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Calculation of the Gamma Function by Stirling's Formula

By Robert Spira

Abstract. In this paper, we derive a simple error estimate for the Stirling formula and also give numerical coefficients.

Stirling's formula is:

$$\begin{aligned} \log \Gamma(s) &= (s - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log 2\pi \\ &\quad + \sum_{k=1}^{m} s^{1-2k}(2k)^{-1}(2k - 1)^{-1}B_{2k} + R_m \end{aligned} \tag{1}$$

where

$$R_m = -\int_0^\infty (s + x)^{-2m}(2m)^{-1}B_{2m}(x - [x])\, dx. \tag{2}$$

Formulas (1) and (2) and a simple estimate for $|R_m|$ are derived in de Bruijn [1, pp. 46–48].

Another form of R_m, developed on the assumption Re $s > 0$, is

$$R_m = \frac{2(-1)^m}{s^{2m-1}} \int_0^\infty \left\{ \int_0^t \frac{u^{2m}\, du}{u^2 + s^2} \right\} \frac{dt}{e^{2\pi t} - 1},$$

(Whittaker and Watson [5, p. 252]), and Whittaker and Watson also estimate this expression, finding

$$|R_m| \leqq \frac{|B_{2m+2}|\, K(s)}{(2m + 1)(2m + 2)\, |s|^{2m+1}}$$

where

$$K(s) = \text{upper bound } |s^2/(u^2 + s^2)|, \qquad u \geqq 0.$$

This is the form given in the NBS *Handbook*, and is clearly poor near the imaginary axis. It follows, however, from this form, that if $|\arg s| \leqq \pi/4$, then the error in taking the first m terms of the asymptotic series is less in absolute value than the absolute value of the $(m + 1)$st term. Another form of the remainder, valid for $|\arg s| \leqq \pi - \delta$, is derived in Whittaker and Watson [5, §13.6], but this remainder involves the Hurwitz zeta function, and has never been used for numerical estimates. An estimate for R_m, as given by (2), may be found in Nielsen [6, p. 208], and, expressed in current notation, is

$$|R_m(s)| < \frac{|B_{2m+2}|}{(2m + 1)(2m + 2)\, |s|^{2m+1} (\cos (\tfrac{1}{2} \arg s))^{2m+2}}.$$

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This gives a uniform estimate in the angle $|\arg s| \leqq \pi - \delta$. We now develop an estimate for R_m which has the advantages of simplicity in application, and uniformity for a set of points whose distance from the negative real axis is $\geqq$ some fixed amount.

THEOREM.

$$|R_m| \leqq 2\,|B_{2m}/(2m-1)|\cdot|\operatorname{Im} s|^{1-2m} \quad \text{for } \operatorname{Re} s < 0,\ \operatorname{Im} s \neq 0, \tag{3}$$

$$|R_m| \leqq |B_{2m}/(2m-1)|\cdot|s|^{1-2m} \quad \text{for } \operatorname{Re} s \geqq 0. \tag{4}$$

Proof. Since $B_{2m}(x-[x])$ varies only slightly over the range of x, and $|B_{2m}(x-[x])| \leqq |B_{2m}|$, the problem of estimating $|R_m|$ reduces to the problem of estimating $\int_0^\infty |s+x|^{-2m}\,dx$. Note that the integrand will be large only when s is near $-x$. By symmetry, we need only consider the case when $\operatorname{Im} s \geqq 0$. First, let $\operatorname{Re} s < 0$ and $\operatorname{Im} s \neq 0$. Then, taking $k = \operatorname{Im} s$,

$$\int_0^\infty |s+x|^{-2m}\,dx = \int_0^{-\operatorname{Re} s} + \int_{-\operatorname{Re} s}^{-\operatorname{Re} s+k} + \int_{-\operatorname{Re} s+k}^\infty .$$

Estimating the integrands of the second integral by $|s+x|^{-2m} \leqq k^{-2m}$, and of the third by $|s+x|^{-2m} \leqq (x+\operatorname{Re} s)^{-2m}$, we obtain

$$\int_0^\infty |s+x|^{-2m}\,dx \leqq \int_0^{-\operatorname{Re} s} |s+x|^{-2m}\,dx + k^{1-2m} + (2m-1)^{-1}(k)^{1-2m}.$$

It remains to estimate $\int_0^{-\operatorname{Re} s}$. If $-\operatorname{Re} s \leqq k$, we approximate the integrand again by k^{-2m}, giving

$$\int_0^{-\operatorname{Re} s} |s+x|^{-2m}\,dx \leqq (-\operatorname{Re} s)\cdot k^{-2m} \leqq k^{1-2m}.$$

If, however, $-\operatorname{Re} s > k$, we break up the range of integration, giving

$$\begin{aligned}\int_0^{-\operatorname{Re} s} |s+x|^{-2m}\,dx &\leqq \int_0^{-\operatorname{Re} s-k} |s+x|^{-2m}\,dx + \int_{-\operatorname{Re} s-k}^{-\operatorname{Re} s} |s+x|^{-2m}\,dx \\ &\leqq \int_0^{-\operatorname{Re} s-k} (-x-\operatorname{Re} s)^{-2m}\,dx + k^{1-2m} \\ &= \frac{1}{2m-1}[k^{1-2m} - (-\operatorname{Re} s)^{1-2m}] + k^{1-2m} \\ &\leqq (1+1/(2m-1))k^{1-2m}.\end{aligned}$$

So that in all cases, if $\operatorname{Re} s < 0$

$$\int_0^\infty |x+s|^{-2m}\,dx \leqq (4m/(2m-1))k^{1-2m},$$

so we have derived (3).

If $\operatorname{Re} s \geqq 0$, then $|s+x|^{-2m} \leqq |ki+x|^{-2m}$ since

$$|s+x|^2 = (\operatorname{Re} s + x)^2 + (\operatorname{Im} s)^2 = 2x \operatorname{Re} s + x^2 + k^2 \geqq |ki+x|^2.$$

Next, estimating as before,

$$\int_0^\infty |ki + x|^{-2m}\, dx \leqq \int_0^k k^{-2m}\, dx + \int_k^\infty x^{-2m}\, dx \leqq k^{1-2m}(1 + 1/(2m - 1)),$$

thus giving (4), and completing the proof.

On taking the exponential, we find

$$\Gamma(s) \sim (2\pi)^{1/2} e^{-s} s^{s-1/2} \exp\left[\sum_{k=1}^{N_1} \frac{A_{2k-1}}{s^{2k-1}}\right] \tag{5}$$

where

$$A_{2k-1} = B_{2k}/2k(2k - 1). \tag{6}$$

A short calculation gives (formally)

$$\exp\left[\sum_{k=1}^{\infty} \frac{A_{2k-1}}{s^{2k-1}}\right] = 1 + \sum_{k=1}^{\infty} s^{-k}\left[\sum_{(\alpha_1 j_1, \cdots, \alpha_n j_n) \in Q(k)} \frac{A_{\alpha_1}^{j_1} A_{\alpha_2}^{j_2}, \cdots, A_{\alpha_n}^{j_n}}{j_1!\, j_2!\,, \cdots, j_n!}\right] = 1 + \sum_{k=1}^{\infty} c_k s^{-k} \tag{7}$$

where the α_i's are distinct and $Q(k)$ is the set of partitions of k into odd parts ($\alpha_i^{j_i}$ means α_i repeated j_i times in the partition).

Wrench [2] found the recurrences

$$(2k - 1)c_{2k-1} = \frac{B_2}{2} c_{2k-2} + \frac{B_4}{4} c_{2k-4} + \cdots + \frac{B_{2k}}{2k}, \tag{8}$$

$$2kc_{2k} = \frac{B_2}{2} c_{2k-1} + \frac{B_4}{4} c_{2k-3} + \cdots + \frac{B_{2k}}{2k} c_1, \tag{9}$$

where $k = 1, 2, 3, \cdots$ and $c_0 = 1$, and these formulas are more suitable for calculation than (7).

Wrench [2] also gave the c_j's for $j = 0(1)20$, in exact form and to 50D, and also found approximations to about 6S for $j = 21(1)30$. We give in Table 1 the exact rational values for $j = 21(1)30$, and in Table 2 their 45D equivalents. The following corrections are necessary in Wrench's tables. In his Table 2, the last ten digits of c_{13} read 01893 93280, and should read 01894 09396. In his Table 3, entries 22, 23, 24, 26, 28, 30 can be corrected from Table 2 of this paper. Dr. Wrench confirmed the correctness of the author's value for c_{13}, and that it is likely that the author's corrections to his Table 3 are also valid. It is of interest to note that while Dr. Wrench's calculations were carried out on a desk calculator, the author's were performed on a Fortran simulator of a large decimal machine (Spira [7]).

A further calculation revealed that entries 3, 4, 7, 8, 11, 12, 15, 16, 17 for c_{n+1}/c_n in Table XII of Spira [3] have errors beyond 16S. These errors did not affect the remaining tables.

Finally, we remark that estimates for the error in using

$$\Gamma(s) \sim (2\pi)^{1/2} e^{-s} s^{s-1/2}\left\{1 + \frac{c_1}{s} + \frac{c_2}{s^2} + \cdots + \frac{c_k}{s^k}\right\} \tag{10}$$

can be obtained from estimating

$$\exp\left\{\sum_{j=1}^{m} A_{2j-1} s^{1-2j} + R_m\right\} - \sum_{j=1}^{k} c_j s^{-j} \tag{11}$$

TABLE 1

Coefficients 21 through 30 in the Stirling asymptotic series for $\Gamma(z)$

	Sign	Numerator	Denominator
c_{21}		34856 85173 42344 01648 33562 31076 88675 64083 96794 47003	2601 64872 18125 16297 62664 73959 14866 28167 68000 00000
c_{22}		9 09773 12459 95425 06852 27522 94225 93983 24288 04521 45053	8 11714 40120 55050 84859 51398 75254 38279 88316 16000 00000
c_{23}	−	15273 35577 85467 70230 23224 27280 09471 25313 62926 72693 90501	97 40572 81446 60610 18314 16785 03052 59358 59793 92000 00000
c_{24}	−	18 38564 55668 17780 20033 16143 79951 80647 19008 29995 86348 26921	1 40264 24852 83112 78663 72401 70443 95734 76381 03244 80000 00000
c_{25}		2 58331 20988 61137 96374 59020 36370 49694 38721 38148 65171 20938 16393	117 82196 87637 81474 07752 81743 17292 41720 16006 72563 20000 00000
c_{26}		5 18013 42908 22682 44375 77104 27952 46758 19182 33549 14089 67023 64013	2827 72725 03307 55377 86067 61836 15018 01283 84161 41516 80000 00000
c_{27}	−	5275 50309 09787 33965 92733 54057 99289 93424 14298 36915 19876 84094 84184 33873	14613 12888 42592 77641 70840 23816 85690 08674 63669 36130 51904 00000 00000
c_{28}	−	21148 66241 53708 11646 13223 32421 55728 12504 64870 36484 82437 46060 29560 15127	7 01430 18644 44453 26802 00331 43209 13124 16382 56129 34264 91392 00000 00000
c_{29}		18039 44129 15538 78214 00157 77241 22802 51037 85450 23572 62351 75126 98174 30990 27459	2609 32029 35733 36615 70345 23292 73796 82188 94312 80115 46547 97824 00000 00000
c_{30}		3226 14019 20539 36286 91281 19490 56082 64758 66044 17173 68772 94520 86326 36420 80203 03641	5589 16406 88340 87030 83679 48893 04472 79248 71618 02007 32705 76939 00800 00000 00000

TABLE 2

45D Values of coefficients 21 through 30 in the asymptotic series for $\Gamma(z)$

c_{21}	13 . 39798	54551	42589	21762	69304	32019	67195	04205	85565
c_{22}	1 . 12080	44642	89911	60686	26394	00139	92394	10087	44581
c_{23}	− 156 . 80141	27040	22726	37282	36984	46041	18986	42959	25353
c_{24}	− 13 . 10786	30226	33865	65902	75053	22267	17265	62139	54267
c_{25}	2192 . 55553	60905	23432	96901	29668	35404	98912	17444	39338
c_{26}	183 . 19073	34845	24338	08866	21120	60475	26830	49008	10167
c_{27}	− 36101 . 11929	32220	75951	91379	10143	10212	31172	74408	12019
c_{28}	− 3015 . 07731	26223	05854	21582	73842	95134	58512	61670	77656
c_{29}	691346 . 37614	18781	21600	20149	42362	07859	56471	17679	20033
c_{30}	57721 . 33636	30407	22716	58721	99716	32365	57540	83996	54732

and using (3) and (4), where $m = [(k+2)/2]$. For example, for $\operatorname{Re} s \geqq 0$ and $|s| \geqq 1$, and $k = m = 2$, we have

$$\Gamma(s) = (2\pi)^{1/2} e^{-s} s^{s-1/2} \exp\left\{\frac{1}{12s} + \frac{1}{360s^3} + R_2\right\},$$

where

$$|R_2| \leqq \frac{1}{90|s|^3},$$

so

$$|\exp R_2 - 1| \leqq |R_2| \{1 + |R_2| + |R_2|^2 + \cdots\} \leqq \frac{1}{89|s|^3}.$$

Next,

$$\left|\exp\left(\frac{1}{12s} + \frac{1}{360s^3}\right) - \left(1 + \frac{1}{12s} + \frac{1}{288s^2}\right)\right|$$

$$\leqq \frac{1}{360|s|^3} + \frac{1}{12\cdot 360|s|^4} + \frac{1}{2\cdot 360^2|s|^6} + \frac{1}{3!}\left|\frac{1}{12s} + \frac{1}{360s^3}\right|^3 + \cdots$$

which estimates as before. Such estimates show the series for $\Gamma(s)$ is an asymptotic series (de Bruijn [1]).

For calculations near the origin, it is best to use the functional equation $\Gamma(s+1) = s\Gamma(s)$ and calculate $\Gamma(s) = \Gamma(s+j)/P(s)$, where $P(s)$ is a polynomial. This formula could also be used for larger $|s|$ for ultraprecise calculations where precisions are needed which are greater than the maximum precision obtainable from the asymptotic formula. For calculations in the left half-plane with small imaginary part, one can use the equation $\Gamma(s)\Gamma(1 - s) = \pi/\sin \pi s$.

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