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Miniaturized Tables of Bessel Functions*

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Abstract. In this report, we discuss the representation of bivariate functions in double series of Chebyshev polynomials. For an application, we tabulate coefficients which are accurate to 20 decimals for the evaluation of $(2z/\pi)^{1/2}e^zK_\nu(z)$ for all $z \geqq 5$ and all ν, $0 \leqq \nu \leqq 1$. Since $K_\nu(z)$ is an even function in ν and satisfies a three-term recurrence formula in ν which is stable when used in the forward direction, we can readily evaluate $K_\nu(z)$ for all $z \geqq 5$ and all $\nu \geqq 0$. Only 205 coefficients are required to achieve an accuracy of about 20 decimals for the z and ν ranges described. Extension of these ideas for the evaluation of all Bessel functions and other important bivariate functions is under way.

1. Introduction. One of the difficulties associated with the development of tables of multiparameter functions is the enormously large number of entries needed to cover the ranges required in these parameters. Even if such values are prepared, their direct use on high-speed computers is limited because of storage, the necessity of table look up and interpolation. From the vantage point of automatic computation, the need is for efficient algorithms and small sets of coefficients to compute entries as they are needed. In the case of a two-parameter function, we can expand $f(z, \nu)$, for example, in the form $\sum C_n(\nu)P_n(z)$ where the $P_n(z)$'s are orthogonal polynomials in z and the $C_n(\nu)$'s depend only on ν. This has the advantage that tabulation of a two-parameter function is made to depend upon the tabulation of two other functions, each of a single parameter. Evaluation of sums involving orthogonal polynomials by backward recurrence is easy, provided the three-term recurrence formula for the orthogonal polynomials is known. This is certainly the case for the classical orthogonal polynomials. For many functions, computation of the $C_n(\nu)$'s for a given ν can also be accomplished by use of backward recurrence techniques as is evidenced by my work on the special functions [1]. We now propose to expand $C_n(\nu)$ in series of orthogonal polynomials. In this manner, $f(z, \nu)$ is expanded in a double series, and by giving a rather small set of coefficients, we can compress previous extensive tabulations of two-parameter functions onto a few pages.

In this report, the above ideas are applied for the evaluation of the modified Bessel function $K_\nu(z)$ in a double series of Chebyshev polynomials valid for $z \geqq 5$ and $0 \leqq \nu \leqq 1$. Further work along these lines is continuing, so that, eventually, we will have coefficients for all the Bessel functions over extensive z and ν values. We will also apply these techniques to other important two-parameter functions. Extension of these concepts to three and more parameter functions is also contemplated.

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2. Chebyshev Expansions for $K_\nu(z)$. From the work in [1], we have

$$K_\nu(z) = (\pi/2z)^{1/2}e^{-z}\sum_{k=0}^{\infty} C_k(\nu,\lambda)T_k^*(\lambda/z), \quad \lambda \text{ fixed}, \lambda/z \leqq 1, |\arg z| < 3\pi/2, \tag{1}$$

where $K_\nu(z)$ is the modified Bessel function of order ν and $T_k^*(x)$ is the "shifted" Chebyshev polynomial of the first kind. The coefficients $C_k(\nu, \lambda)$ satisfy the recurrence formula

$$\begin{aligned}\frac{2C_k(\nu,\lambda)}{\epsilon_k} = 2(k+1)\Big\{1 &- \frac{(2k+3)(k+3/2+\nu)(k+3/2-\nu)}{2(k+2)(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)} \\ &- \frac{4\lambda}{(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)}\Big\}C_{k+1}(\nu,\lambda) \\ &+ \Big\{1 - \frac{2(k+1)(2k+3-4\lambda)}{(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)}\Big\}C_{k+2}(\nu,\lambda) \\ &- \frac{(k+1)(k+5/2+\nu)(k+5/2-\nu)}{(k+2)(k+\frac{1}{2}+\nu)(k+\frac{1}{2}-\nu)}C_{k+3}(\nu,\lambda), \\ &\epsilon_0 = 1, \quad \epsilon_k = 2, \quad k > 0.\end{aligned} \tag{2}$$

As the expansion formula (1) converges, for λ and ν fixed,

$$\lim_{k\to\infty} C_k(\nu,\lambda) = 0. \tag{3}$$

Further, since

$$\lim_{z\to\infty} (2z/\pi)^{1/2}e^z K_\nu(z) = 1, \tag{4}$$

we have

$$\sum_{k=0}^{\infty} (-1)^k C_k(\nu,\lambda) = 1. \tag{5}$$

In the reference cited, it is shown that if $|\arg \lambda| < \pi$, the coefficients $C_k(\nu, \lambda)$ are readily evolved by use of (2) in the backward direction. This is very efficient. Further, no prior values of $K_\nu(z)$ need be known. To aid in the application of (2), we have the asymptotic formula [2],

$$C_k(\nu,\lambda) = \frac{4(-1)^k\pi^{1/2}k^{-2/3}(2\lambda)^{1/6}}{3^{1/2}\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\exp(-3k^{2/3}\{2\lambda\}^{1/3})[1+O(k^{-1/3})]. \tag{6}$$

Next, we consider the expansion of the coefficients $C_k(\nu, \lambda)$ in series of Chebyshev polynomials. To this end, it is easier to consider a more general situation. Suppose that $f'(x)$ is continuous in $[-1, 1]$ except for a finite number of bounded jumps. Then, $f(x)$ can be expanded in a convergent series as

$$\begin{aligned} f(x) &= \tfrac{1}{2}c_0 + \sum_{r=1}^{\infty} c_r T_r(x), \qquad -1 \leqq x \leqq 1, \\ c_r &= (2/\pi)\int_{-1}^{1} (1-x^2)^{-1/2} f(x) T_r(x)\,dx. \end{aligned} \tag{7}$$

Here, $T_r(x)$ is the "unshifted" Chebyshev polynomial of the first kind. Simple and easy to evaluate forms for c_k are not often available. If $f(x) = (2z/\pi)^{1/2}e^z K_\nu(z)$,

$z(x + 1) = 2\lambda$, the situation is considerably simplified, in view of the recurrence formula (2) and the normalization relation (5). For the case when $f(x)$ is $C_k(\nu, \lambda)$, no recurrence formula is known, and if the integral representation in (7) is used, we must resort to numerical integration. Now, the Chebyshev polynomials satisfy two orthogonality relations with respect to summation. Use of expansion formulas based on these conditions is equivalent to the evaluation of (7) by trapezoidal-type numerical integration formulas. Indeed, we have the following results. If

$$f(x) = f_n(x) + R_n(x), \tag{8}$$

$$f_n(x) = \tfrac{1}{2}d_0 + \sum_{r=1}^{n-1} d_r T_r(x), \tag{9}$$

then

$$d_r = (2/n) \sum_{\alpha=0}^{n-1} f(x_\alpha)T_r(x_\alpha), \qquad x_\alpha = \cos\theta_\alpha, \qquad \theta_\alpha = (\pi/2n)(2\alpha + 1), \tag{10}$$

and

$$d_r = c_r + \sum_{s=1}^{\infty} (-1)^s\{c_{2sn-r} + c_{2sn+r}\}, \qquad r = 1, 2, \cdots, n - 1. \tag{11}$$

Again, if

$$f(x) = f_n(x) + S_n(x), \tag{12}$$

$$f_n(x) = \tfrac{1}{2}e_0 + \sum_{r=1}^{n-1} e_r T_r(x) + \tfrac{1}{2}T_n(x), \tag{13}$$

then

$$e_r = \frac{2}{n}\left[\tfrac{1}{2}\{f(1) + (-1)^r f(-1)\} + \sum_{\alpha=1}^{n-1} f(x_\alpha)T_r(x_\alpha)\right], \qquad x_\alpha = \cos\varphi_\alpha, \qquad \varphi_\alpha = \alpha\pi/n, \tag{14}$$

and

$$\begin{aligned} e_r &= c_r + \sum_{s=1}^{\infty} \{c_{2sn-r} + c_{2sn+r}\}, \qquad r = 0, 1, \cdots, n - 1, \\ e_n &= 2c_n + 2\sum_{s=1}^{\infty} c_{(2s+1)n}. \end{aligned} \tag{15}$$

Thus, d_r and e_r are approximations to c_r, and from (11) and (15)

$$c_r = \tfrac{1}{2}(d_r + e_r) + \sum_{s=1}^{\infty} \{c_{4sn-r} + c_{4sn+r}\}, \qquad r = 0, 1, \cdots, n - 1. \tag{16}$$

Proof of (8)–(16) is given in [3]. There, we also develop some general information concerning the asymptotic behavior of c_r for large r. In most cases of interest, (7) converges quite rapidly, so that for n sufficiently large, c_r is well approximated by d_r or e_r, and an improved approximation is simply the arithmetic mean of the latter two quantities.

For application of (8)–(16) to $C_k(\nu, \lambda)$ valid for $0 \leqq \nu \leqq 1$, we write

$$C_k(\nu, \lambda) = \sum_{r=0}^{\infty} L_{r,k}(\lambda)T_r^*(\nu), \qquad 0 \leqq \nu \leqq 1. \tag{17}$$

Coefficients in the Expansion of

$$K_\nu(z) = (\pi/2z)^{\frac{1}{2}}e^{-z} \sum_{k=0}^{\infty} C_k(\nu)T_k^*(5/z) \text{ , } z \geq 5 \text{ ,}$$

$$C_k(\nu) = \sum_{r=0}^{\infty} L_{r,k}T_r^*(\nu) \text{ , } 0 \leq \nu \leq 1 \text{ .}$$

r	$L_{r,k}$, k = 0
0	1.00607 66859 78294 33189
1	0.02367 20980 36879 87423
2	0.00608 97384 81944 49510
3	0.00009 90242 93398 39463
4	0.00001 30722 50003 18810
5	0.00000 02196 13551 61367
6	0.00000 00197 68831 32376
7	0.00000 00003 26575 97550
8	0.00000 00000 22498 49793
9	0.00000 00000 00360 09504
10	0.00000 00000 00020 21390
11	0.00000 00000 00000 31189
12	0.00000 00000 00000 01484
13	0.00000 00000 00000 00022
14	0.00000 00000 00000 00001

r	$L_{r,k}$, k = 1
0	0.00602 15036 04662 74176
1	0.02325 81816 55108 78143
2	0.00603 86674 65067 42659
3	0.00012 93931 45178 04993
4	0.00001 71930 09049 19380
5	0.00000 03226 32525 19147
6	0.00000 00291 84000 17904
7	0.00000 00005 11729 27069
8	0.00000 00000 35388 74773
9	0.00000 00000 00587 87945
10	0.00000 00000 00033 10263
11	0.00000 00000 00000 52390
12	0.00000 00000 00000 02499
13	0.00000 00000 00000 00038
14	0.00000 00000 00000 00002

r	$L_{r,k}$, k = 2
0	-0.00005 37236 98553 04638
1	-0.00039 80431 75117 81456
2	-0.00004 97465 49910 04795
3	0.00002 89205 14415 32315
4	0.00000 39880 78060 13616
5	0.00000 01186 02610 05376
6	0.00000 00109 46041 18670
7	0.00000 00002 37314 19859
8	0.00000 00000 16641 85427
9	0.00000 00000 00313 12644
10	0.00000 00000 00017 82002
11	0.00000 00000 00000 30624
12	0.00000 00000 00000 01473
13	0.00000 00000 00000 00024
14	0.00000 00000 00000 00001

r	$L_{r,k}$, k = 3
0	0.00000 13961 41734 51613
1	0.00001 49927 12492 36296
2	0.00000 12681 70364 79510
3	-0.00000 13610 18292 01200
4	-0.00000 01265 51139 39276
5	0.00000 00141 24509 01269
6	0.00000 00014 24282 69365
7	0.00000 00000 55693 42227
8	0.00000 00000 04058 21783
9	0.00000 00000 00100 45462
10	0.00000 00000 00005 85827
11	0.00000 00000 00000 11875
12	0.00000 00000 00000 00581
13	0.00000 00000 00000 00010

r	$L_{r,k}$, k = 4
0	-0.00000 00589 09792 34012
1	-0.00000 08184 31644 38871
2	-0.00000 00530 50266 39835
3	0.00000 00808 85729 57198
4	0.00000 00057 60293 40591
5	-0.00000 00013 30534 35272
6	-0.00000 00000 98969 45233
7	0.00000 00000 02855 70119
8	0.00000 00000 00264 02704
9	0.00000 00000 00015 12912
10	0.00000 00000 00000 94488
11	0.00000 00000 00000 02703
12	0.00000 00000 00000 00138
13	0.00000 00000 00000 00003

r	$L_{r,k}$, k = 5
0	0.00000 00033 63643 06632
1	0.00000 00568 52171 72673
2	0.00000 00030 14215 93849
3	-0.00000 00058 78142 93966
4	-0.00000 00003 42384 83406
5	0.00000 00001 16340 34571
6	0.00000 00000 07004 16824
7	-0.00000 00000 00591 12471
8	-0.00000 00000 00038 11696
9	-0.00000 00000 00000 13843
10	0.00000 00000 00000 00871
11	0.00000 00000 00000 00232
12	0.00000 00000 00000 00014

Coefficients in the Expansion of

$$K_\nu(z) = (\pi/2z)^{\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} C_k(\nu) T_k^*(5/z) \ , \ z \ge 5 \ ,$$

$$C_k(\nu) = \sum_{r=0}^{\infty} L_{r,k} T_r^*(\nu) \ , \ 0 \le \nu \le 1 \ .$$

r	$L_{r,k}$, k = 6
0	-0.00000 00002 37980 67584
1	-0.00000 00047 09153 37960
2	-0.00000 00002 12579 69017
3	0.00000 00005 00775 96094
4	0.00000 00000 24837 13998
5	-0.00000 00000 10973 23314
6	-0.00000 00000 00559 83984
7	0.00000 00000 00075 24545
8	0.00000 00000 00003 99813
9	-0.00000 00000 00000 12459
10	-0.00000 00000 00000 00773
11	-0.00000 00000 00000 00020
12	-0.00000 00000 00000 00001

r	$L_{r,k}$, k = 7
0	0.00000 00000 19822 45732
1	0.00000 00004 47354 03547
2	0.00000 00000 17667 05974
3	-0.00000 00000 48496 56295
4	-0.00000 00000 02104 49873
5	0.00000 00000 01133 89565
6	0.00000 00000 00050 47884
7	-0.00000 00000 00009 13484
8	-0.00000 00000 00000 41875
9	0.00000 00000 00000 02557
10	0.00000 00000 00000 00126

r	$L_{r,k}$, k = 8
0	-0.00000 00000 01880 59807
1	-0.00000 00000 47495 63125
2	-0.00000 00000 01673 32073
3	0.00000 00000 05221 69434
4	0.00000 00000 00202 16650
5	-0.00000 00000 00127 72511
6	-0.00000 00000 00005 06442
7	0.00000 00000 00001 13970
8	0.00000 00000 00000 04625
9	-0.00000 00000 00000 00409
10	-0.00000 00000 00000 00017

r	$L_{r,k}$, k = 9
0	0.00000 00000 00198 60992
1	0.00000 00000 05533 54983
2	0.00000 00000 00176 49151
3	-0.00000 00000 00614 91767
4	-0.00000 00000 00021 55533
5	0.00000 00000 00015 55073
6	0.00000 00000 00000 55762
7	-0.00000 00000 00000 14899
8	-0.00000 00000 00000 00545
9	0.00000 00000 00000 00062
10	0.00000 00000 00000 00002

r	$L_{r,k}$, k = 10
0	-0.00000 00000 00022 96222
1	-0.00000 00000 00697 91868
2	-0.00000 00000 00020 38398
3	0.00000 00000 00078 21592
4	0.00000 00000 00002 51096
5	-0.00000 00000 00002 02942
6	-0.00000 00000 00000 06659
7	0.00000 00000 00000 02049
8	0.00000 00000 00000 00068
9	-0.00000 00000 00000 00009

r	$L_{r,k}$, k = 11
0	0.00000 00000 00002 86971
1	0.00000 00000 00094 30265
2	0.00000 00000 00002 54535
3	-0.00000 00000 00010 64136
4	-0.00000 00000 00000 31573
5	0.00000 00000 00000 28180
6	0.00000 00000 00000 00854
7	-0.00000 00000 00000 00296
8	-0.00000 00000 00000 00009
9	0.00000 00000 00000 00001

Coefficients in the Expansion of

$$K_\nu(z) = (\pi/2z)^{\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} C_k(\nu) T_k^*(5/z) \ , \ z \geq 5 \ ,$$

$$C_k(\nu) = \sum_{r=0}^{\infty} L_{r,k} T_r^*(\nu) \ , \ 0 \leq \nu \leq 1 \ .$$

r	$L_{r,k}$, k = 12
0	-0.00000 00000 00000 38387
1	-0.00000 00000 00013 53879
2	-0.00000 00000 00000 34024
3	0.00000 00000 00001 53647
4	0.00000 00000 00000 04245
5	-0.00000 00000 00000 04137
6	-0.00000 00000 00000 00117
7	0.00000 00000 00000 00045
8	0.00000 00000 00000 00001

r	$L_{r,k}$, k = 13
0	0.00000 00000 00000 05453
1	0.00000 00000 00002 05154
2	0.00000 00000 00000 04830
3	-0.00000 00000 00000 23394
4	-0.00000 00000 00000 00606
5	0.00000 00000 00000 00639
6	0.00000 00000 00000 00017
7	-0.00000 00000 00000 00007

r	$L_{r,k}$, k = 14
0	-0.00000 00000 00000 00817
1	-0.00000 00000 00000 32632
2	-0.00000 00000 00000 00724
3	0.00000 00000 00000 03736
4	0.00000 00000 00000 00091
5	-0.00000 00000 00000 00103
6	-0.00000 00000 00000 00003
7	0.00000 00000 00000 00001

r	$L_{r,k}$, k = 15
0	0.00000 00000 00000 00129
1	0.00000 00000 00000 05424
2	0.00000 00000 00000 00114
3	-0.00000 00000 00000 00623
4	-0.00000 00000 00000 00014
5	0.00000 00000 00000 00017

r	$L_{r,k}$, k = 16
0	-0.00000 00000 00000 00021
1	-0.00000 00000 00000 00938
2	-0.00000 00000 00000 00019
3	0.00000 00000 00000 00108
4	0.00000 00000 00000 00002
5	-0.00000 00000 00000 00003

r	$L_{r,k}$, k = 17
0	0.00000 00000 00000 00004
1	0.00000 00000 00000 00168
2	0.00000 00000 00000 00003
3	-0.00000 00000 00000 00019
4	-0.00000 00000 00000 00000
5	0.00000 00000 00000 00001

r	$L_{r,k}$, k = 18
0	-0.00000 00000 00000 00001
1	-0.00000 00000 00000 00031
2	-0.00000 00000 00000 00001
3	0.00000 00000 00000 00004

r	$L_{r,k}$, k = 19
0	0.00000 00000 00000 00000
1	0.00000 00000 00000 00006
2	0.00000 00000 00000 00000
3	-0.00000 00000 00000 00001

r	$L_{r,k}$, k = 20
0	-0.00000 00000 00000 00000
1	-0.00000 00000 00000 00001

$$d_{r,k} = (2/n) \sum_{\alpha=0}^{n-1} C_k(\mu_\alpha, \lambda) \cos r\theta_\alpha, \qquad \mu_\alpha = \tfrac{1}{2}(1 + \cos \theta_\alpha), \tag{18}$$

$$e_{r,k} = (2/n)\left[\tfrac{1}{2}\{C_k(1, \lambda) + (-1)^r C_k(0, \lambda)\} + \sum_{\alpha=1}^{n-1} C_k(\eta_\alpha, \lambda) \cos r\varphi_\alpha \right], \qquad \eta_\alpha = \tfrac{1}{2}(1 + \cos \varphi_\alpha), \tag{19}$$

where θ_α and φ_α are defined by (10) and (14), respectively. Then omitting remainder terms, we have

$$L_{r,k}(\lambda) = \tfrac{1}{2}(d_{r,k} + e_{r,k}), \qquad k = 0, 1, \cdots, n-1. \tag{20}$$

Finally, it is of interest to record a useful asymptotic estimate of $L_{r,k}(\lambda)$ for k sufficiently large. Since $T_r^*(\nu) = T_r(2\nu - 1)$, from (7) and (17),

$$L_{r,k}(\lambda) = \pi^{-1} \int_0^1 [\nu(1 - \nu)]^{-1/2} C_k(\nu, \lambda) T_r^*(\nu)\, d\nu. \tag{21}$$

If λ is fixed, we can insert (6) in (21), as the $O(k^{-1/3})$ term in (6) is uniform for ν such that $0 \leqq \nu \leqq 1$. Now,

$$\Gamma(\tfrac{1}{2} + \nu)\Gamma(\tfrac{1}{2} - \nu) = \pi \sec \nu\pi. \tag{22}$$

So

$$\begin{aligned} L_{r,k}(\lambda) &= \beta_k A_r + \delta_k, \\ \beta_k &= 4(-1)^k(\pi/3)^{1/2} k^{-2/3} (2\lambda)^{1/6}, \qquad \delta_k = \beta_k O(k^{-1/3}), \\ A_r &= \pi^{-1} \int_0^1 [\nu(1 - \nu)]^{1/2} T_r^*(\nu) \cos \nu\pi\, d\nu. \end{aligned} \tag{23}$$

It can be readily shown that

$$\begin{aligned} A_r &= 0 && \text{if } r \text{ is even}, \\ A_r &= (-1)^{r+1} J_{2r+1}(\pi/2) && \text{if } r \text{ is odd}, \end{aligned} \tag{24}$$

where $J_\mu(z)$ is the Bessel function of the first kind. Now,

$$J_\mu(z) = \frac{(z/2)^\mu}{\Gamma(\mu + 1)} [1 + O(\mu^{-1})]. \tag{25}$$

Thus, we can easily estimate the number of coefficients needed in our double Chebyshev series expansions to achieve a given level of accuracy.

3. Numerical Results. We have from (1) and (17), with a slight change of notation,

$$\begin{aligned} K_\nu(z) &= (\pi/2z)^{1/2} e^{-z} \sum_{k=0}^{\infty} C_k(\nu) T_k^*(5/z), \qquad z \geqq 5, \\ C_k(\nu) &= \sum_{r=0}^{\infty} L_{r,k} T_r^*(\nu), \qquad 0 \leqq \nu \leqq 1. \end{aligned} \tag{26}$$

In the tables (pp. 326–328), we present the coefficients $L_{r,k}$ which were computed by the techniques previously enunciated.

In the computations for $d_{r,k}$ and $e_{r,k}$, we chose n an even integer, actually $n = 20$. Then, as a check on the coefficients, we computed $C_k(\nu)$ with the aid of the values of $d_{r,k}$ for $\nu = 0, 1/4, 1/3, 1/2, 2/3, 3/4$ and 1, and $C_k(\nu)$ with the aid of the values of $e_{r,k}$ for $\nu = 1/4, 1/3, 2/3$ and $3/4$. Note that none of these ν values were used to produce the respective coefficients, in view of (10) and (14). Since

$$(2z/\pi)^{1/2}e^z K_{1/2}(z) = 1, \quad \text{all } z, \tag{27}$$

$$C_0(\tfrac{1}{2}, \lambda) = 1, \qquad C_k(\tfrac{1}{2}, \lambda) = 0, \quad k > 0, \tag{28}$$

and as

$$T^*_{2m}(\tfrac{1}{2}) = (-1)^m, \qquad T^*_{2m+1}(\tfrac{1}{2}) = 0, \quad m = 0, 1, \cdots, \tag{29}$$

we have

$$\begin{aligned} \sum_{r=0}^{\infty} (-1)^r d_{2r,k} &= 1, \quad \text{if } k = 0, \nu = \tfrac{1}{2}, \\ &= 0 \quad \text{if } k > 0, \nu = \tfrac{1}{2}. \end{aligned} \tag{30}$$

The computations were designed to make the coefficients accurate to about 23S. In view of their application in (26), it is convenient to round the coefficients to 20D for presentation. Then only 205 coefficients are needed to produce values of $(2z/\pi)^{1/2}e^z K_\nu(z)$ for all $z \geqq 5$ and all ν, $0 \leqq \nu \leqq 1$, to 20D except possibly for round off. Actually, we have in effect coefficients for the evaluation of the above transcendental for all $z \geqq 5$ and all $\nu \geqq 0$, since $K_\nu(z) = K_{-\nu}(z)$ and use of the recurrence formula for $K_\nu(z)$, namely,

$$K_{\nu+1}(z) = K_{\nu-1}(z) + (2\nu/z)K_\nu(z) \tag{31}$$

is stable in the forward direction. The space required for these coefficients to compute $K_\nu(z)$ for all ν (and $z \geqq 5$) in comparison with that required for existing conventional tables for only a few values of ν manifests the miniaturization described.

4. Concluding Remarks. To the best of our knowledge, except for some coefficients reported by Clenshaw and Picken [5], use of double Chebyshev series has not received much attention. These authors give coefficients to 6D for the evaluation of the Bessel functions $J_\nu(z)$ and $I_\nu(z)$, based on their ascending series representations valid for $0 < z \leqq 8$ and $-1 \leqq \nu \leqq 1$. As previously remarked, we are extending our work to cover all the Bessel functions.

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