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Optimal Quadrature Formulas Using Generalized Inverses. Part I: General Theory and Minimum Variance Formulas*

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Abstract. This paper is the first of two papers concerning the derivation of optimal quadrature formulas. In Part I, we develop results concerning generalized inverses and use these results to derive some minimum variance quadrature formulas. The formulas are obtained by inverting appropriate systems of numerical differentiation formulas. The second paper, Part II, will use the same results concerning generalized inverses to derive Sard "best" quadrature formulas.

1. Introduction. In an earlier paper [1], a method was described and used to study the truncation error for interpolatory quadrature formulas by inverting systems of differentiation formulas. With the help of generalized inverses (see [6]–[8]), the same idea can be used to derive quadrature formulas without the restriction that the formulas be interpolatory. In this paper, Part I, we will develop some results concerning generalized inverses and use these results to derive some minimum variance formulas as described by Sard in [5]. In a following paper, Part II, the results developed here will be used to formulate a procedure to derive Sard "best" quadrature formulas (see [4]).

We assume that the formulas under consideration have the form

$$(1.1)\qquad \int_a^b f(x)\,dx = \sum_{i=0}^{N} w_i f(x_i) + E(f).$$

We also assume that (1.1) has $E(f) = 0$ for all $f \in \mathcal{P}_n$ where $0 \leq n \leq N$. $\mathcal{P}_n$ denotes the set of all polynomials of degree n or less.

Assuming the function $f(x)$ is not known exactly at the points $x_0, x_1, \cdots, x_N$, but only that $\bar{f}(x_i)$ is known such that $f(x_i) = \bar{f}(x_i) + \rho_i$, we get what we call here the inherent error

$$(1.2)\qquad I(f) = \sum_{i=0}^{N} w_i f(x_i) - \sum_{i=0}^{N} w_i \bar{f}(x_i) = \sum_{i=0}^{N} w_i \rho_i.$$

If we ignore computer round-off error, the expression $\sum_{i=0}^{N} w_i\, \bar{f}(x_i)$ is what is computed to approximate $\int_a^b f(x)\,dx$. Thus

$$(1.3)\qquad \int_a^b f(x)\,dx = \sum_{i=0}^{N} w_i \bar{f}(x_i) + T(f),$$

where $T(f) = I(f) + E(f)$.

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From Krylov [2], we know that if $|\rho_i| \leqq \rho$, a bound on the inherent error $I(f)$ is

$$|I(f)| \leqq \rho \sum_{i=0}^{N} |w_i|,$$

and this bound is minimized if the w_i's are all positive. In [3] an analysis is given which indicates that the round-off error itself is likely to be smaller if the weights w_i are positive. In [2] and [5] we find that under reasonable statistical assumptions on the ρ_i's, the variance of $I(f)$ is minimized by minimizing $\sum_{i=0}^{N} w_i^2$.

In this paper, Part I, we develop a procedure for solving

Problem I. *Determine the quadrature formula of the form of* (1.1) *with* $E(f) = 0$ *for* $f \in \mathcal{P}_n$, $n < N$, *such that* $\sum_{i=0}^{N} w_i^2$ *is a minimum.*

A computer program has been written employing this procedure and some tables of minimum variance formulas have been computed.

2. Inverting Numerical Differentiation Formulas. We assume $f \in C^1[c, d]$ where $[c, d]$ is a finite interval containing the $n + 2$ distinct points $z_0, \cdots, z_{n+1}$ and the $N + 1$ distinct points $x_0, \cdots, x_N$. The notation $C^k[c, d]$ is used to denote the set of functions with k continuous derivatives on $[c, d]$. We permit the possibility that $z_i = x_j$ for some i and some j.

At this point we define

$$g(x) = \int_{z_0}^{x} f(t)\, dt, \tag{2.1}$$

which implies that $g \in C^2[c, d]$. From the Lagrange interpolation formula, we get

$$g(x) = l_0(x)g(z_0) + l_1(x)g(z_1) + \cdots + l_{n+1}(x)g(z_{n+1}) + R(g; x) \tag{2.2}$$

where

$$R(g; x) = \omega_{n+1}(x)g[z_0, \cdots, z_{n+1}, x].$$

Here, $g[z_0, \cdots, z_{n+1}, x]$ denotes the $(n + 2)$nd divided difference of g at $z_0, \cdots, z_{n+1}, x$, $\omega_{n+1}(x) = (x - z_0)(x - z_1) \cdots (x - z_{n+1})$, and

$$l_i(x) = \omega_{n+1}(x)/(x - z_i)\omega'_{n+1}(z_i)$$

for $i = 0, \cdots, n + 1$.

By differentiating (2.2) and noting that $g'(x) = f(x)$, we get

$$f(x) = l'_1(x)S_{0,1} + l'_2(x)S_{0,2} + \cdots + l'_{n+1}(x)S_{0,n+1} + R'(g; x) \tag{2.3}$$

where

$$S_{0,i} = \int_{z_0}^{z_i} f(t)\, dt \tag{2.4}$$

and

$$R'(g; x) = \omega'_{n+1}(x)g[z_0, \cdots, z_{n+1}, x] + \omega_{n+1}(x)g[z_0, \cdots, z_{n+1}, x, x]. \tag{2.5}$$

Another representation of (2.5) is

$$R'(g; x) = [\omega'_{n+1}(x) - \omega_n(x)]g[z_0, \cdots, z_{n+1}, x] + \omega_n(x)g[z_0, \cdots, z_n, x, x] \tag{2.6}$$

where $\omega_n(x) = (x - z_0) \cdots (x - z_n)$. The form (2.6) is worthwhile when the $(n + 2)$nd divided difference can be related to the $(n + 2)$nd derivative.

The system of equations

$$\begin{bmatrix} l_1'(x_0) & \cdots & l_{n+1}'(x_0) \\ l_1'(x_1) & \cdots & l_{n+1}'(x_1) \\ \vdots & & \vdots \\ l_1'(x_N) & \cdots & l_{n+1}'(x_N) \end{bmatrix} \begin{bmatrix} S_{0,1} \\ \vdots \\ S_{0,n+1} \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} - \begin{bmatrix} R'(g;\, x_0) \\ R'(g;\, x_1) \\ \vdots \\ R'(g;\, x_N) \end{bmatrix} \tag{2.7}$$

is obtained by evaluating (2.3) at the points $x_0, x_1, \cdots, x_N$. From Lemma 2.2 of [1], we learn that for $N \geqq n$ the columns of the $(N + 1) \times (n + 1)$ matrix $L = (l_{ij}) = (l_j'(x_i))$ are linearly independent. Hence, the rank of L is $n + 1$.

We will now indicate how generalized inverses can be used to derive numerical integration formulas from the system of equations (2.7). As done by Rohde in [7], we define the generalized inverses of L to be the set of $(n + 1) \times (N + 1)$ matrices B such that $LBL = L$. Because L has rank $n + 1$ for our case, we may define the generalized inverses of L to be the set

$$\mathcal{B} = \{B \in \mathfrak{M}(n + 1,\, N + 1) \mid BL = I\}. \tag{2.8}$$

$\mathfrak{M}(n + 1, N + 1)$ denotes the set of all real $(n + 1) \times (N + 1)$ matrices, and I the $(n + 1) \times (n + 1)$ identity matrix. The set $\mathcal{B}$ is nonempty since $(L^TL)^{-1}L^T \in \mathcal{B}$.

If we let $B = (b_{ij}) \in \mathcal{B}$, (2.7) produces the numerical integration formulas

$$\int_{x_0}^{x_i} f(t)\, dt = \sum_{j=0}^{N} b_{ij} f(x_j) + E_i(f) \tag{2.9}$$

for $i = 1, 2, \cdots, n + 1$. The error term is given by

$$E_i(f) = -\sum_{j=0}^{N} b_{ij} R'(g;\, x_j), \tag{2.10}$$

which can be placed in either of the following two forms

$$\begin{aligned} E_i(f) = -\sum_{j=0}^{N} b_{ij} \{\omega_{n+1}'(x_j) g[z_0, \cdots, z_{n+1}, x_j] \\ + \omega_{n+1}(x_j) g[z_0, \cdots, z_{n+1}, x_j, x_j]\}, \end{aligned} \tag{2.11}$$

or

$$\begin{aligned} E_i(f) = -\sum_{j=0}^{N} b_{ij} \{[\omega_{n+1}'(x_j) - \omega_n(x_j)] g[z_0, \cdots, z_{n+1}, x_j] \\ + \omega_n(x_j) g[z_0, \cdots, z_n, x_j, x_j]\} \end{aligned} \tag{2.12}$$

using (2.5) or (2.6), respectively. Since $g(x) = \int_{x_0}^{x} f(t)\, dt$, we see from either (2.11) or (2.12) that (2.9) has $E_i(f) = 0$ for $f \in \mathcal{P}_n$.

If in (2.12) the terms $b_{ij}[\omega_{n+1}'(x_j) - \omega_n(x_j)]$ and $b_{ij}\omega_n(x_j)$ all have the same sign (some may be zero) for $j = 0, 1, \cdots, N$, then

$$E_i(f) = -\left(\sum_{j=0}^{N} b_{ij}\omega_{n+1}'(x_j)\right) g[z_0, \cdots, z_n, \bar{x}, \bar{y}]$$

for some $\bar{x}$ and $\bar{y}$ in $[c, d]$. If $g[z_0, \cdots, z_n, \bar{x}, \bar{y}]$ is replaced by $(n+2)!^{-1}f^{(n+1)}(\xi)$ for some $\xi \in [c, d]$, then we get

$$E_i(f) = -\left(\frac{1}{(n+2)!}\sum_{j=0}^{N} b_{ij}\omega'_{n+1}(x_j)\right)f^{(n+1)}(\xi). \tag{2.13}$$

In any event (even if terms change sign),

$$|E_i(f)| \leqq C_i(n, N) \max_{x\in[c,d]} |f^{(n+1)}(x)|, \tag{2.14}$$

where

$$C_i(n, N) = \frac{1}{(n+2)!}\sum_{j=0}^{N} |b_{ij}|\,(|\omega'_{n+1}(x_j) - \omega_n(x_j)| + |\omega_n(x_j)|). \tag{2.15}$$

Note that (2.13) and (2.14) both require that $f \in C^{n+1}[c, d]$.

3. Minimization by Generalized Inverses. Let $\mathfrak{M}^*(n, m)$ designate the set of all $n \times m$ complex matrices. Clearly, $\mathfrak{M}^*(n, m)$ is a complex vector space of dimension nm with respect to matrix addition and multiplication by scalars. Let $P = (p_{ij})$ be an $m \times m$ positive definite Hermitian matrix. For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathfrak{M}^*(n, m)$, define

$$(A, B) = \operatorname{Tr}(APB^H) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m} a_{ik}p_{kj}\bar{b}_{ij}, \tag{3.1}$$

where Tr stands for the trace. It is easy to check that (A, B) is an inner product for $\mathfrak{M}^*(n, m)$. Let L be an $m \times n$ matrix. The operator $\mathfrak{L}: \mathfrak{M}^*(n, m) \to \mathfrak{M}^*(n, n)$ define by $\mathfrak{L}(A) = AL$ is a linear operator and if L has rank n the mapping is onto $\mathfrak{M}^*(n, n)$.

THEOREM 3.1. *Let L be an $m \times n$ matrix with $m > n$ and rank $L = n$, and let*

$$\mathfrak{B} = \{B \in \mathfrak{M}^*(n, m) \mid BL = I\}.$$

Then $W = (L^HP^{-1}L)^{-1}L^HP^{-1}$ is the unique matrix in $\mathfrak{B}$ such that

$$(W, W) = \min_{B\in\mathfrak{B}} (B, B).$$

Proof. Since the linear mapping $\mathfrak{L}: \mathfrak{M}^*(n, m) \to \mathfrak{M}^*(n, n)$ defined by $\mathfrak{L}(A) = AL$ is onto, the dimension of $\ker(\mathfrak{L}) = \{A \in \mathfrak{M}^*(n, m) \mid AL = 0\}$ is $p = n(m - n)$. Let $A_1, A_2, \cdots, A_p$ be an orthogonal basis for $\ker(\mathfrak{L})$. The set $\mathfrak{B}$ can now be represented by

$$\mathfrak{B} = \left\{B \mid B = W + \sum_{i=1}^{p} \alpha_i A_i\right\}$$

where the α_i's are arbitrary complex numbers. Now consider $B \in \mathfrak{B}$

$$\begin{aligned}(B, B) &= \left(W + \sum_{i=1}^{p}\alpha_i A_i,\ W + \sum_{i=1}^{p}\alpha_i A_i\right)\\ &= (W, W) + \sum_{i=1}^{p}[\alpha_i(A_i, W) + \bar{\alpha}_i(W, A_i)] + \sum_{i=1}^{p}|\alpha_i|^2 (A_i, A_i).\end{aligned}$$

Since

$$(A_i, W) = \operatorname{Tr}(A_iPP^{-1}L(L^HP^{-1}L)^{-1}) = 0,$$

we get

$$(B, B) = (W, W) + \sum_{i=1}^{p} |\alpha_i|^2 (A_i, A_i). \tag{3.2}$$

Clearly, (B, B) is a minimum if and only if $\alpha_i = 0$ for all i. This proves Theorem 3.1.

Let $w_{(i)} = (w_{i1}, w_{i2}, \cdots, w_{im})$ be the ith row of W defined in Theorem 3.1. The following corollary is useful as a minimization procedure for integration formulas.

COROLLARY 3.2. *Let* $\mathcal{B}_i = \{b \in C^m \mid b^H L = e_i\}$ *where* e_i *is the row vector of all zeros except for a* 1 *in the ith position. Then* $w_{(i)}^H \in \mathcal{B}_i$ *is the unique vector such that*

$$w_{(i)} P w_{(i)}^H = \sum_{j=1}^{m} \sum_{k=1}^{m} w_{ik} p_{kj} \bar{w}_{ij} = \min_{b \in \mathcal{B}_i} b^H P b.$$

Proof. Assume there exists a $b \in \mathcal{B}_i$ such that $b^H P b \leqq w_{(i)} P w_{(i)}^H$. Then the matrix V formed from W by replacing $w_{(i)}$ with b^H would still have $VL = I$. If $b^H \neq w_{(i)}$, we would have $(V, V) \leqq (W, W)$ and $V \neq W$. This contradicts Theorem 3.1, thus proving the desired corollary.

4. Minimum Variance Integration Formulas. In the introduction, we stated

Problem I. *Determine the quadrature formulas of the form*

$$\int_a^b f(x)\, dx = \sum_{i=0}^{N} w_i f(x_i) + E(f) \tag{4.1}$$

with $E(f) = 0$ *for* $f \in \mathcal{P}_n$, $0 \leqq n \leqq N$, *such that* $\sum_{i=0}^{N} w_i^2$ *is a minimum.*

As mentioned in the introduction, formulas for which $\sum_{i=0}^{N} w_i^2$ is a minimum minimize the variance of the inherent error. Using Corollary 3.2 and Eqs. (2.7) and (2.9), we get

THEOREM 4.1. *The vector* $w^T = (w_0, \cdots, w_N)$ *solving Problem* I *is the* $(n + 1)$*st row in the* $(n + 1) \times (N + 1)$ *matrix* $(L^T L)^{-1} L^T$ *where* $L = (l_{ij})$ *with*

$$l_{ij} = l_j'(x_i) = \frac{d}{dx} \left(\frac{\omega_{n+1}(x)}{(x - z_j)\omega_{n+1}'(z_j)} \right) \Bigg|_{x=x_i}. \tag{4.2}$$

Here $\omega_{n+1}(x) = (x - z_0)(x - z_1) \cdots (x - z_{n+1})$, *where* $z_0 = a$ *and* $z_{n+1} = b$. *(The* n *other points* $z_1, \cdots, z_n$ *can be arbitrary.)*

The proof of this theorem is an immediate consequence of Eq. (2. 9) and Corollary 3.2 with $P = I$ and L the matrix of the system of equations in (2.7).

A FORTRAN program has been written to solve Problem I given n, N, a, b, and $x_0, x_1, \cdots, x_N$. The program also requires $z_1, z_2, \cdots, z_n$, $z_0 = a$ and $z_{n+1} = b$ for defining $\omega_{n+1}(x) = (x - z_0)(x - z_1) \cdots (x - z_{n+1})$. The points $z_1, \cdots, z_n$ can be arbitrary distinct numbers not equal to a or b. The program computes the $(n + 1)$st row in $W = (L^T L)^{-1} L$ using the elementary unitary Hermitian matrices of Householder (see Hanson and Lawson [9]).

Table I gives the minimum variance weights $w_0, w_1, \cdots, w_N$ for the formula

$$\int_0^N f(x)\, dx = \sum_{i=0}^{N} w_i f(i) + E(f) \tag{4.3}$$

having $E(f) = 0$ for $f \in \mathcal{P}_n$, i.e. having algebraic degree of precision n. For formulas of the form of (4.3), we get $w_i = w_{N-i}$. Thus the table only lists w_i for $i =$

TABLE I

ALGEBRAIC DEGREE OF PRECISION 3

N	2	3	4	5
W(0)	.333333333333	.375000000000	.419047619048	.461309523810
W(1)	1.333333333333	.112500000000	.990476190476	.907738095238
W(2)			1.180952380952	1.130952380952
C	.111111111111	.375000000000	.306031746032	.896654885913
VAR	.500000000000	.312500000000	.231746031746	.185267857143

N	6	7	8	9
W(0)	.500000000000	.534722222222	.565656565657	.325360733017
W(1)	.857142857143	.826388888889	.808080808081	1.312430226024
W(2)	1.071428571429	1.020833333333	.981240981241	.876187874625
W(3)	1.142857142857	1.118055555556	1.085137085137	.868247689810
W(4)			1.119769119769	1.117773476523
C	2.148214285714	4.738975694444	8.591822991823	.361037620649
VAR	.154761904762	.133101851852	.116883116883	.113563197336

ALGEBRAIC DEGREE OF PRECISION 5

N	4	5	6	7
W(0)	.311111111111	.329861111111	.348051948052	.366130050505
W(1)	1.422222222222	1.302083333333	1.211688311688	1.139488636364
W(2)	.533333333333	.868055555556	1.020779220779	1.093087121212
W(3)			.838961038961	.901294191919
C	.008465608466	.043300182048	.182578849722	.547389230092
VAR	.282716049383	.204619984568	.165735930736	.140394132295

N	8	9
W(0)	.384149184149	.402010489510
W(1)	1.080341880342	1.031381118881
W(2)	1.123853923854	1.131687062937
W(3)	.968453768454	1.021547202797
W(4)	.886402486402	.913374125874
C	1.351446212681	2.961937593656
VAR	.122141068808	.108244099650

ALGEBRAIC DEGREE OF PRECISION 7

N	6	7	8	9
W(0)	.292857142857	.304224537037	.314987728321	.593181818182
W(1)	1.542857142857	1.449016203704	1.374278067611	.797727272727
W(2)	.192857142857	.535937500000	.743465916799	.951136363636
W(3)	1.942857142857	1.210821759259	.951163404497	1.053409090909
W(4)			1.232209765543	1.104545454545
C	.006428571429	.032011796898	.119802533348	15.660383522727
VAR	.243928571429	.161041586291	.131390041274	.104261363636

$0, 1, \cdots, N_m$, where $N_m = N/2$ or $(N-1)/2$. The constant $C(n, N)$ giving the bound

$$|E(f)| \leqq C(n, N) \max_{x \in [0, N]} |f^{(n+1)}(x)|$$

is computed by (2.15). This $C(n\ N)$ is denoted by C in the table. The parameter

$$\sigma^2(n, N) = \left(\sum_{i=0}^{N} w_i^2 \right) \Big/ N^2$$

is given in Table I and is denoted there by VAR. This VAR parameter supplies a comparison of the variance when evaluating $\int_0^1 f(x)\, dx$ approximately using (4.3). The results were computed to 15 decimal digits, but only 12 digits are given in the tables. Table I, for algebraic degree of precision $n = 3$, agrees with the results given by Sard in [5]. For example, for $N = 6$ with degree of precision 3, we have

$$\int_0^6 f(x)\,dx = \tfrac{1}{2}f(0) + \tfrac{8}{7}f(1) + \tfrac{15}{14}f(2) + \tfrac{8}{7}f(3) + \tfrac{15}{14}f(4) + \tfrac{6}{7}f(5) + \tfrac{1}{2}f(6) + E(f)$$

with

$$|E(f)| < 2.1482 \max |f^{(4)}(x)| \quad \text{and} \quad \sigma^2(3, 6) = 0.154761904762.$$

TABLE II

ALGEBRAIC DEGREE OF PRECISION 3

N	3	4	5	6
W(0)	-.375000000000	-.326785714286	-.294642857143	-.270833333333
W(1)	1.541666666667	.932142857143	.616071428571	.427579365079
W(2)	-2.458333333333	-.419047619048	.107142857143	.254960317460
W(3)	2.291666666666	-1.151190476190	-.654761904762	-.253968253968
W(4)		1.964880952381	-.502976190476	-.564484126984
W(5)			1.729166666667	-.141865079365
W(6)				1.548611111111
C	1.01453993055	1.701468253968	3.132722594246	5.212268518519
VAR	13.81250000000	6.337276785714	4.149553571429	3.122643849206

N	7	8	9
W(0)	-.251893939394	-.236111111111	-.222552447552
W(1)	.304924242424	.220328282828	.159537684538
W(2)	.283639971140	.271194083694	.245056332556
W(3)	-.032918470418	.081259018759	.136956099456
W(4)	-.361922799423	-.184704184704	-.061810411810
W(5)	-.420544733045	-.361922799423	-.248290598291
W(6)	.074044011544	-.285624098124	-.319531857032
W(7)	1.404671717172	.208964646465	-.172581585082
W(8)		1.286616161616	.295512820513
W(9)			1.187703962704
C	7.776189630682	12.736264430014	19.371175962348
VAR	2.524395743146	2.130174512987	1.849113733489

ALGEBRAIC DEGREE OF PRECISION 5

N	5	6	7	8
W(0)	-.329861111111	-.295747655123	-.270092754468	-.250145687646
W(1)	1.997916666667	1.444624819625	1.093300796426	.855562839938
W(2)	-5.068055555556	-2.438298160173	-1.217718045843	-.595182595183
W(3)	6.931944444444	.846897546898	-.509064199689	-.717426670552
W(4)	-5.502083333333	2.495729617605	1.425667735043	.428982128982
W(5)	2.970138888889	-3.727597402597	.643665015540	1.050449203574
W(6)		2.674391233766	-2.608641705517	-.169308469308
W(7)			2.442883158508	-1.858204642580
W(8)				2.255273892774
C	1.789155158934	3.259265228819	5.230551707291	8.407463013524
VAR	116.932166280864	36.112958109117	18.229763142914	11.518823366544

N	9
W(0)	-.234178321678
W(1)	.686567599068
W(2)	-.256784188034
W(3)	-.634212315462
W(4)	-.073436285936
W(5)	.663160450660
W(6)	.601607420357
W(7)	-.520575951826
W(8)	-1.331303418803
W(9)	2.099155011655
C	15.995918276361
VAR	8.251303971877

TABLE II (CONT.)

ALGEBRAIC DEGREE OF PRECISION 7

N	7	8	9
W(0)	-.304224537037	-.278836535764	-.258261982095
W(1)	2.445163690476	1.926467749072	1.557693114688
W(2)	-8.612127976190	-5.362259310905	-3.468338099129
W(3)	17.379654431217	7.002718026572	2.672557909819
W(4)	-22.027752976191	-2.138903072236	1.780086833102
W(5)	18.054538690476	-6.412906973428	-3.745866516381
W(6)	-9.525206679894	10.247115689095	-.788534918921
W(7)	3.589955357143	-7.294514393785	5.885357769273
W(8)		3.311118821379	-5.716246725429
W(9)			3.081552615073
C	2.669934918147	5.939372462114	10.850298655461
VAR	1297.098124735910	296.458022717346	116.295933670209

Table II gives the minimum variance weights $w_0, \cdots, w_N$ for the Adams-Bashforth type predictor formulas, and Table III gives the minimum variance weights for the Adams-Moulton type corrector formulas. The symmetry occurring in Table I does not occur in Tables II or III. The C again denotes $C(n, N)$ computed by formula (2.15), and VAR here denotes

$$\sigma^2(n, N) = \sum_{i=0}^{N} w_i^2.$$

No division by N^2 is made since the integral is already over the unit interval $[N, N+1]$ or else the unit interval $[N-1, N]$. As an example, consider the case $N = 6$ with algebraic degree of precision 5. From Table II, the predictor formula is

$$\int_6^7 f(x)\,dx = -0.295747655123\,f(0) + 1.444624819625\,f(1) - 2.438298160173\,f(2) + 0.846897546898\,f(3) + 2.495729617605\,f(4) - 3.727597402597\,f(5) + 2.674391233766\,f(6) + E(f)$$

TABLE III

ALGEBRAIC DEGREE OF PRECISION 3

N	3	4	5	6
W(0)	.041666666667	.060119047619	.059193121693	.048611111111
W(1)	-.208333333333	-.198809523810	-.129298941799	-.070436507937
W(2)	.791666666667	.152380952381	-.033068783069	-.064484126984
W(3)	.375000000000	.551190476190	.199735449735	.031744603175
W(4)		.435119047619	.420965608466	.183531746032
W(5)			.482473544974	.356150793651
W(6)				.514880952381
C	.309895833333	.857698412698	1.836419753086	3.462880291005
VAR	.812500000000	.559499007937	.471202601411	.438120039683

N	7	8	9
W(0)	.034722222222	.020622895623	.007634032634
W(1)	-.030753968254	-.006523569024	.007051282051
W(2)	-.050595238095	-.027807840308	-.007371794872
W(3)	-.024801587302	-.032918470418	-.023203185703
W(4)	.046626984127	-.011544011544	-.028010878011
W(5)	.163690476190	.046626984127	-.009362859363
W(6)	.326388888889	.151905964406	.045172882673
W(7)	.534722222222	.314604377104	.148028360528
W(8)		.545033670034	.311635586636
W(9)			.548426573427
C	5.709604414683	8.688378827962	12.845799319315
VAR	.426752645503	.423745139891	.423414432789

TABLE III (CONT.)

ALGEBRAIC DEGREE OF PRECISION 5

N	5	6	7	8
W(0)	.018750000000	.024391233766	.029766414141	.033673271173
W(1)	-.120138888889	-.127597402597	-.128693181818	-.121763306138
W(2)	.334722222222	.245729617605	.165814393939	.098601398601
W(3)	-.554166666667	-.153102453102	.033112373737	.098467851593
W(4)	.990972222222	-.188298160173	-.216887626263	-.093162393162
W(5)	.329861111111	.844624819625	.015814393939	-.166377719503
W(6)		.354252344877	.721306818182	.125252525253
W(7)			.379766414141	.620287004662
W(8)				.405021367521
C	.404374329779	1.009714105339	2.265906728588	4.842623076417
VAR	1.524758873457	.975041442450	.757835209736	.636225593947

N	9
W(0)	.035763403263
W(1)	-.109380341880
W(2)	.048112859363
W(3)	.104561965812
W(4)	.000118492618
W(5)	-.128925796426
W(6)	-.097831196581
W(7)	.178897630148
W(8)	.539819347319
W(9)	.428863636364
C	9.192981616763
VAR	.560017258590

ALGEBRAIC DEGREE OF PRECISION 7

N	7	8	9
W(0)	.011367394180	.013000643261	.014967857918
W(1)	-.093840939153	-.092637751909	-.093437457491
W(2)	.343080357143	.270177072156	.220024169050
W(3)	-.732035383598	-.384955665477	-.195489618633
W(4)	1.017964616402	.178009644676	-.082272316463
W(5)	-1.006919642857	.289928593783	.271142982834
W(6)	1.156159060847	-.642901631547	-.036038089770
W(7)	.304224537037	1.052153914758	-.390111142262
W(8)		.317225180298	.960683671924
W(9)			.330529942892
C	.511719802279	1.295734473541	2.885010646645
VAR	4.141910744139	1.966665490798	1.361517280271

with

$$|E(f)| \leqq 3.2593 \max |f^{(6)}(x)| \quad \text{and} \quad \sigma^2(5, 6) = 36.112958109117.$$

Note that in Table III for $N = 6$ with degree of precision 5, the corresponding corrector formula has the smaller error bound

$$|E(f)| \leqq 1.0097 \max |f^{(6)}(x)|$$

as well as the smaller variance

$$\sigma^2(5, 6) = 0.975041442450.$$

As can be seen by comparing Tables II and Tables III, this holds true in all cases considered. Hence, the corrector formulas would appear to give considerable improvement over the predictor formulas.

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