A Closed Form Evaluation of the Elliptic Integral

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Abstract. The complete elliptic integral of the first kind $K(k)$ is evaluated in terms of gamma functions for the moduli $k = \sqrt{2} - 1$ and $(2\sqrt{2} - 2)^{1/2}$.

We have found a new identity relating to the complete elliptic integral of the first kind $K(k)$ for modulus $k = \tan \pi/8 = \sqrt{2} - 1$. By combining this with Special Case I of Abel's [1] theorem, we are able to express K and K' of this modulus directly in terms of the gamma function. While such evaluations are interesting in themselves, they should also be useful for performing cross checks on the accuracy of numerical procedures.

The case $a = b = 1$ of a result due to one of the authors [2] gives

$$
(1) \quad I = \int_0^\infty K\left[\frac{(x^2 + 1)^{1/2} - 1}{(x^2 + 1)^{1/2} + 1}\right] \frac{[(x^2 + 1)^{1/2} - 1]^{1/2}}{(x^2 + 1)^{1/2} + 1} \frac{dx}{(x^2 + 1)^{1/2}}
$$

$$
= 2(\sqrt{2} - 1) K(\sqrt{2} - 1) K'(\sqrt{2} - 1),
$$

where $K'(k) = K(k')$ and $k' = (1 - k^2)^{1/2}$ is the complementary modulus. The substitution $k = [(x^2 + 1)^{1/2} - 1]/[(x^2 + 1)^{1/2} + 1]$ gives

$$
(2) \quad I = \frac{1}{\sqrt{2}} \int_0^1 (1 - k)^{-1/2} K(k) \, dk.
$$

By using the facts that

$$
K(k) = (\pi/2) \, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) \quad \text{and} \quad \int_0^1 k^{2n}(1 - k)^{-1/2} \, dk = B(\tfrac{1}{2}, 2n + 1),
$$

we find

$$
(3) \quad I = 2^{-3/2}\pi \sum_{n=0}^\infty \frac{(\tfrac{1}{2})_n(\tfrac{1}{2})_n}{(1)_n n!} \frac{\Gamma(1/2)\Gamma(2n + 1)}{\Gamma(2n + 3/2)}
$$

$$
= \frac{\pi^2}{4\Gamma(3/4)\Gamma(5/4)} \sum_{n=0}^\infty \frac{(\tfrac{1}{2})_n(\tfrac{1}{2})_n(\tfrac{1}{2})_n}{(3/4)_n(5/4)_n n!}.
$$

The duplication formula for the gamma function has been used to obtain the last expression in (3). The sum in (3) is simply ${}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{4}, \tfrac{5}{4}; 1)$ and by Whipple's theorem [3] we obtain

$$
(4) \quad I = \frac{\pi^3}{4[\Gamma(5/8)\Gamma(7/8)]^2}.
$$

From (1) and (4) we have

(5) $$K(\sqrt{2} - 1)K'(\sqrt{2} - 1) = \frac{\pi^3(\sqrt{2} + 1)}{8[\Gamma(5/8)\Gamma(7/8)]^2} \cdot$$

We can now combine (5) with Abel's result

$$K(\sqrt{2} - 1)/K'(\sqrt{2} - 1) = 1/\sqrt{2}$$

and we find

(6) $$K(\sqrt{2} - 1) = \frac{\pi^{3/2}(2 + \sqrt{2})^{1/2}}{4\Gamma(5/8)\Gamma(7/8)} = 1.645568395 \cdots = K'(2\sqrt{2} - 2)^{1/2},$$

(7) $$K'(\sqrt{2} - 1) = \frac{\pi^{3/2}(4 + 2\sqrt{2})^{1/2}}{4\Gamma(5/8)\Gamma(7/8)} = 2.327185142 \cdots = K(2\sqrt{2} - 2)^{1/2}.$$

To the authors' knowledge (6) and (7) together with Legendre's results $K(1/\sqrt{2}) = K'(1/\sqrt{2}) = [\Gamma(\frac{1}{4})]^2/4\pi^{1/2}$ and $K[\sqrt{2}(\sqrt{3} - 1)/4] = \pi^{1/2}\Gamma(\frac{1}{3})/2 \cdot 3^{3/4}\Gamma(\frac{2}{3}) = K'[\sqrt{2}(\sqrt{3} - 1)/4]/\sqrt{3}$ appear to represent the only real cases where the elliptic integral has been expressed explicitly in terms of gamma functions (except for the trivial cases $K(0) = \pi/2$, $K(1) = \infty$). In the complex case only the value $K(e^{i\pi/3}) = e^{-i\pi/3}\pi^{1/2}\Gamma(\frac{1}{6})/2(3^{3/4})\Gamma(\frac{2}{3})$ appears to be known. The list can be extended somewhat by use of Landen's transformation.

The identity (5) is rather mysterious, having been obtained outside the proper theory of elliptic functions, and it is an interesting question whether it is unique or hints at something deeper. The fact that $K(\sqrt{2} - 1)$ should be expressible in terms of gamma functions is stated by Ramanujan [4], but he does not give any details.

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1. E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, New York, 1962, sect. 22.81. MR 31 #2375.
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