MATHEMATICS OF COMPUTATION, VOLUME 25, NUMBER 115, JULY, 1971

Chebyshev Approximations for the Riemann Zeta Function

By W. J. Cody,* K. E. Hillstrom,* and Henry C. Thacher, Jr.**

Abstract. This paper presents well-conditioned rational Chebyshev approximations, involving at most one exponentiation, for computation of either $\zeta(s)$ or $\zeta(s) - 1$, $.5 \leqq s \leqq 55$, for up to 20 significant figures. The logarithmic error is required in one case. An algorithm for the Hurwitz zeta function, and an example of nearly double degeneracy are also given.

1. Introduction. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} \qquad (\mathrm{Re}(s) > 1) \tag{1.1}$$

or by the power series expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n \qquad (\mathrm{Re}(s) > 0) \tag{1.2}$$

where

$$\gamma_n = \lim_{m\to\infty} \left\{ \sum_{k=1}^{m} \frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1} \right\}.$$

It is an analytic function of s, regular throughout the complex plane except for a simple pole of residue 1 at $s = 1$. The zeta function satisfies the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s)\zeta(s). \tag{1.3}$$

Evaluation of the function for real s usually involves taking a partial sum of (1.1) and applying the Euler-Maclaurin summation formula to the remainder. While this procedure is theoretically valid for all $s > -2n - 1$, where n terms of the Euler-Maclaurin summation formula are used, there is serious cancellation error for $s < 1.5$. However, the reflection formula, Eq. (1.3), can be used for $s < .5$, while Thacher [7] has recently used Eq. (1.2) as a basis for expansions in Chebyshev polynomials valid both for $\frac{1}{2} \leqq s \leqq \frac{3}{2}$ and for $1 \leqq s \leqq 2$. For $s \geqq 2$, it is still necessary to evaluate a partial sum of the series (1.1). The process involves an exponentiation for each new term added to the sum, and is therefore quite slow. This paper presents rational Chebyshev approximations for evaluating $\zeta(s)$ or $\zeta(s) - 1$ for up to 20S without any exponentiation for $.5 \leqq s \leqq 11$, and with only one exponentiation for $11 \leqq s \leqq 55$.

Received August 7, 1970, revised December 21, 1970.

AMS 1969 *subject classifications*. Primary 3304, 3345, 6525; Secondary 4117, 4140, 6520.

Key words and phrases. Rational Chebyshev approximations, Riemann zeta function, Hurwitz zeta function, logarithmic error, near degeneracy.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

** Work supported in part by the U. S. Atomic Energy Commission and in part by the University of Notre Dame.

The approximation forms and the values of s for which they are used are

$$\zeta(s) \simeq R_{lm}(s)/(s-1), \qquad .5 \leqq s \leqq 5,$$

$$\zeta(s) - 1 \simeq R_{lm}(s), \qquad 5 \leqq s \leqq 11,$$

$$\simeq 2^{-s+(1/s)R_{lm}(1/s)}, \qquad 11 \leqq s \leqq 25,\ 25 \leqq s \leqq 55,$$

where the $R_{lm}(s)$ are rational functions of degree l in the numerator and m in the denominator. The maximum error was computed relative to $\zeta(s)$ for the first interval, and relative to $\zeta(s) - 1$ for the others.

2. Computational of Reference Values. Reference function values for the generation of the approximations were calculated for $.5 \leqq s \leqq 1.5$ from the coefficients given by Thacher [7], and for the other s from a modification of the above-described technique based on the Euler-Maclaurin summation formula. The modification involves a method for estimating the number of terms needed in the partial sum of (1.1).

The Euler-Maclaurin formula applied to the Dirichlet series for the Hurwitz zeta function,

$$\zeta(s;\alpha) = \sum_{k=0}^{\infty} (k+\alpha)^{-s}, \qquad \alpha > 0,\ \mathrm{Re}(s) > 1, \tag{2.1}$$

gives

$$(s-1)\zeta(s;\alpha) = \frac{2\alpha + s - 1}{2\alpha^s} + \sum_{k=1}^{[n/2]} \frac{B_{2k}}{(2k)!} \frac{\Gamma(s+2k-1)}{\Gamma(s-1)\alpha^{s+2k-1}} + R_n, \tag{2.2}$$

where

$$R_n = \frac{(-1)^{n+1}\Gamma(s+n)}{n!\,\Gamma(s-1)} \int_0^{\infty} \frac{\hat{B}_n(-t)\,dt}{(\alpha+t)^{s+n}} \tag{2.3}$$

and $\hat{B}_n(x)$ is the periodic extension of the nth Bernoulli polynomial. Since $(\alpha + t)^{s+n} > 0$ for $t > -\alpha$, the mean value theorem can be used to obtain

$$R_n = (-1)^{n+1} \frac{\Gamma(s+n-1)}{n!\,\Gamma(s-1)} \frac{B_n(\xi)}{\alpha^{s+n-1}}, \qquad 0 \leqq \xi \leqq 1. \tag{2.4}$$

Thus

$$|R_n| \leqq \frac{1}{n!} \prod_{k=1}^{n} \frac{(s+k-2)M_n}{|\alpha|^{s+n-1}}, \tag{2.5}$$

where

$$M_n = \max_{0 \leq x \leq 1} |B_n(x)|.$$

Letting G and A denote the geometric and arithmetic means of the quantities $\{(s+k-2)\}$, and using the arithmetic-geometric mean inequality, we have

$$\prod_{k=1}^{n} (s+k-2) = G^n \leqq A^n = \left\{\sum_{k=1}^{n} \frac{(s+k-2)}{n}\right\}^n = \left(s + \frac{n-3}{2}\right)^n, \tag{2.6}$$

so that

$$|R_n| \leqq \frac{1}{n!}\left(s + \frac{n-3}{2}\right)^n \frac{M_n}{|\alpha|^{s+n-1}}. \tag{2.7}$$

Lehmer [4] discusses the extrema of the Bernoulli polynomials and shows that $M_n = |B_n|$ for n even. For n odd, he gives 11D values of M_n, for $n \leqq 13$, as well as relatively sharp asymptotic formulas for larger n. We thus see that $[n/2]$ terms of (2.2) will approximate $(s - 1)\zeta(s; \alpha)$ with an absolute error less than 10^{-D} provided that

$$|\alpha| \geqq \left\{\left(s + \frac{n-3}{2}\right)^n \frac{M_n}{n!} 10^D\right\}^{1/(s+n-1)}. \tag{2.8}$$

In principle, then, one can obtain $(s - 1)\zeta(s; a)$ to any desired accuracy by selecting n and m so that (2.8) is satisfied for $\alpha = a + m$, evaluating $(s - 1)\zeta(s; \alpha)$ by (2.2), and finally computing $(s - 1)\zeta(s; a)$ by adding $(s - 1)$ times the appropriate partial sum of the Dirichlet series for $\zeta(s; a)$. The procedure is valid for $\mathrm{Re}(s) > 0$.

The most efficient value of n depends upon s and D. In our calculations, we did not vary this parameter, but gave it the constant value 13, for which

$$m = [\{8.40798 \times 10^{D-11}(s + 5)^{13}\}^{1/(s+12)} + 1 - a],$$

where [ ] denotes "the integer part of." The variable order of the partial sum prevented the use of Markman's economical method [5] of reducing the necessary number of exponentiations in (2.2).

All computations were carried out on a CDC 3600 in 25S arithmetic. Extensive checking against tables and by overlapping of methods shows that our master routines were accurate to roughly a minimum of 23S.

3. Generation of the Approximations. The various approximations were generated in 25S floating-point arithmetic on a CDC 3600 using standard versions of the Remes algorithm [2]. With two exceptions, the computations were straightforward.

The first exception was the analysis relating to the approximation form for the last two intervals. The quantity

$$\delta(s) = \frac{\zeta(s) - 1 - 2^{-s+(1/s)R_{lm}(1/s)}}{\zeta(s) - 1} \tag{3.1}$$

is the error of approximation relative to $\zeta(s) - 1$. However, the basic Remes algorithm is limited to error expressions of the form

$$\Delta(s) = \frac{f(s) - R_{lm}(\phi(x))}{g(x)}. \tag{3.2}$$

Hence, we must modify (3.1). By letting

$$\delta(s) = 1 - 2^{-d(s)}, \tag{3.3}$$

we find that $d(s)$ has the form (3.2) with

$$f(s) = s[\ln(\zeta(s) - 1)/\ln 2 + s], \qquad g(s) = s,$$

and

$$\phi(s) = 1/s.$$

We note that the local extrema of $d(s)$ and $\delta(s)$ occur for the same values of s, and that

$$\hat{\delta}(s) = (-\ln 2)\, d(s) = \ln(1 - \delta(s)) \simeq \delta - \tfrac{1}{2}\delta^2,$$

where $\hat{\delta}(s)$ is the "logarithmic error" discussed by King and Phillips [3] and Sterbenz and Fike [6]. The logarithmic error has been associated primarily with obtaining starting values for various Newton iteration schemes. However, we can use $\hat{\delta}(s)$ in the Remes algorithm since it approximates the Chebyshev error $\delta(s)$ to within terms of order $\delta^2(s)$, an error that is swamped by normal roundoff in the Remes algorithm itself whenever $\delta(s)$ is small.

The second anomaly occurred in the computation of $R_{88}(1/s)$ for the interval [11, 25]. Although the error curves for R_{66} and R_{77} appear to be standard, R_{88} is nearly doubly degenerate. The method of artificial poles [2] determined the Chebyshev error for R_{88} as approximately 5.2×10^{-17}, with the error curve still not leveled. At this point, the denominator had among its zeros the values

$$s_1 = .0371111862 \quad \text{and} \quad s_2 = .13063202.$$

Corresponding zeros in the numerator were $s_1 + (1 \times 10^{-10})$ and $s_2 + (2 \times 10^{-8})$. To our knowledge, this is the first case of nearly double degeneracy that has occurred in practice.

4. Results. Table I lists the values of

$$E_{lm} = -100 \log_{10} \delta_{lm}$$

for selected segments of the L_∞ Walsh arrays. The minimax error δ_{lm} of approximation by R_{lm} is the error relative to $\zeta(s)$ for the interval [.5, 5], and relative to $\zeta(s) - 1$ for the other intervals.

Tables II–V present the approximations giving accuracies most appropriate for computers in use today. The coefficients are given to accuracy slightly greater than that justified by the approximation errors, but reasonable additional rounding should not greatly affect the overall accuracies. Each approximation listed, using the coefficients just as they appear here, was tested for random arguments against the master function routines, and the stated accuracies were all verified.

There are a few anomalies present in the Walsh array. Nonstandard error curves are flagged in Table I. Usually, a nearly degenerate case is signalled by the presence of a nonstandard error curve for the approximation that is one degree lower in both numerator and denominator. Although as previously mentioned, R_{77} for the interval [11, 25] has a standard error curve, R_{88} is nearly doubly degenerate. This troublesome approximation is not given in Table IV. Instead, the nondiagonal element R_{79} is given.

With a little care, computer subroutines returning almost full machine precision values of $\zeta(s)$ and of $\zeta(s) - 1$ can be written using these approximations. One troublesome computation is that for $\zeta(s) - 1$ for $.5 \leqq s \leqq 5$. If one uses

$$(4.1) \qquad (s - 1)\zeta(s) \simeq R_{lm}(s) \equiv \sum_{i=0}^{l} p_i s^i \Big/ \sum_{i=0}^{m} q_i s^i, \qquad .5 \leqq s \leqq 5,$$

TABLE I

$$E_{\ell m} = -100 \log_{10} \delta_{\ell m}$$

$.5 \leq s \leq 5.$

m \ ℓ	1	2	3	4	5	6	7	8	9
0		214	367	417				796	
1	191	322	414						
2	266	429	590	717					
3	342	536	722	782					
4	418	633	792	862*					
5					1142*				
6						1370			
7							1658		
8								1906*	
9									2193*

$5 \leq s \leq 11$

m \ ℓ	1	2	3	4	5	6	7	8	9
0	13†	47†	97	156	219	283	348		
1	71	146	227	316					
2	147	244	342	443					
3			456	570					
4	315	444	569	692					
5					951*				
6						1211*			
7							1571		
8								1738*	
9									1865*

†Nonstandard error curve.

*Coefficients for these approximations only are given in Tables II-V.

TABLE I (cont'd)

$11 \leq s \leq 25$

$m \backslash \ell$	1	2	3	4	5	6	7	8	9
0		447†	457	561	626	676	772		
1	361	457		615					
2	416	525	683	721					
3	466	584	719	778					
4	514	640	781	939*					
5					1165*				
6						1438			
7							1572	1585	
8							1585	1630	
9							1803*		1961*

$25 \leq s \leq 55$

$m \backslash \ell$	1	2	3	4	5	6	7	8	9
0		555	624	741	795	882	948		
1	584	638	707	788					
2	646	705	779	867					
3	699	763	843*	935					
4		817	902	998					
5					1169*				
6						1354			
7							1552		
8								1760*	
9									1979*

†Nonstandard error curve.

*Coefficients for these approximations only are given in Tables II-V.

TABLE II

$$(s-1)\zeta(s) \approx \sum_{j=0}^{n} p_j s^j \Big/ \sum_{j=0}^{n} q_j s^j, \qquad .5 \le s \le 5$$

n	j	p_j		q_j	
4	0	-1.32899 37437	(04)	-2.65799 37266	(04)
	1	-1.70341 74205	(04)	-1.17972 24222	(04)
	2	-7.70056 02483	(03)	-1.03940 13777	(03)
	3	-1.42561 64640	(03)	-1.18419 54886	(02)
	4	-8.36940 23543	(01)	1.00000 00000	(00)
5	0	-3.44793 47840 721	(06)	-6.89586 96520 340	(06)
	1	-3.25983 44394 057	(06)	-7.41777 18287 314	(05)
	2	-9.63006 98255 869	(05)	-1.42686 20411 978	(05)
	3	-1.08872 52505 125	(05)	-6.21717 54627 536	(03)
	4	-6.95142 48803 854	(03)	-7.03261 35254 848	(02)
	5	-6.52319 89744 728	(02)	1.00000 00000 000	(00)
8	0	1.28716 81214 82446 39280 9	(10)	2.57433 62429 64846 24466 7	(10)
	1	1.37539 69320 37025 11182 5	(10)	5.93816 56486 79590 16000 3	(09)
	2	5.10665 59183 64406 10368 3	(09)	9.00633 03732 61233 43908 9	(08)
	3	8.56147 10024 33314 86246 9	(08)	8.04253 66342 83289 88858 7	(07)
	4	7.48361 81243 80232 98482 4	(07)	5.60971 17595 41920 06281 4	(06)
	5	4.86010 65854 61882 51153 5	(06)	2.24743 12028 99137 52354 3	(05)
	6	2.73957 49902 21406 08772 8	(05)	7.57457 89093 41537 56011 5	(03)
	7	4.63171 08431 83427 12306 1	(03)	-2.37383 57813 73772 62308 6	(01)
	8	5.78758 10040 96660 65910 9	(01)	1.00000 00000 00000 00000 0	(00)
9	0	9.53904 31383 75296 85073 071	(11)	1.90780 86276 75059 16848 384	(12)
	1	1.09086 19179 76949 25970 267	(12)	5.83214 73972 70833 55022 737	(11)
	2	4.57302 66750 30644 70069 336	(11)	1.04516 64692 48187 07276 116	(11)
	3	9.44640 69371 30822 59763 084	(10)	1.22272 73758 61580 59174 256	(10)
	4	1.13392 40859 96679 77378 270	(10)	1.07248 18832 75479 65542 864	(09)
	5	9.69200 64774 06284 28141 723	(08)	6.86833 53570 00418 28875 818	(07)
	6	6.71953 10507 96081 75927 677	(07)	3.42258 54978 70618 48187 183	(06)
	7	3.27425 31821 83494 05394 001	(06)	1.13430 57824 29430 01348 947	(05)
	8	1.09513 05859 34055 20435 868	(05)	2.77837 18457 74528 05958 463	(03)
	9	2.84456 26751 69802 16448 194	(03)	1.00000 00000 00000 00000 000	(00)

TABLE III

$$\zeta(s) \approx 1 + \sum_{j=0}^{n}{}' p_j T_j(t) \Big/ \sum_{j=0}^{n}{}' q_j T_j(t), \qquad t = \frac{s-8}{3}, \qquad 5 \le s \le 11$$

n	j	p_j		q_j	
5	0	1.84900 99918 7	(03)	4.58161 73101 5	(05)
	1	-8.39681 24248 2	(02)	2.17592 06213 4	(05)
	2	1.84197 92969 4	(02)	4.75387 98338 2	(04)
	3	-2.32222 78218 6	(01)	6.01111 09082 7	(03)
	4	1.68560 34490 2	(00)	4.36801 89581 7	(02)
	5	-5.72149 34762 5	(-02)	1.51875 00000 0	(01)
6	0	7.65350 39697 5206	(04)	1.87400 70944 9945	(07)
	1	-3.51659 51754 0597	(04)	8.84015 00526 2110	(06)
	2	8.00110 63791 5310	(03)	1.94507 16063 8845	(06)
	3	-1.09338 72046 0591	(03)	2.57600 76199 8960	(05)
	4	9.43060 49292 2886	(01)	2.12069 28052 1806	(04)
	5	-4.90452 46402 1741	(00)	1.04683 24143 9803	(03)
	6	1.21767 64406 2363	(-01)	2.27812 50000 0000	(01)
8	0	-1.37639 45864 32697 9078	(07)	-2.59451 24986 97831 0818	(09)
	1	7.48218 91630 53159 7222	(06)	-9.48715 40757 99078 1663	(08)
	2	-2.07584 50481 02110 1368	(06)	-1.05496 19347 40052 0329	(08)
	3	3.55302 55709 62142 9466	(05)	4.67774 48821 19930 4847	(06)
	4	-4.06706 44955 18548 8897	(04)	3.12936 04057 38135 3370	(06)
	5	3.19804 86402 71469 1139	(03)	4.59581 80383 93050 6974	(05)
	6	-1.69820 93703 37228 5303	(02)	3.88176 10961 03968 3366	(04)
	7	5.61485 84239 42890 4752	(00)	1.92561 54483 44914 2325	(03)
	8	-8.93888 70592 61549 4375	(-02)	5.12578 12500 00000 0000	(01)
9	0	1.31731 C0778 41255 95241	(08)	2.32735 50216 37673 34965	(10)
	1	-7.49969 32583 63955 41082	(07)	8.06195 84801 09344 64523	(09)
	2	2.38862 72960 74183 05175	(07)	1.20398 93324 79128 14480	(09)
	3	-4.95116 97536 05156 15979	(06)	1.93082 33109 74192 31735	(08)
	4	7.10728 21386 88157 86508	(05)	3.66527 98303 89832 19232	(07)
	5	-7.25277 77641 34448 98223	(04)	6.22570 65496 50081 68318	(06)
	6	5.26209 82645 20380 54349	(03)	7.20159 25663 83949 26869	(05)
	7	-2.63045 60896 79592 12803	(02)	5.71179 75640 89075 79564	(04)
	8	8.28410 63650 33952 88601	(00)	2.80036 61256 59096 99239	(03)
	9	-1.26433 19164 40679 13509	(-01)	7.68867 18750 00000 00000	(01)

TABLE IV

$$\zeta(s) \approx 1 + 2^{-s + (1/s) \sum_{j=0}^{n} p_j s^{-j} / \sum_{j=0}^{n} q_j s^{-j}}, \qquad 11 \le s \le 25$$

n	j	p_j		q_j	
4	0	2.88915 56312 2	(-06)	8.72078 04947 2	(-05)
	1	-3.60673 21549 6	(-04)	-9.58659 06458 3	(-04)
	2	1.74441 55297 5	(-02)	2.61239 70760 9	(-02)
	3	-3.88297 10518 9	(-01)	-1.39152 31614 9	(-02)
	4	3.36332 63994 4	(00)	1.00000 00000 0	(00)
5	0	-6.95395 38811 340	(-08)	1.61059 22487 913	(-06)
	1	1.16861 20068 219	(-05)	9.59892 63880 291	(-06)
	2	-8.12298 42080 718	(-04)	6.07669 74338 932	(-04)
	3	2.92431 24478 121	(-02)	1.07138 06527 427	(-02)
	4	-5.46064 90480 737	(-01)	8.47868 52767 957	(-02)
	5	4.23648 33879 757	(00)	1.00000 00000 000	(00)
*9	0	1.66156 48051 57746 75916	(-11)	-6.99562 63351 91916 54964	(-10)
	1	-4.68068 82766 06545 26862	(-09)	-1.77757 96189 51492 56941	(-08)
	2	5.83519 72731 91470 47318	(-07)	-9.82231 82573 40780 36442	(-07)
	3	-4.17644 01264 31456 02124	(-05)	-2.84927 28275 90964 87594	(-05)
	4	1.85468 42284 35979 59483	(-03)	-5.81727 90938 80480 93531	(-04)
	5	-5.11288 80022 04902 40591	(-02)	-1.15848 74916 97665 85807	(-02)
	6	8.10450 23175 11003 53193	(-01)	-1.28149 12405 19781 95742	(-01)
	7	-5.69951 94876 84789 22618	(00)	-1.11913 05734 90977 09324	(00)
	8			-7.67928 76160 46288 12537	(-01)
	9			1.00000 00000 00000 00000	(00)
9	0	6.54074 87262 07601 13319 6	(-13)	-9.41483 96988 23587 07316 4	(-11)
	1	-1.91182 33182 41692 93750 0	(-10)	8.05123 81296 05720 22212 8	(-10)
	2	2.45909 27987 80779 03147 0	(-08)	-1.48342 86157 21743 05569 6	(-07)
	3	-1.80047 53535 42409 75813 5	(-06)	-2.21243 85714 34347 20342 1	(-07)
	4	8.06024 99146 89416 87391 8	(-05)	-7.12213 00202 26622 79049 2	(-05)
	5	-2.17335 07154 33717 13164 9	(-03)	-4.25218 58071 71304 01282 1	(-04)
	6	3.09201 82991 73503 66849 2	(-02)	-1.27791 21921 98850 98331 8	(-02)
	7	-1.10135 65806 72497 58178 8	(-01)	-5.86508 77210 99441 18425 8	(-02)
	8	-1.73589 22656 22043 62653 2	(00)	-3.00031 31173 95834 98139 1	(-01)
	9	3.80279 09938 36744 53166 3	(00)	1.00000 00000 00000 00000 0	(00)

*Denominator is of degree 2 greater than numerator.

TABLE V

$$\zeta(s) \approx 1 + 2^{-s + (1/s) \sum_{j=0}^{n} p_j s^{-j} / \sum_{j=0}^{n} q_j s^{-j}}, \qquad 25 \le s \le 55$$

n	j	p_j		q_j	
3	0	4.45020 40561	(-09)	-9.08360 84927	(-05)
	1	-6.39481 80674	(-07)	5.83123 29118	(-03)
	2	3.06819 07594	(-05)	-1.30562 75204	(-01)
	3	-4.92460 44511	(-04)	1.00000 00000	(00)
5	0	6.30581 82031 926	(-12)	-2.45066 98983 435	(-07)
	1	-1.69754 12776 425	(-09)	2.27263 46889 318	(-05)
	2	1.84569 49027 429	(-07)	-9.29701 78224 265	(-04)
	3	-1.01420 20734 917	(-05)	1.98757 34180 426	(-02)
	4	2.81957 70988 437	(-04)	-2.20338 90078 903	(-01)
	5	-3.17620 79765 651	(-03)	1.00000 00000 000	(00)
8	0	1.03144 87718 88597 1168	(-15)	5.93959 41728 84190 5020	(-11)
	1	-5.12584 61396 46882 4062	(-13)	-6.04755 35907 99918 0572	(-09)
	2	1.12948 79419 48735 4786	(-10)	3.64680 20866 83885 6275	(-07)
	3	-1.44234 66537 31309 5228	(-08)	-1.29456 90556 80118 1241	(-05)
	4	1.16824 67698 44580 9766	(-06)	3.20189 49847 02292 5001	(-04)
	5	-6.14975 16799 03148 0614	(-05)	-5.07801 55709 99940 7748	(-03)
	6	2.05594 67798 88303 2750	(-03)	5.49628 90788 15872 6560	(-02)
	7	-3.99339 42939 46688 6853	(-02)	-3.24517 61115 59724 1852	(-01)
	8	3.45234 97673 61784 5708	(-01)	1.00000 00000 00000 0000	(00)
9	0	2.32320 68054 88716 51963 3	(-16)	-1.41965 98040 97653 26071 0	(-11)
	1	-1.35449 79553 19349 35076 5	(-13)	1.34852 03552 59192 68792 1	(-09)
	2	3.55929 73750 95400 68585 2	(-11)	-8.97874 17847 37323 82249 4	(-08)
	3	-5.53602 70696 83907 98449 3	(-09)	3.35733 07823 35633 51346 7	(-06)
	4	5.61986 24870 48476 40107 6	(-07)	-9.80017 32529 00364 84028 5	(-05)
	5	-3.86350 56607 15340 11238 7	(-05)	1.76658 33281 80539 21937 9	(-03)
	6	1.79968 78444 35254 07412 3	(-03)	-2.58993 67889 18623 48909 1	(-02)
	7	-5.48022 94365 91701 99314 2	(-02)	1.97161 45596 61601 18167 1	(-01)
	8	9.90367 46668 06121 96654 9	(-01)	-1.36202 98660 70096 90749 2	(00)
	9	-8.09626 06414 79518 69042 8	(00)	1.00000 00000 00000 00000 0	(00)

the computation

$$\zeta(s) - 1 = \frac{R_{lm}(s)}{s - 1} - 1$$

can lead to considerable subtraction error. Instead, one should use the form

$$\zeta(s) - 1 \simeq \frac{\sum_{i=0}^{M} \hat{p}_i s^i}{(s - 1) \sum_{i=0}^{m} q_i s^i}, \qquad .5 \leqq s \leqq 5, \tag{4.2}$$

where

$$M = \max(l, m + 1), \qquad \sum_{i=0}^{M} \hat{p}_i s^i \equiv \sum_{i=0}^{l} p_i s^i - (s - 1) \sum_{i=0}^{m} q_i s^i,$$

and the $\hat{p}_i$ are determined explicitly. It is not difficult to show that if δ_{lm} is the relative error in using (4.1) as an approximation to $\zeta(s)$, then the relative error in using (4.2) as an approximation to $\zeta(s) - 1$ is bounded by $30\delta_{lm}$. Thus, if the relative error in the machine is bounded by 10^{-D}, one should choose an $R_{lm}(s)$, $.5 \leqq s \leqq 5$, such that $30\delta_{lm} < 10^{-D}$.

In the second interval, the $R_{lm}(s)$ are poorly conditioned when expressed in the usual way. They are therefore presented instead as well-conditioned ratios of sums of Chebyshev polynomials. In the last two intervals, better accuracy in $\zeta(s) - 1$ will be obtained by setting $n = [s]$ and computing $2^{-s+(1/s)R(1/s)}$ as $2^{-n} \times 2^{-(s-n)+(1/s)R(1/s)}$, where 2^{-n} can be done exactly on most computers by modifying the floating point exponent. Use of the new self-contained exponentiation routines [1] would also help.

Subroutines for both the CDC 3600 and the IBM 360 have been written using these techniques. In each case, essentially machine precision was achieved for $\zeta(s)$ and $\zeta(s) - 1$ for s in the respective ranges.

Argonne National Laboratory
9700 South Cass Avenue
Argonne, Illinois 60439

Department of Computer Science
University of Notre Dame
Notre Dame, Indiana 46556

Applied Mathematics Division
Argonne National Laboratory
Argonne, Illinois 60439

1. N. W. Clark & W. J. Cody, *Self-Contained Exponentiation, Proc. FJCC*, AFIPS Press, Montvale, N. J., 1969, pp. 701–706.
2. W. J. Cody, "A survey of practical rational and polynomial approximation of functions," *SIAM Rev.*, v. 12, 1970, pp. 400–423.
3. R. F. King & D. L. Phillips, "The logarithmic error and Newton's method for the square root," *Comm. ACM*, v. 12, 1969, pp. 87–88.
→ D. H. Lehmer, "On the maxima and minima of Bernoulli polynomials," *Amer. Math. Monthly*, v. 47, 1940, pp. 533–538. MR **2**, 43.
5. B. Markman, "The Riemann zeta function," *BIT*, v. 5, 1965, pp. 138–141.
6. P. H. Sterbenz & C. T. Fike, "Optimal starting approximations for Newton's method," *Math. Comp.*, v. 23, 1969, pp. 313–318. MR **39** #6511.
7. H. C. Thacher, Jr., "On expansions of the Riemann zeta function," *Math. Comp.* (To appear.)