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The Triangular Decomposition of Hankel Matrices

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Abstract. An algorithm for determining the triangular decomposition $H = R^*DR$ of a Hankel matrix H using $O(n^2)$ operations is derived. The derivation is based on the Lanczos algorithm and the relation between orthogonalization of vectors and the triangular decomposition of moment matrices. The algorithm can be used to compute the three-term recurrence relation for orthogonal polynomials from a moment matrix.

1. Introduction. Let H be a Hankel matrix of order n,

$$H = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \\ h_2 & h_3 & \cdots & h_{n+1} \\ \vdots & \vdots & & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n-1} \end{bmatrix} = (h_{i+j-1}). \tag{1.1}$$

Assume all leading principal minors of H are nonzero. Then H has a unique decomposition of the form

$$H = R^*DR, \tag{1.2}$$

where R is unit upper triangular and D is diagonal. In the following sections, we derive an algorithm for determining this decomposition in $O(n^2)$ operations. An algorithm for determining the inverse of H in $O(n^2)$ operations has appeared previously [4].

It will be convenient to interpret H as a moment matrix. Let B be an $n \times n$ Hermitian matrix and v a vector such that

$$F = (v, Bv, B^2v, \cdots, B^{n-1}v) \tag{1.3}$$

is nonsingular, and let

$$A = (F^{-1})^*HF^{-1}. \tag{1.4}$$

Then

$$\langle x, y\rangle = x^*Ay \tag{1.5}$$

defines a Hermitian bilinear form on $\mathcal{E}^n$. With respect to this form, H is the moment matrix

$$H = (\langle B^{i-1}v, B^{j-1}v\rangle). \tag{1.6}$$

The matrix $Q = FR^{-1}$ satisfies

$$F = QR, \qquad Q^*AQ = D. \tag{1.7}$$

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Thus the columns $q_1, \cdots, q_n$ of Q are orthogonal with respect to the bilinear form given in (1.5) and

$$q_i = B^{i-1}v + y_i, \quad \text{where } y_i \in \operatorname{span}(v, Bv, \cdots, B^{i-2}v). \tag{1.8}$$

The algorithm given below is based on applying the Lanczos procedure [3] to orthogonalize the columns of F with respect to (1.5). The factor DR is then determined as a by-product of the orthogonalization.

We note that it is not always possible to orthogonalize an ordered linearly independent set of vectors with respect to a nondefinite Hermitian form (1.5). For example, if $A = [\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}]$ and $F = [\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}]$, there is no unit upper triangular matrix R and matrix Q such that (1.7) holds. A sufficient condition for such an orthogonalization is that F^*AF have all leading principal minors nonzero and thus admit a unique triangular decomposition $F^*AF = R^*DR$. This condition is always satisfied if F is nonsingular and if (1.5) defines a true inner product on $\mathcal{E}^n$, for then F^*AF is positive definite.

2. The Lanczos Algorithm. The Lanczos algorithm is a method for orthogonalizing a set of vectors of the type given by the columns of F in (1.3). Orthogonal vectors $q_1, q_2, \cdots, q_n$ satisfying (1.8) are determined by choosing $q_1 = v$ and

$$q_{i+1} = (B - a_i I)q_i - b_i q_{i-1}, \qquad i = 1, \cdots, n-1, \tag{2.1}$$

where a_i and b_i are chosen so q_{i+1} is orthogonal to q_i and q_{i-1}, and hence also to $q_1, \cdots, q_{i-2}$. This implies

$$\begin{aligned} &a_i = \langle Bq_i, q_i\rangle/\langle q_i, q_i\rangle, && i = 1, \cdots, n; \\ &b_1 = 0, \quad b_i = \langle q_i, q_i\rangle/\langle q_{i-1}, q_{i-1}\rangle, && i = 2, \cdots, n. \end{aligned} \tag{2.2}$$

Thus a tridiagonal matrix T with

$$t_{i,i-1} = 1, \qquad t_{ii} = a_i, \qquad t_{i,i+1} = b_{i+1} \tag{2.3}$$

is determined such that

$$BQ = QT \tag{2.4}$$

where Q is the same matrix as appears in (1.7). The a_i and b_i in (2.2) are all well-defined since $\langle q_i, q_i\rangle = q_i^*Aq_i = d_{ii} \neq 0$ for each i.

3. Derivation of Decomposition Algorithm. Let E denote the $n \times 2n$ matrix

$$E = (F, G) = (v, Bv, B^2v, \cdots, B^{2n-1}v),$$

and N the nilpotent matrix $N = (e_2, e_3, \cdots, e_n, 0)$. Then the columns of E are such that

$$(BE)_j = (EN)_j, \qquad j = 1, \cdots, 2n-1. \tag{3.1}$$

From (1.7) and (2.4), we have $BE = BQ(R, Q^{-1}G) = QT(R, Q^{-1}G)$, while $EN = Q(R, Q^{-1}G)N$. Hence, (3.1) implies

$$[T(R, Q^{-1}G)]_j = [(R, Q^{-1}G)N]_j, \qquad j = 1, \cdots, 2n-1,$$

or

(3.2) $$(PC)_j = (CN)_j, \qquad j = 1, \cdots, 2n-1,$$

where

(3.3) $$P = DTD^{-1}, \qquad C = (DR, DQ^{-1}G).$$

Comparing the (i, j)th elements on both sides of (3.2), we have

$$p_{i,i-1}c_{i-1,j} + p_{ii}c_{ij} + p_{i,i+1}c_{i+1,j} = c_{i,j+1}.$$

Using (2.3) and (3.3), this can be expressed as

(3.4) $$c_{i+1,j} = c_{i,j+1} - a_i c_{ij} - b_i c_{i-1,j}.$$

Now $e_1^T C = e_1^T R^* C = e_1^T(R^*DR, R^*DQ^{-1}G) = e_1^T(H, F^*AB^nF)$, so the first row of C is given by $c_{1j} = h_j$, $j = 1, \cdots, 2n-1$. Other rows of C can be generated using (3.4). Fig. 1 illustrates the matrix C when $n = 3$. For obtaining DR, only the labeled elements (elements $c_{ii}, \cdots, c_{i,2n-i}$ of row i) need to be computed.

$$\left[\begin{array}{ccc|ccc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & x \\ 0 & c_{22} & c_{23} & c_{24} & x & x \\ 0 & 0 & c_{33} & x & x & x \end{array}\right]$$

FIGURE 1

Equation (3.4) could be used to compute the elements of C in order by columns instead of by rows. However, if C is generated by rows, the a_i and b_i can be computed simultaneously with the elements of C. From (2.2), we have $b_i = d_{ii}/d_{i-1,i-1} = c_{ii}/c_{i-1,i-1}$. The a_i are given by

(3.5) $$a_i = c_{i,i+1}/c_{ii} - c_{i-1,i}/c_{i-1,i-1}.$$

To see this, note from (1.7) that by expanding q_i in terms of the $B^{i-1}v$, we obtain $Bq_i = B^i v + (R^{-1})_{i-1,i}B^{i-1}v + \cdots$, while (2.4) implies $Bq_i = q_{i+1} + a_i q_i + b_{i-1}q_i = B^i v + (R^{-1})_{i,i+1}B^{i-1}v + a_i B^{i-1}v + \cdots$. Comparing coefficients of $B^{i-1}v$ in these expressions, we obtain $a_i = (R^{-1})_{i-1,i} - (R^{-1})_{i,i+1}$, which implies (3.5) since $(R^{-1})_{i-1,i} = -(D^{-1}C)_{i-1,i}$. Summarizing, we have:

Algorithm. If the Hankel matrix (1.1) admits a triangular decomposition (1.2), then to find the elements of DR, set

$$c_{1j} = h_j, \qquad j = 1, \cdots, 2n-1,$$

$$a_1 = c_{12}/c_{11}, \qquad b_1 = 0;$$

for $i = 1, \cdots, n-1$, form

$$c_{i+1,j} = c_{i,j+1} - a_i c_{ij} - b_i c_{i-1,j}, \qquad j = i+1, \cdots, 2n-i-1,$$

and if $i \neq n-1$,

$$a_{i+1} = c_{i+1,i+2}/c_{i+1,i+1} - c_{i,i+1}/c_{ii}, \qquad b_{i+1} = c_{i+1,i+1}/c_{ii}.$$

Then

$$(DR)_{ij} = c_{ij}, \qquad i = 1, \cdots, n; \quad j = i, \cdots, n.$$

4. Relation to Orthogonal Polynomials. The a_i and b_i computed in the algorithm have another significance. Corresponding to (2.1), define the polynomials

$$\begin{aligned} &p_0(x) \equiv 1, \qquad p_1(x) = (x - a_1)p_0(x), \\ &p_i(x) = (x - a_i)p_{i-1}(x) - b_i p_{i-2}(x), \qquad i = 2, \cdots, n. \end{aligned} \tag{4.1}$$

These can be considered as generalized Lanczos polynomials determined by H (see e.g. [2, p. 23]). The p_i are orthogonal with respect to the bilinear form defined on the set of polynomials of degree $< n$ by the moments

$$\langle x^{i-1}, x^{j-1} \rangle = h_{i+j-1}, \qquad i, j = 1, \cdots, n.$$

Thus the algorithm provides a technique for obtaining the coefficients for the three-term recurrence relation between orthogonal polynomials from the moments.

The coefficients a_i and b_i in the Lanczos algorithm are usually defined using (2.2), possibly substituting $\langle B^{i-1}v, q_i \rangle$ for $\langle q_i, q_i \rangle$. The corresponding formulas in the polynomial case (see e.g. [1, Appendix]) are

$$a_i = \langle xp_{i-1}, p_{i-1} \rangle / \langle p_{i-1}, p_{i-1} \rangle, \qquad b_i = \langle p_{i-1}, p_{i-1} \rangle / \langle p_{i-2}, p_{i-2} \rangle.$$

The derivation of the algorithm given herein provides a different formula for the a_i. From (1.7), $B^{j-1}v = \sum_k r_{kj} q_k$, so

$$\langle B^{i-1}v, q_i \rangle = \sum_k r_{ki} \langle q_k, q_i \rangle = r_{ii} \langle q_i, q_i \rangle = c_{ii},$$

and with (3.5) this implies

$$a_i = \langle B^i v, q_i \rangle / \langle B^{i-1}v, q_i \rangle - \langle B^{i-1}v, q_{i-1} \rangle / \langle B^{i-2}v, q_{i-1} \rangle.$$

Similarly, for the polynomials (4.1) we have

$$a_i = \langle x^i, p_{i-1} \rangle / \langle x^{i-1}, p_{i-1} \rangle - \langle x^{i-1}, p_{i-2} \rangle / \langle x^{i-2}, p_{i-2} \rangle.$$

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