 \leq G'_h(x, y) \leq g_h(x, y), \quad x, y \in \Omega_h.$$

From (5.20), we see that all of the inequalities proved for g_h hold for G'_h . In particular, the difficult inequality (5.10) does, from which we prove the key inequality

$$(5.21) \quad \max_{\Omega_h} |W| \leq C \left[\max_{\Omega'_h} |\Delta_h W| + \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |W| \right],$$

for all W defined on Ω_h . To prove this, let

$$\begin{aligned} W^*(x) &= W(x), & x \in \Omega'_h, \\ &= 0, & x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}. \end{aligned}$$

Then, by (5.18),

$$\begin{aligned} W^*(x) &= h^2 \sum_{y \in \Omega_h'} G'_h(x, y) [-\Delta_h W^*(y)] \\ &= h^2 \sum_{y \in \Omega_h'} G'_h(x, y) [-\Delta_h W(y)] + h^2 \sum_{y \in \Omega_h''} G'_h(x, y) [\Delta_h W(y) - \Delta_h W^*(y)], \end{aligned}$$

and (5.21) follows from (4.6), (5.10), and (5.20).

6. Convergence of $\mu_h^{(n)}$ to $\lambda^{(n)}$. In this section, we show that the eigenvalue $\mu_h^{(n)}$ of

$$(6.1) \quad \Delta_h V_h^{(n)}(x) + \mu_h^{(n)} V_h^{(n)}(x) = 0, \quad x \in \Omega'_h, \quad V_h^{(n)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)},$$

converges to $\lambda^{(n)}$ of (1.1) for each n . We will use the variational principles associated with (1.1) and (6.1), and a technique of Weinberger [9].

The n th eigenvalue of (1.1) can be characterized by

$$(6.2) \quad \lambda^{(n)} = \min \max D(u) / \int_{\Omega} u^2 dx,$$

where $u = \alpha_1 u_1 + \dots + \alpha_n u_n$, the max is with respect to the scalars $\alpha_1, \dots, \alpha_n$, the min is with respect to choices of linearly independent u_1, \dots, u_n , continuous, piecewise differentiable functions vanishing on $\partial\Omega$, and $D(u)$ is the Dirichlet integral.

Similarly, the n th eigenvalue of (6.1) can be characterized by

$$(6.3) \quad \mu_h^{(n)} = \min \max \frac{h^2 \sum \left[U_{x_1}^2 + U_{x_2}^2 + \frac{h^2}{12} U_{x_1 \bar{x}_1}^2 + \frac{h^2}{12} U_{x_2 \bar{x}_2}^2 \right]}{h^2 \sum U^2},$$

where $U = \alpha_1 U_1 + \dots + \alpha_n U_n$, the max is with respect to the scalars $\alpha_1, \dots, \alpha_n$, the min is with respect to choices of linearly independent mesh functions U_1, \dots, U_n vanishing on $\Omega_h^{(2)} \cup \Omega_h^{(3)}$, the sum is over the mesh points of Ω'_h , and subscript x_i (\bar{x}_i)

denotes forward (backward) difference quotient in the x_i direction, $i = 1, 2$, i.e., $U_{x_1}(y_1, y_2) = [U(y_1 + h, y_2) - U(y_1, y_2)]/h$, etc.

First, we show

$$(6.4) \quad \mu_h^{(n)} \leq \lambda^{(n)} + O(h).$$

Let $u^{(1)}, \dots, u^{(n)}$ be eigenfunctions associated with $\lambda^{(1)}, \dots, \lambda^{(n)}$ in (1.1), $u = \alpha_1 u^{(1)} + \dots + \alpha_n u^{(n)}$, and define

$$\begin{aligned} u(x) &= h^{-1} \int_{Q_h(x)} u(y) dy, & x \in \Omega'_h, \\ &= 0, & x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}, \end{aligned}$$

where $Q_h(x) = \{(y_1, y_2) : |x_1 - y_1| \leq \frac{1}{2}h, |x_2 - y_2| \leq \frac{1}{2}h\}$ is the square of side h centered at x . Put this U in (6.3). Employing inequalities (2.14), (2.22) and (8.6) of Weinberger [9], we see that

$$\mu_h^{(n)} \leq \max_{\alpha} \frac{D(u) + \frac{h^2}{12} \int_{\Omega} \left\{ \left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right\} dx}{\int_{\Omega} u^2 dx - (h^2/\pi^2)D(u)},$$

and Hubbard [5, pp. 568–569], has shown

$$\int_{\Omega} \left\{ \left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right\} dx \leq C(\lambda^{(n)})^2.$$

From these, (6.4) follows.

Next, we show

$$(6.5) \quad \lambda^{(n)} \leq \mu_h^{(n)} + O(h).$$

Let $V_h^{(1)}, \dots, V_h^{(n)}$ be eigenvectors associated with $\mu_h^{(1)}, \dots, \mu_h^{(n)}$ in (6.1), $U = \alpha_1 V_h^{(1)} + \dots + \alpha_n V_h^{(n)}$, and define u to be the continuous, piecewise linear function interpolating U (see [9, Section 6]). Then, by (6.4), (6.7) of [9] we see that

$$\begin{aligned} \lambda^{(n)} &\leq \max_{\alpha} \frac{h^2 \sum (U_{x_1}^2 + U_{x_2}^2)}{h^2 \sum U^2 - \frac{1}{4}h^4 \sum (U_{x_1}^2 + U_{x_2}^2)} \\ &\leq \max_{\alpha} \frac{h^2 \sum \left[U_{x_1}^2 + U_{x_2}^2 + \frac{h^2}{12} U_{x_1 x_1}^2 + \frac{h^2}{12} U_{x_2 x_2}^2 \right]}{h^2 \sum U^2 - \frac{1}{4}h^2 \sum \left[U_{x_1}^2 + U_{x_2}^2 + \frac{h^2}{12} U_{x_1 x_1}^2 + \frac{h^2}{12} U_{x_2 x_2}^2 \right]} \\ &= \frac{\mu_h^{(n)}}{1 - \frac{1}{4}h^2 \mu_h^{(n)}} \end{aligned}$$

and we obtain (6.5). Combining (6.4) and (6.5), we have

$$(6.6) \quad |\mu_h^{(n)} - \lambda^{(n)}| \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

for each $n = 1, 2, \dots$.

7. Convergence of $\lambda_h^{(n)}$ to $\lambda^{(n)}$ by Perturbation. Next, we will show that the $\lambda_h^{(n)}$ are a perturbation of the $\mu_h^{(n)}$, and that as h tends to zero, $\lambda_h^{(n)}$ tends to $\mu_h^{(n)}$, hence to $\lambda^{(n)}$, by Section 6. We employ the following theorem of Wielandt:

THEOREM 7.1. *If A, B are $\nu \times \nu$ matrices and A has an orthonormal basis of eigenvectors, then the eigenvalues of B lie in the union of the ν discs $|\mu^{(i)} - z| \leq \|A - B\|_2$, where the $\mu^{(i)}$ are the eigenvalues of A . If k discs are disjoint from the others, they contain exactly k eigenvalues of B .*

In the theorem, $\|\cdot\|_2$ is the spectral norm of a matrix, defined by

$$\|M\|_2 = \sup_{\xi} \|M\xi\|_2 / \|\xi\|_2, \quad \text{where } \|\xi\|_2 = \left(\sum_{i=1}^{\nu} |\xi_i|^2 \right)^{1/2}$$

for a ν -vector $\xi = (\xi_1, \dots, \xi_{\nu})$. For a proof of the theorem, see [6].

We apply the theorem as follows. For A , we take the matrix $[h^2 G'_h(x, y)]_{x, y \in \Omega_h}$. Note that the minor $[h^2 G'_h(x, y)]_{x, y \in \Omega'_h}$ is symmetric, while $h^2 G'_h(x, y) = 0$ for $x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}$, so that A has an orthonormal basis of eigenvectors, and the eigenvalues are simply $[\mu_h^{(i)}]^{-1}$ plus some zeros. For B , we take the matrix $[h^2 G_h(x, y)]$ whose eigenvalues are $[\lambda_h^{(i)}]^{-1}$. Thus, we must estimate $\|h^2(G_h - G'_h)\|_2$. However, for any matrix,

$$\|M\|_2 \leq [\rho(MM^T)]^{1/2} \leq \|MM^T\|_1^{1/2},$$

where $\|\cdot\|_1$ is the maximum of the absolute row sums of the matrix. This is a consequence of the Gerschgorin circle theorem (see, e.g., [7, p. 146]). Thus, we need to estimate

$$(7.1) \quad h^4 \max_{x \in \Omega_h} \sum_{y \in \Omega_h} \left| \sum_{z \in \Omega_h} [G_h(x, z) - G'_h(x, z)][G_h(y, z) - G'_h(y, z)] \right|.$$

Let x_0 be the point where the max is attained and put

$$\sigma(y) = \operatorname{sgn} \sum_{z \in \Omega_h} [G_h(x_0, z) - G'_h(x_0, z)][G_h(y, z) - G'_h(y, z)].$$

Then, let

$$W(x) = h^4 \sum_{y, z \in \Omega_h} [G_h(x, z) - G'_h(x, z)][G_h(y, z) - G'_h(y, z)]\sigma(y)$$

in (4.9). Then, (7.1) is bounded by

$$(7.2) \quad Ch^4 \max_{z \in \Omega_h^{(3)}} \sum_{y \in \Omega_h} |G_h(y, z) - G'_h(y, z)| + Ch^4 \max_{z \in \Omega_h^{(2)'}} \sum_{z \in \Omega_h'} G'_h(x, z) \sum_{y \in \Omega_h} |G_h(y, z) - G'_h(y, z)|.$$

Now,

$$h^2 \sum_{y \in \Omega_h} |G_h(y, z) - G'_h(y, z)| \leq C \max_{y, z \in \Omega_h} [|G_h(y, z)| + G'_h(y, z)] \leq C|\log h|,$$

by (4.11), (5.15) and (5.20). Using this in (7.2) and also (4.10) and (5.20), we have (7.2) bounded by $Ch|\log h|$, which tends to zero as h tends to zero. Thus, the radii of the discs in Theorem 7.1 tend to zero as h does. Since the $\mu_h^{(n)}$ tend to the $\lambda^{(n)}$, which have no finite accumulation point, the disc associated with $[\mu_h^{(n)}]^{-1}$ for any

fixed n eventually becomes disjoint from the remaining discs. Consequently, for any fixed n and $\epsilon > 0$, there is h sufficiently small that

$$(7.3) \quad |\lambda_h^{(n)} - \lambda^{(n)}| < \epsilon.$$

8. Main Theorem. We are now ready to state and prove our main theorem:

THEOREM 8.1. *Let $\lambda^{(n)}$ be the n th eigenvalue of (1.1), let $\lambda_h^{(n)}$ be the n th eigenvalue of (2.6) with associated eigenvector $U_h^{(n)}$. For each $n = 1, 2, \dots$, there are constants C_n, h_n such that for $h < h_n$*

$$(8.1) \quad |\lambda_h^{(n)} - \lambda^{(n)}| < C_n h^4,$$

and there is an eigenfunction $u^{(n)}$ associated with $\lambda^{(n)}$ such that

$$(8.2) \quad \max_{\Omega_h} |U_h^{(n)} - u^{(n)}| < C_n h^4.$$

Proof. With the machinery generated in the previous sections, our proof will have exactly the form of the proof of the corresponding Theorem 5.1 of [6]. For this reason, we only sketch the proof.

By (7.3)

$$(8.3) \quad |\lambda_h^{(n)}| \leq C_n.$$

By (5.11), (2.6) is equivalent to

$$(8.4) \quad U_h^{(n)}(x) = \lambda_h^{(n)} h^2 \sum_{y \in \Omega_h} G_h(x, y) U_h^{(n)}(y), \quad x \in \Omega_h.$$

Let us use the notations

$$\begin{aligned} \langle U, V \rangle_h &\equiv h^2 \sum_{y \in \Omega_h} U(y) \overline{V(y)}, & \|U\|_h &\equiv \langle U, U \rangle_h^{1/2}, \\ \langle U, V \rangle'_h &\equiv h^2 \sum_{y \in \Omega_h} U(y) \overline{V'(y)}, & \|U\|'_h &\equiv \langle U, U \rangle'^{1/2}. \end{aligned}$$

If $U_h^{(n)}$ is normalized by requiring $\|U_h^{(n)}\|_h = 1$, then (8.4), (8.3), the Schwarz inequality, and (5.16) show

$$(8.5) \quad \max_{\Omega_h} |U_h^{(n)}| \leq C_n.$$

From (8.4), (8.5) and (5.17), we see that for $|x - \partial\Omega| \leq Ch$

$$(8.6) \quad |U_h^{(n)}(x)| \leq C_n h.$$

Let us suppose that $\lambda^{(n)} = \lambda^{(n+1)} = \dots = \lambda^{(n+m)}$ is an eigenvalue of multiplicity $m + 1$. Since Δ_h restricted to Ω'_h is symmetric, the eigenvectors $V_h^{(n)}$ of (6.1) are a complete orthonormal basis on Ω'_h :

$$\langle V_h^{(i)}, V_h^{(j)} \rangle'_h = \delta(i, j).$$

If we set

$$\tilde{V}_h^{(i)} = \sum_{j=n}^{n+m} \langle U_h^{(i)}, V_h^{(j)} \rangle'_h V_h^{(j)}, \quad i = n, \dots, n + m,$$

then

$$(8.7) \quad \|U_h^{(i)} - \tilde{V}_h^{(i)}\|_h \leq C_n h, \quad i = n, \dots, n + m.$$

This follows from Parseval's identity:

$$\begin{aligned} \|U_h^{(i)}\|_h^2 &= \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle'_h + \sum_{j \neq n, \dots, n+m} |\langle U_h^{(i)}, V_h^{(j)} \rangle'_h|^2 \\ &= \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle'_h + \sum_{j \neq n, \dots, n+m} \left| \frac{\mu_h^{(j)}}{\mu_h^{(j)} - \lambda_h^{(i)}} \langle H_h^{(i)}, V_h^{(j)} \rangle'_h \right|^2, \end{aligned}$$

where $H_h^{(i)}$ is uniquely defined by

$$\Delta_h H_h^{(i)}(x) = 0, \quad x \in \Omega'_h, \quad H_h^{(i)}(x) = U_h^{(i)}(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

It follows from our hard-won inequality (5.21) that

$$\max_{\Omega_h} |H_h^{(i)}| \leq \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)}| \leq C_i h,$$

by (8.6), and so

$$\|U_h^{(i)} - \tilde{V}_h^{(i)}\|_h^2 \equiv \|U_h^{(i)}\|_h^2 - \langle U_h^{(i)}, \tilde{V}_h^{(i)} \rangle'_h \leq C_i h^2.$$

In a very similar manner, we show that if

$$\tilde{\tilde{V}}_h^{(i)} = \sum_{j=n}^{n+m} \langle U^{(i)}, V_h^{(j)} \rangle'_h V_h^{(j)}, \quad i = n, \dots, n + m,$$

then

$$(8.8) \quad \|u^{(i)} - \tilde{\tilde{V}}_h^{(i)}\|_h \leq C_n h, \quad i = n, \dots, n + m.$$

From (8.8), we can conclude that the $(m + 1) \times (m + 1)$ matrix $[\langle u^{(i)}, V_h^{(j)} \rangle'_h]$, $i, j = n, \dots, n + m$, is nonsingular. In particular then, there are eigenvectors

$$u_h^{(i)} = \sum_{j=n}^{n+m} a_{ij}(h) u^{(j)}, \quad i = n, \dots, n + m,$$

in the eigenmanifold associated with $\lambda^{(n)}$ such that

$$(8.9) \quad \langle u_h^{(i)}, V_h^{(j)} \rangle'_h = \langle U_h^{(i)}, V_h^{(j)} \rangle'_h, \quad i, j = n, \dots, n + m.$$

Moreover, the coefficients $a_{ij}(h)$ are bounded independently of h .

Then, it follows from (8.9) and Parseval's identity that

$$\begin{aligned} \|U_h^{(i)} - u_h^{(i)}\|_h^2 &= h^2 \sum_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}|^2 + \sum_{j \neq n, \dots, n+m} |\langle U_h^{(i)} - u_h^{(i)}, V_h^{(j)} \rangle'_h|^2 \\ &= h^2 \sum_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}|^2 \\ &\quad + \sum_{j \neq n, \dots, n+m} \left| \frac{\mu_h^{(j)}}{\mu_h^{(j)} - \lambda_h^{(i)}} \langle H_h^{(i)}, V_h^{(j)} \rangle'_h - \frac{\mu_h^{(j)}}{\mu_h^{(j)} - \lambda^{(i)}} \langle \tilde{H}_h^{(i)}, V_h^{(j)} \rangle'_h \right|^2, \end{aligned}$$

where $\tilde{H}_h^{(i)}$ is defined by

$$\Delta_h \tilde{H}_h^{(i)}(x) = 0, \quad x \in \Omega'_h, \quad \tilde{H}_h^{(i)}(x) = u_h^{(i)}(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

Since $|u_h^{(i)}(x)| \leq C_i h$ for $|x - \partial\Omega| \leq Ch$, we see that

$$(8.10) \quad \|U_h^{(i)} - u_h^{(i)}\|_h \leq C_i h.$$

From (8.10), we also have

$$(8.11) \quad |\langle U_h^{(i)}, u_h^{(i)} \rangle_h| \geq 1 - C_i h^2.$$

Inequality (8.11) is the key inequality needed to prove the first half of Theorem 8.1, for now

$$(8.12) \quad \begin{aligned} (\lambda_h^{(i)} - \lambda^{(i)}) \langle U_h^{(i)}, u_h^{(i)} \rangle &= \langle U_h^{(i)}, \Delta u_h^{(i)} - \Delta_h^* u_h^{(i)} \rangle_h \\ &= \langle U_h^{(i)}, \tau_h u_h^{(i)} \rangle_h - \langle u_h^{(i)}, \tau_h u_h^{(i)} \rangle_h + \langle \tau_h u_h^{(i)}, u_h^{(i)} \rangle_h \\ &\quad + \langle U_h^{(i)} - u_h^{(i)}, \Delta_h u_h^{(i)} - \Delta_h^* u_h^{(i)} \rangle_h, \end{aligned}$$

obtained by adding and subtracting terms. We have used the notations

$$\tau_h u_h^{(i)} \equiv \Delta u_h^{(i)} - \Delta_h u_h^{(i)}$$

for the truncation error, and Δ_h^* for the adjoint of Δ_h defined by

$$\Delta_h^* V(x) = \sum_{y \in \Omega_h} l_h(y, x) V(y).$$

Recall by (2.6) and our smoothness assumption on $u^{(i)}$ that

$$\begin{aligned} |\tau_h u_h^{(i)}| &\leq C_i h^4, && \text{on } \Omega'_h, \\ &\leq C_i h^2, && \text{on } \Omega_h^{(2)} \cup \Omega_h^{(3)}. \end{aligned}$$

However, on $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ both $U_h^{(i)}$ and $u_h^{(i)}$ are bounded by $C_i h$, while the number of points in $\Omega_h^{(2)} \cup \Omega_h^{(3)}$ is only proportional to h^{-1} . From these considerations, we see that the first three terms on the right side of (8.12) are bounded by $C_i h^4$. As for the remaining term,

$$\Delta_h u_h^{(i)}(x) - \Delta_h^* u_h^{(i)}(x)$$

vanishes for $x \notin \Omega'_h \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$, and is bounded by

$$C h^{-2} \max_{\Omega_h' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |u_h^{(i)}| \leq C_i h^{-1}$$

for $x \in \Omega_h' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$. Again noting that the number of points in $\Omega_h' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}$ is only proportional to h^{-1} , the last term on the right of (8.12) is bounded by

$$C_i \max_{\Omega_h' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}|.$$

Thus, using (8.11) we have the inequality

$$(8.13) \quad |\lambda_h^{(i)} - \lambda^{(i)}| \leq C_i \left[\max_{\Omega_h' \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| + h^4 \right].$$

We next employ the discrete Green's function to write

$$(8.14) \quad \begin{aligned} U_h^{(i)}(x) - u_h^{(i)}(x) &= h^2 \sum_{y \in \Omega_h} G_h(x, y) \Delta_h [u_h^{(i)}(y) - U_h^{(i)}(y)] \\ &= -h^2 \sum_{y \in \Omega_h} G_h(x, y) \tau_h u_h^{(i)}(y) + \lambda^{(i)} h^2 \sum_{y \in \Omega_h} G_h(x, y) [U_h^{(i)}(y) - u_h^{(i)}(y)] \\ &\quad + (\lambda_h^{(i)} - \lambda^{(i)}) h^2 \sum_{y \in \Omega_h} G_h(x, y) U_h^{(i)}(y). \end{aligned}$$

Using inequalities (5.13) and (5.14), we see that the first term on the right of (8.14) is bounded by $C_i h^4$. By (5.14) and (8.5) the last term on the right is bounded by

$C_i |\lambda_h^{(i)} - \lambda^{(i)}|$, or if $|x - \partial\Omega| \leq Ch$, (5.17) shows the last term bounded by $C_i h |\lambda_h^{(i)} - \lambda^{(i)}|$. Using (8.3), (5.16) and Schwarz's inequality bound the middle term on the right by $\|U_h^{(i)} - u_h^{(i)}\|_h$, or, if $|x - \partial\Omega| \leq Ch$, (5.17) bounds it by $C_i h \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}|$. In summary,

$$(8.15) \quad \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}| \leq C_i [\|U_h^{(i)} - u_h^{(i)}\|_h + |\lambda_h^{(i)} - \lambda^{(i)}| + h^4],$$

$$(8.16) \quad \max_{\Omega_h \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| \leq C_i \left[h \max_{\Omega_h} |U_h^{(i)} - u_h^{(i)}| + h |\lambda_h^{(i)} - \lambda^{(i)}| + h^4 \right].$$

Finally, we use Parseval's identity and (8.9) to conclude that

$$\|U_h^{(i)} - u_h^{(i)}\|_h^2 = h^2 \sum_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}|^2 + \sum_{j \neq n, \dots, n+m} |\langle U_h^{(i)} - u_h^{(i)}, V_h^{(j)} \rangle_h|^2,$$

and by a straightforward computation

$$(\mu_h^{(i)} - \lambda^{(i)}) \langle U_h^{(i)} - u_h^{(i)}, V_h^{(i)} \rangle_h = \langle (\lambda_h^{(i)} - \lambda^{(i)}) U_h^{(i)} - \tau_h u_h^{(i)} + \tilde{H}_h^{(i)}, V_h^{(i)} \rangle_h,$$

where $\tilde{H}_h^{(i)}$ is defined by

$$\Delta_h \tilde{H}_h^{(i)}(x) = 0, \quad x \in \Omega_h', \quad \tilde{H}_h^{(i)}(x) = U_h^{(i)}(x) - u_n^{(i)}(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

It follows that

$$(8.17) \quad \|U_h^{(i)} - u_h^{(i)}\|_h \leq C_i \left[\max_{\Omega_h \cup \Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h^{(i)} - u_h^{(i)}| + |\lambda_h^{(i)} - \lambda^{(i)}| + h^4 \right].$$

Combining (8.13), (8.15), (8.16), and (8.17) yields the proof of Theorem 8.1.

Let us observe some simple consequences of Theorem 8.1. Since the $\lambda^{(i)}$ are real, we have

$$(8.18) \quad |\operatorname{Re} \lambda_h^{(i)} - \lambda^{(i)}| \leq Ch^4.$$

Also, when $\lambda^{(i)}$ is simple, $\lambda_h^{(i)}$ will be real for h sufficiently small. This is because the matrix $[l_h(x, y)]_{x, y \in \Omega_h}$ is real. Thus, if $\lambda_h^{(i)}$ were complex, its conjugate $[\lambda_h^{(i)}]^-$ would also be a distinct eigenvalue of Δ_h converging to $\lambda_h^{(i)}$. But this is impossible, since $[\lambda_h^{(i)}]^-$ must converge to some $\lambda^{(i)} \neq \lambda^{(i)}$.

We normalized $U_h^{(i)}$ by requiring $\|U_h^{(i)}\|_h = 1$. This determines $U_h^{(i)}$ only up to a multiplicative constant of modulus 1. If we specify this constant by requiring that $\langle U_h^{(i)}, V_h^{(i)} \rangle_h \geq 0$, then when $\lambda^{(i)}$ is simple, $u_h^{(i)}$ is a real multiple of $u^{(i)}$, as can be seen from (8.9).

Theorem 8.1 shows that $U_h^{(i)}$ approximates to $O(h^4)$ an eigenfunction $u_h^{(i)}$ which depends on h . Properly normalized, however, $U_h^{(i)}$ will approximate to $O(h^4)$ an eigenfunction $u_h^{(i)}$ such that $\int_{\Omega} |u_h^{(i)}|^2 dx = 1$, independently of h . In particular, when $\lambda^{(i)}$ is simple, $U_h^{(i)}$ will approximate the unique normalized eigenfunction $u^{(i)}$. This normalization is

$$h^2 \sum_{y \in \Omega_h} \alpha_h(y) |U_h^{(i)}(y)|^2 = 1,$$

where α_h is given in the appendix of [6]. For a proof, see [6, Corollary 6.2].

9. Forced Vibration Problems. Let us remark that all of the results of the previous sections hold for the problem

$$(9.1) \quad \Delta u(x) + (q(x) + \lambda)u(x) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega,$$

where q is nonpositive and smooth on Ω , and for the discrete Green's function G_h defined by

$$(9.2) \quad (\Delta_{h,x} + q(x))G_h(x, y) = -h^{-2}\delta(x, y), \quad x, y \in \Omega_h.$$

The proofs require only that the additional term q be carried along throughout. We make this remark because we next wish to consider the problem

$$(9.3) \quad \Delta u(x) + r(x)u(x) = F(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega,$$

for F and r given smooth functions on Ω . Problem (9.3) is a forced vibration problem and an $O(h^2)$ analogue of it was studied by Bramble in [1].

Let us rewrite (9.3) in the form

$$(9.4) \quad \Delta u(x) + q(x)u(x) + \left(\sup_{\Omega} r\right)u(x) = F(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial\Omega,$$

where $q(x) \equiv r(x) - \sup_{\Omega} r \leq 0$ on Ω . A unique solution u of (9.3) or (9.4) exists if and only if $\sup r$ is not an eigenvalue of the operator $\Delta + q$. Now, we consider the difference approximation

$$(9.5) \quad \Delta_h U_h(x) + r(x)U_h(x) = F(x), \quad x \in \Omega_h,$$

where Δ_h is the difference operator defined in Section 2. We prove:

THEOREM 9.1. *If (9.3) has a unique solution $u \in C^6(\bar{\Omega})$, there are constants C, h_0 such that for $h < h_0$, (9.5) has a unique solution U_h for which*

$$\max_{\Omega_h} |U_h - u| < Ch^4.$$

Proof. Let G_h be the discrete Green's function defined in (9.2). Then, for $x \in \Omega_h$,

$$\begin{aligned} |U_h(x) - u(x)| &= \left| h^2 \sum_{y \in \Omega_h} G_h(x, y) [\Delta_h u(y) + q(y)u(y) - \Delta_h U_h(y) - q(y)U_h(y)] \right| \\ &\leq \sup_{\Omega} |q| h^2 \sum_{y \in \Omega_h} |G_h(x, y)| |U_h(y) - u(y)| + h^2 \sum_{y \in \Omega_h} |G_h(x, y)| |\tau_h u(y)|. \end{aligned}$$

Therefore, using (5.13) and (5.14) for G_h of (9.2) and (2.5),

$$(9.6) \quad |U_h(x) - u(x)| \leq C \left[h^2 \sum_{y \in \Omega_h} |G_h(x, y)| |U_h(y) - u(y)| + h^4 \right].$$

Employing (5.17), this yields

$$(9.7) \quad \max_{\Omega_h^{(*)} \cup \Omega_h^{(s)}} |U_h - u| \leq C \left[h \max_{\Omega_h} |U_h - u| + h^4 \right],$$

while (5.16) and Schwarz's inequality yield

$$(9.8) \quad \max_{\Omega_h} |U_h - u| \leq C [\|U_h - u\|_h + h^4].$$

From (9.7) and (9.8), we see

$$\begin{aligned} \|U_h - u\|_h &\leq \|U_h - u\|'_h + Ch^{1/2} \max_{\Omega_h^{(*)} \cup \Omega_h^{(s)}} |U_h - u| \\ &\leq \|U_h - u\|'_h + Ch^{1/2} [h \|U_h - u\|_h + h^4], \end{aligned}$$

which implies

$$(9.9) \quad \|U_h - u\|_h \leq C[\|U_h - u\|'_h + h^4].$$

Finally, we complete the proof by using Parseval's identity to estimate

$$(9.10) \quad \|U_h - u\|'_h = \left[\sum_i |\langle U_h - u, V_h^{(i)} \rangle'_h|^2 \right]^{1/2},$$

where $V_h^{(i)}$ is the eigenvector associated with $\mu_h^{(i)}$ in the symmetric problem

$$\Delta_h V_h^{(i)}(x) + (q(x) + \mu_h^{(i)}) V_h^{(i)}(x) = 0, \quad x \in \Omega'_h, \quad V_h^{(i)}(x) = 0, \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

Define H_h by

$$\Delta_h H_h(x) + q(x)H_h(x) = 0, \quad x \in \Omega'_h, \quad H_h(x) = U_h(x) - u(x), \quad x \in \Omega_h^{(2)} \cup \Omega_h^{(3)}.$$

From (5.21), we have

$$\max_{\Omega_h'} |H_h| \leq C \max_{\Omega_h^{(2)} \cup \Omega_h^{(3)}} |U_h - u|,$$

or, employing (9.7), (9.8), (9.9),

$$(9.11) \quad \max_{\Omega_h'} |H_h| \leq C[h \|U_h - u\|'_h + h^4].$$

Then, we have

$$\begin{aligned} \mu_h^{(i)} \langle U_h - u, V_h^{(i)} \rangle'_h &= \langle H_h + u - U_h, (\Delta_h + q) V_h^{(i)} \rangle'_h + \mu_h^{(i)} \langle H_h, V_h^{(i)} \rangle'_h \\ &= \langle (\Delta_h + q)(H_h + u - U_h), V_h^{(i)} \rangle'_h + \mu_h^{(i)} \langle H_h, V_h^{(i)} \rangle'_h \\ &= (\sup r) \langle U_h - u, V_h^{(i)} \rangle'_h - \langle \tau_h u, V_h^{(i)} \rangle'_h + \mu_h^{(i)} \langle H_h, V_h^{(i)} \rangle'_h. \end{aligned}$$

Now, since $\sup r$ is not an eigenvalue $\lambda^{(i)}$ of $\Delta + q$ and $\mu_h^{(i)} \rightarrow \lambda^{(i)}$ as $h \rightarrow 0$, there are constants C, h_0 such that for $h < h_0$,

$$\max_i |\mu_h^{(i)} - \sup r|^{-1} < C, \quad \max_i \mu_h^{(i)} / |\mu_h^{(i)} - \sup r| < C,$$

and so

$$|\langle U_h - u, V_h^{(i)} \rangle'_h| \leq C[|\langle \tau_h u, V_h^{(i)} \rangle'_h| + |\langle H_h, V_h^{(i)} \rangle'_h|].$$

Using this in (9.10), we see that

$$\|U_h - u\|'_h \leq C[\|\tau_h u\|'_h + \|H_h\|'_h] \leq C[h^4 + h \|U_h - u\|'_h],$$

by (9.11), from which it follows that

$$\|U_h - u\|'_h \leq Ch^4,$$

completing the proof.

Let us remark that by employing the results of [6], the above technique of proof will show that a unique solution of the forced vibration problem

$$(9.12) \quad \begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(a_i(x) \frac{\partial u(x)}{\partial x_i} \right) + r(x)u(x) &= F(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

can be approximated to $O(h^2)$ by using the symmetric difference scheme given in [6] at the beginning of Section 7.

Applied Physics Laboratory
The Johns Hopkins University
Silver Spring, Maryland 20910

1. J. H. BRAMBLE, "Error estimates for difference methods in forced vibration problems," *SIAM J. Numer. Anal.*, v. 3, 1966, no. 1, pp. 1–12. MR 34 #969.
2. J. H. BRAMBLE & B. E. HUBBARD, "New monotone type approximations for elliptic problems," *Math. Comp.*, v. 18, 1964, pp. 349–367. MR 29 #2982.
3. J. H. BRAMBLE & V. C. THOMÉE, "Point-wise bounds for discrete Green's functions," *SIAM J. Numer. Anal.*, v. 6, 1969, pp. 583–590.
4. R. COURANT & D. HILBERT, *Methoden der Mathematischen Physik*. Vol. I, Springer, Berlin, 1931; English transl., Interscience, New York, 1953. MR 16, 426.
5. B. E. HUBBARD, "Bounds for eigenvalues of the free and fixed membrane by finite difference methods," *Pacific J. Math.*, v. 11, 1961, pp. 559–590. MR 25 #4633.
6. J. R. KUTTLER, "Finite difference approximations for eigenvalues of uniformly elliptic operators," *SIAM J. Numer. Anal.*, v. 7, 1970.
7. M. MARCUS & H. MINC, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, Mass., 1964. MR 29 #112.
8. H. S. PRICE, "Monotone and oscillation matrices applied to finite difference approximations," *Math. Comp.*, v. 22, 1968, pp. 489–516. MR 38 #875.
9. H. F. WEINBERGER, "Lower bounds for higher eigenvalues by finite difference methods," *Pacific J. Math.*, v. 8, 1958, pp. 339–368; erratum, 941. MR 21 #6097.