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An Elliptic Integral Identity

By M. L. Glasser

Abstract. The definite integral

$$\int_0^\infty \left[\frac{(x^2+a^2)^{1/2}-a}{x^2+a^2}\right]^{1/2} K\left[\frac{(x^2+b^2)^{1/2}-b}{(x^2+b^2)^{1/2}+b}\right] \frac{dx}{(x^2+b^2)^{1/2}+b}$$

is evaluated in closed form.

The following interesting integral does not appear to fit into the standard theory of elliptic integrals:

$$(1) \qquad I = \int_0^\infty \left[\frac{(x^2+a^2)^{1/2}-a}{x^2+a^2}\right]^{1/2} K\left[\frac{(x^2+b^2)^{1/2}-b}{(x^2+b^2)^{1/2}+b}\right] \frac{dx}{(x^2+b^2)^{1/2}+b}$$

where $K(k) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}\,d\theta$ is the complete elliptic integral of the first kind.

To evaluate (1), we note that [1] for Re $a \geqq 0$,

$$\left[\frac{(x^2+a^2)^{1/2}-a}{x^2+a^2}\right]^{1/2} = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty e^{-ay}\,y^{-1/2}\sin xy\,dy.$$

Thus, since the order of integration can be reversed, after a simple change of variables, we have

$$(2) \qquad I = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty dy\,e^{-ay}\,y^{-1/2}\int_0^\infty dx\,\frac{\sin(xyb)}{(x^2+1)^{1/2}+1}\,K\left[\frac{(x^2+1)^{1/2}-1}{(x^2+1)^{1/2}+1}\right].$$

Next, we note that

$$[(x^2+1)^{1/2}-1]/[(x^2+1)^{1/2}+1] = [(z-1)/(z+1)]^{1/2}$$

where

$$z = (x^2+2)/2(x^2+1)^{1/2}.$$

Since [2]

$$K([(z-1)/(z+1)]^{1/2}) = 2^{-3/2}\pi(z+1)^{1/2}P_{-1/2}(z),$$

we obtain

$$(3) \qquad I = 2^{-3/2}\pi^{1/2}\int_0^\infty dy\,e^{-ay}\,y^{-1/2}\int_0^\infty dx\,\frac{\sin(xyb)}{(x^2+1)^{1/4}}\,P_{-1/2}[(x^2+2)/2(x^2+1)^{1/2}].$$

Now the x-integration can be rewritten as a tabulated Hankel transform [3] and we find

$$\int_0^\infty (x^2+1)^{-1/4}\,P_{-1/2}[(x^2+2)/2(x^2+1)^{1/2}]\sin(xt)\,dx = I_0(t/2)\,K_0(t/2)$$

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so (3) becomes

$$I = \tfrac{1}{2}(\pi/2)^{1/2} \int_0^\infty dy\, y^{-1/2}\, e^{-ay} I_0(by/2)\, K_0(by/2). \tag{4}$$

The integral in (4) is of a type considered by Bailey [4], and from his results we have

$$\int_0^\infty t^{-1/2} I_0(\lambda t)\, K_0(\lambda t) e^{-t}\, dt = 2(c/\pi)^{1/2} \operatorname{sech}^2 \alpha\, K(\operatorname{sech} \alpha)\, K(\tanh \alpha),$$

where

$$c = (2\lambda^2)^{-1}[1 - (1 - 4\lambda^2)^{1/2}], \qquad \cosh \alpha = 2^{-1/2}[1 + (1 + \lambda^2 c^2)^{1/2}]^{1/2}.$$

Thus

$$I = b^{-1}[a - (a^2 - b^2)^{1/2}]^{1/2} \operatorname{sech}^2 \alpha\, K(\operatorname{sech} \alpha)\, K(\tanh \alpha),$$

where

$$\cosh \alpha = (2b)^{-1/2}\{b + [2a^2 - 2a(a^2 - b^2)^{1/2}]^{1/2}\}^{1/2},$$

and the result is valid for $\operatorname{Re} a \geqq \operatorname{Re} b > 0$.

Battelle Memorial Institute
505 King Avenue
Columbus, Ohio 43201

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