

Exponential Chebyshev Approximation on Finite Subsets of $[0, 1]$

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Abstract. In this note the convergence of best exponential Chebyshev approximation on finite subsets of $[0, 1]$ to a best approximation on the interval is proved when the function to be approximated is continuous and when the union of the finite subsets is dense in $[0, 1]$.

1. Introduction. In this note we study the convergence of best exponential Chebyshev approximation on finite subsets of $[0, 1]$ to a best approximation on the interval. This problem has been considered for linear approximation [1] and, recently, for generalized rational approximation in [2].

Let X_r be a set of r distinct points in $[0, 1]$, containing the endpoints (a common computational situation). We assume that the sequence of subsets $\{X_r\}$ fills up the interval in the sense that, given $x \in [0, 1]$, there is an $x_r \in X_r$ such that $\{x_r\} \rightarrow x$.

Following Rice [3, Chapter 8], we approximate $f \in C([0, 1])$ by exponential functions of the form

$$(1) \quad E(A, x) = \sum_{i=1}^k \left(\sum_{j=0}^{m_i} p_{ij} x^j \right) e^{t_i x},$$

where $|p_{ij}| < \infty$, $|t_i| < \infty$ and $\sum_{i=1}^k (m_i + 1) \leq n$, n a fixed positive integer.

For each set X_r , we define the usual seminorm on $C([0, 1])$, corresponding to X_r , by

$$\|f\|_{X_r} = \sup_{x \in X_r} |f(x)|$$

for all f in $C([0, 1])$. We denote by $E(A_r, x)$ and $E_r(x)$ the best approximation to $f(x)$ on X_r , i.e. the best approximation to f with respect to the seminorm corresponding to X_r . Norm signs without subscripts denote the usual Chebyshev norm.

It is known [3] that best approximation need not exist on finite point sets. However, we assume existence, a reasonable assumption in many computational situations. Moreover, Rice solves a special case of this problem through the use of pseudo functions [3, pp. 65-69]. An extension of this technique to handle general exponential approximation is under investigation.

2. The Convergence Theorem. The main result of this note is

THEOREM 1. *Let E^* be a best exponential approximation to f on $[0, 1]$. Then*

$$\|f - E_r\|_{X_r} \rightarrow \|f - E^*\|.$$

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Proof. First, suppose that $\{\|E_r\|\}$ is a bounded sequence. Then by a theorem in Rice [3, Chapter 8], $\{E_r(x)\}$, or a subsequence thereof, converges pointwise to a possibly discontinuous function

$$\begin{aligned} E(x) &= E_0(x), & 0 < x < 1, \\ &= e_0, & x = 0, \\ &= e_1, & x = 1, \end{aligned}$$

where $E_0(x)$ is an exponential of form (1). We claim that E_0 is a best approximation to f on $[0, 1]$. For if not, there exists a point x such that $|f(x) - E_0(x)| > \|f - E^*\|$. By continuity, we may assume that $x \in [\delta, 1 - \delta]$ where $\delta > 0$ is sufficiently small. Let $\{x_r\} \rightarrow x$ with $x_r \in X_r$, $\epsilon > 0$, and let y_1, \dots, y_n be in $[\delta, 1 - \delta]$. By the definition of varisolvence, there is a $\delta(\epsilon)$ such that the inequality $|Y_i - E_0(y_i)| < \delta(\epsilon)$ implies that there is a function $E(A, y)$ satisfying $E(A, y_i) = Y_i$ and $|E(A, y) - E_0(y)| < \epsilon$. But since $E_r \rightarrow E_0$ pointwise on $[\delta, 1 - \delta]$, for r large enough, $|E_r(y_i) - E_0(y_i)| < \delta(\epsilon)$. Hence, there are functions $\{E(A_r, y)\}$ such that $E(A_r, y_i) = E_r(y_i)$. But since these interpolating functions are unique (m is the degree of solvence), $E(A_r, y) = E_r(y)$ for all y . Therefore, $|E_r(y) - E_0(y)| < \epsilon$ for y in $[\delta, 1 - \delta]$. Therefore, by a standard inequality, $|f(x_r) - E_r(x_r)| \rightarrow |f(x) - E_0(x)|$. Hence, for sufficiently large r , $|f(x_r) - E_r(x_r)| > \|f - E^*\|$. This contradicts the fact that E_r is a best approximation on X_r , and hence E_0 is a best approximation.

Suppose now that $\{\|E_r\|\}$ is an unbounded sequence. Following Dunham [2], define $B_r(x) = E_r(x)/\|E_r\|$. Then $\{\|B_r\|\}$ is a bounded sequence and, by the aforementioned result of Rice, we may assume that $\{B_r(x)\}$ converges pointwise to a function of the form

$$\begin{aligned} B(x) &= B_0(x), & 0 < x < 1, \\ &= b_0, & x = 0, \\ &= b_1, & x = 1, \end{aligned}$$

where $B_0(x)$ is an exponential of form (1). Now, assume that $b_0 = B_0(0)$ and $b_1 = B_0(1)$. The $B(x)$ is an exponential and, using varisolvence as before, it follows that $\{B_r\}$ converges uniformly to B . Since $\|B_r\| = 1$, there exists $y \in [0, 1]$ and a neighborhood N of y such that $m = \inf_{x \in N} B(x) > 0$ and $B_r(x) > m/2$ for r sufficiently large. Hence, $\inf_{x \in N} E_r(x) \rightarrow \infty$ as $r \rightarrow \infty$ and, for large enough r , there exists $x_r \in X_r$ such that $E_r(x_r) > 2\|f\|$. This contradicts E_r being a best approximation to f on X_r , since then

$$\|f - E_r\|_{x_r} > \|f\| \geq \|f\|_{x_r} = \|f - 0\|_{x_r}.$$

It remains to consider the case where $B(x)$ has an endpoint discontinuity. Assume without loss of generality that $B(x) = B_0(x)$, $0 < x \leq 1$, and $b_0 > B_0(0)$. If $B_0 \not\equiv 0$, there exists $y \in [\delta, 1 - \delta]$, $\delta > 0$, such that $B_0(y) \neq 0$ and $\{B_r(y)\} \rightarrow B_0(y)$. Using the varisolvence of B_0 on this interval, we again contradict E_r being best on X_r . If $B_0 \equiv 0$, we reach the same contradiction on E_r since then $\{E_r(0)\}$ is unbounded. This concludes the proof.

If we cannot assume that the endpoints are in X_r , the proof goes through without change except for the last case. If $B_0 \equiv 0$ and if $0 \in X_r$ for only a finite set of r values

with $b > 0$, it follows from the properties of exponentials and the boundedness of $\|E_r\|_x$, that $\{E_r(x)\}$ is bounded on $[\delta, 1 - \delta]$, $\delta > 0$. Hence, $\{E_r(x)\}$ or a subsequence thereof converges pointwise to $E_0(x)$, an exponential, on $(0, 1)$. From this reasoning, as before using varisolence, E_0 is a best approximation to f on $[0, 1]$.

From the proof of Theorem 1 we have

COROLLARY. *The sequence $\{E_r(x)\}$ has a subsequence which converges pointwise except possibly at the endpoints to a best exponential approximation, $E_0(x)$, to f on $[0, 1]$. If the subsequence converges to $E_0(0)$ and $E_0(1)$ also, then the convergence is uniform.*

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