Homomorphisms of Knot Groups on Finite Groups

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Abstract. We describe trial and error computer programs for finding certain homomorphisms of a knot group on a special projective group $LF(2, p)$, p prime, and programs to evaluate $H_1(\mathfrak{M}; Z)$ where $\mathfrak{M}$ is a finitely sheeted branched covering space of S^3 associated with such a homomorphism. These programs have been applied to several collections of examples, in particular to the Kinoshita-Terasaka knots, and we state numerous conjectures based on these experiments.

About forty years ago a universal method for obtaining algebraic invariants of knot type was proposed and became standard. The method, as applied to a knot k of type K and having group $\pi K = \pi_1(S^3 - k; *)$, begins with the determination of the homomorphisms of πK on a given group G. These homomorphisms fall into equivalence classes under the action of the automorphisms of G, and a crude preliminary invariant of K is the number of homomorphism classes. In the next stage of the method, we fix a transitive permutation representation of G, perhaps of infinite degree. Each homomorphism class of πK on G is associated with a covering space $\mathfrak{U}$ of $S^3 - k$ such that the number of sheets in the covering is the degree of the permutation representation, and the group $H_1(\mathfrak{U}; Z)$ is an algebraic invariant of the knot type K. In this paper, we shall discuss the means and results of implementing the universal method on a computer when the group G is chosen to be one of the special projective groups $L_p = LF(2, p) = PSL(2, p)$, where p is a prime integer.

The universal method has been most thoroughly examined in the case where the group G is cyclic. The determination of the homomorphism classes becomes completely trivial, and all the homology invariants can be deduced from a single matrix, the Alexander matrix. These "cyclic invariants" have been applied with good effect to just about every problem in knot theory, but, alas, when the Alexander polynomial $\Delta(x)$ of the knot reduces to the constant 1 these invariants degenerate and are worthless. It is no good choosing a solvable group for G in such a case; to get useful results from the universal method, we must use nonsolvable groups. Of course, simple groups receive first consideration in this context, and of the families of classical finite simple groups, the family $\{L_p\}$ is the most manageable. As further encouragement for this study, the group L_5 is isomorphic to the alternating group A_5, and R. H. Fox has shown in several papers ([5] is a good example) that the homomorphisms of a knot group on A_5 have some interesting applications. There are, however, very few clues to suggest what the best method for finding homomorphisms on the special projective groups could be. In addition, almost nothing is known about the homology invariants of homomorphisms on noncyclic groups G. It therefore seems reasonable

to explore the subject initially by applying the most humble methods to a large number of examples. This leads directly to the development of computer programs to do the work and to large tables of output data. These are the subject of this paper.

Our method for finding the homomorphisms of a knot group πK on a finite group G is simple trial and error. Suppose that πK is given to us in the form

(0) $$\pi K = |x_1, \cdots, x_n : r_1(\mathbf{x}) = r_2(\mathbf{x}) = \cdots = 1|.$$

Let $\mathbf{a} = (a_1, \cdots, a_n)$ be a set of elements of G which together generate G. The assignment $x_i \rightarrow a_i$, for $i = 1, \cdots, n$, can be extended to a homomorphism $\theta : \pi K \rightarrow G$ such that $(x_i)\theta = a_i$ if and only if $r_1(\mathbf{a}) = r_2(\mathbf{a}) = \cdots = 1$. Hence the determination of the homomorphisms (and later their arrangement into classes) is a finite process. But before we actually carry out this process even on a computer, we should find ways of reducing the number of experiments to the irreducible minimum. When the presentation (0) is an over-presentation (see [5] or [7]), the generators x_i are all conjugate in πK. This is such a great aid to the classification that we use only over-presentations for knot groups. In Section 4, we discuss a version of the over-presentation that is especially well adopted to our methods and programs.

In Section 2, we determine the irreducible set of experiments needed to determine the homomorphism classes of πK on A_5 when n in (0) is 2 or 3. It turns out that there are three possible kinds of such homomorphisms, and that the search for one kind is much easier to describe and implement in a computer program than the others. The characteristic property of these homomorphisms is that the image of an over-generator of (0) is an element of order 5 in A_5. In addition, we found that such homomorphisms are more numerous than the others for our test cases. Hence, when we generalize the A_5 case to L_p in Section 3, we restrict ourselves to the homo-morphisms which carry an over-generator of πK to an element of order p in L_p. We call such homomorphisms "reps" for brevity, and note that the property of being a rep is invariant, that is, it does not depend on the over-presentation (0) used in the definition. In Section 5, we discuss how to implement the second stage of the universal method on a computer when G is an arbitrary finite group. Our tools are now ready.

In Section 6, we discuss the outcome of applying our programs to various collec-tions of test cases. We have concentrated most of our effort on the groups $G = L_p$ where $p \leq 11$, with the greatest emphasis on $p = 5$ because the results are all new and the computation time per knot is still reasonable. The first set of test knots that we discuss are the prime knots with at most 9 crossing points, conveniently drawn on pp. 70–72 of Reidemeister [13]. We call these *the classical knots* and we have two tables in the microfiche section for them. The first table is of over-presentations for their groups which may be used as a standard reference by anyone doing computation in knot theory. The second table is of the invariants of these knots associated with their homomorphisms on $A_5 = L_5$. We state a large number of conjectures about the homology invariants associated with reps of an arbitrary knot group on L_p that are backed up by numerous tables of data which are not all discussed here. We hope that the reader shares the author's faith that only a small amount of wit and dis-crimination is needed to extract good conjectures from modest amounts of numerical data in this subject. The homology invariants for a certain special sort of reps are also discussed in Conjecture B, and we believe that this may be the best clue to the

whole subject. The section ends with a discussion of three particular knots having Alexander polynomial $\Delta(x) \equiv 1$, namely Seifert's knot, the 11-crossing Kinoshita-Terasaka knot [10], and a third 11-crossing knot discovered by J. Conway [2]. These three knots are shown to have different types.

In Section 7, we demonstrate the power and limitations of our programs by trying them on an infinite collection of knots that can be described by two integral parameters. These knots were discovered by Kinoshita and Terasaka [10] and shown to be nontrivial with $\Delta(x) \equiv 1$. We shall show that a homomorphism of the group of one of these knots on A_5 determines a corresponding homomorphism for the groups of an infinity of other knots whose integral parameters vary in certain arithmetic progressions. If we vary one of these integral parameters in its progression, we get interesting formulas for the corresponding homology invariants. If these formulas were proven (but it is not worth the effort to do this), they would show that these knots determine infinitely many types. As it is, we have only proven that they determine at least 200 types.

The work described in this paper has led to a later paper, "Parabolic representations of knot groups" that we are submitting for publication elsewhere. In it, we prove that many of the knot groups discussed here have representations on subgroups of $LF(2, \mathbf{C})$, from each of which one can derive an infinity of reps on the groups L_p.

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1. Throughout this paper we shall use a standard notation for groups and homomorphisms taken from Huppert [9]. In particular, we write operators on the right so that products of permutations are read from left to right, when x, y are elements of a group $x^y \equiv y^{-1}xy$, and $G = \langle x_1, \cdots, x_n \rangle$ means that G is the group generated by $x_1, \cdots, x_n$.

All knots and knot types discussed in this paper are tame. We shall use the notation k for a knot in S^3, $\bar{K}$ for its isotopy type, $\bar{K}'$ for the isotopy type of a mirror image of k and K for the type of k, i.e. two knots k_1, k_2 in S^3 have the same *type* if there exists an autohomeomorphism of S^3 such that $\alpha(k_1) = k_2$. This notation is obviously consistent with subscripts and we use it with the understanding that mentioning any of k_i, $\bar{K}_i$, $\bar{K}'_i$ or K_i makes clear what the other symbols mean. The group of a knot, $\pi_1(S^3 - k; *)$, depends only on K and we denote it πK. If T is the boundary of a nice tubular neighbourhood of k and $*_T$ is a point on T the group $\pi_1(T; *_T)$ determines a conjugacy class of subgroups of πK, and πK with this conjugacy class specified will be called the *marked group* of $\bar{K}$. The simplest way to mark πK is to name a longitude commuting with a chosen over-generator of πK, in practice always with the first named over generator.

Consider an over-presentation

$$(1) \qquad \pi K = |x_1, \cdots, x_n : r_1, \cdots, r_{n-1}|$$

of the knot group πK. Each relation $r_i = 1$ can be written as $x_* = W^{-1}x_*W$ where $W \in \pi K$. We can use such a relation to eliminate the generator x_* whenever W can be written without using x_*. By this means the original presentation (1) can be changed to a new presentation of the same form as (1), but with a smaller number of

generators n. Of course, the remaining generators are still over-generators of πK so they are still conjugate. We shall call such a presentation a *normal presentation* and we shall use only normal presentations in this paper. In Section 4 we will describe a modified version of a normal presentation which is more convenient in practice.

Among all the normal presentations of πK, there must be some written on the least number of generators. If this least number is ν, then ν depends only on K and we denote it by $\nu K = \nu$. It is not hard to see that νK is the minimal bridge number [14] of K, but we make no use of its known properties. We do not know how to calculate νK; however, it is usually possible to find its true value by inspection even when there is no proof available. νK must not be confused with the minimal number of generators for πK. For example, when K is a torus knot, this minimal number of generators is 2, but νK can be arbitrarily large [14, Satz 10].

The only knot K with $\nu K = 1$ is the trivial knot $K = 0$. The class of knots K with $\nu K = 2$ has been completely classified in [15] and differently in [2]. These knots are also known as *rational knots* (Viergeflechte) and they are all of alternating type. There is probably no reasonable scheme to classify the knots with $\nu K = 3$. A wild knot K would have $\nu K = \infty$.

Let πK be a knot group, G a group, and consider the various homomorphisms $\theta_1, \theta_2, \cdots$ of πK on G. We shall say that θ_1 *is equivalent to* θ_2, in symbols $\theta_1 \equiv \theta_2$, when there exists an automorphism ω of G such that $\theta_2 = \theta_1 \omega$. We shall also say that θ_1 is *weakly equivalent to* θ_2 if there is an automorphism α of πK such that $\theta_2 \equiv \alpha \theta_1$. Because the determination of the automorphism group of a knot group is an unsolved and presumably very knotty problem, we are forced to use equivalence much more than weak equivalence.

Let G be a group of permutations of the numbers $1, \cdots, n$. We will write the elements g of G as permutations in two different ways. In the first way, we write $[a_1, \cdots, a_n]$ for the element $g \in G$ such that $i \cdot g = a_i$, for $i = 1, \cdots, n$. In the second way, we write g as a product of disjoint cycles.

$$(2) \qquad g = (b_{11}, b_{12}, \cdots, b_{1\ r_1})(b_{21}, b_{22}, \cdots, b_{2\ r_2}) \cdots (b_{s1}, b_{s2}, \cdots, b_{s\ r_s})$$

where $r_1 + r_2 + \cdots + r_s = n$, $b_{ij}g = b_{i\ j+1}$ (second index (mod r_i)), and each of $1, \cdots, n$ occurs in exactly one cycle. We shall omit cycles of length 1 from (2), and when $s = 1$ we shall also omit the parentheses. In fact, we shall omit the commas from both representations wherever possible, e.g. $[23451] = 12345$. We shall write E for the identity $[123 \cdots n]$.

2. Let πK be a knot group presented by a normal presentation (1) on $n = 2$ or 3 generators. In this section, we shall reduce the problem of classifying the equivalence classes of homomorphisms of πK on A_5 to a reasonable number of experiments, viz. 3 when $n = 2$, and 93 when $n = 3$. We begin with some assertions about A_5 which are best proven by direct calculation.

The group A_5 has 60 elements, viz. 1 of order 1, 15 of order 2, 20 of order 3 and 24 of order 5. All the elements of a given order except 5 are conjugate and the elements of order 5 lie in two conjugacy classes. If $C \in A_5$ has order 5, then α is conjugate to C, C^{-1} but not to C^2, C^{-2}. The automorphism group of A_5 is the symmetric group S_5 acting by conjugation, viz. let $a \in S_5$, $g \in A_5$, then $g\alpha = g^\alpha$. All the elements of A_5 of a given order are equivalent under the action of S_5. The maximal

subgroups of A_5 are the stabilizers of a symbol 1, $\cdots$, 5 (these subgroups are isomorphic to A_4), and dihedral groups of order 10. If two elements of A_5 commute, they either lie in a cyclic subgroup or they have order 2 and lie in the stabilizer of a symbol.

In this section, θ always means a homomorphism of πK on A_5. Since the over-generators of πK in any normal presentation (1) are all conjugate, the elements $x_i\theta \in A_5$ all have the same order, say r. If y_1 is a generator in a second normal presentation of πK, then y_1 is conjugate to x_1 or x_1^{-1}, therefore $y_1\theta$ has order r. This means that the number r depends only on θ and not on the presentation, and this allows us to define the *order of θ* as r. Equivalent homomorphisms obviously have the same order, so that the problem of classifying the various homomorphisms is split into three cases. From now on, we assume that the number of generators n in the normal presentation (1) is 2 or 3.

If θ has order 5, then $x_1\theta$ is a 5-cycle in A_5. For some $\alpha \in S_5$, $\alpha^{-1}(x_1\theta)\alpha = 12345 = C$ so that θ is equivalent to a homomorphism θ_1 such that $x_1\theta_1 = C$. We may assume $\theta = \theta_1$. Next, $x_2\theta$ is one of the 12 elements of A_5 conjugate to C. These are C, C^{-1}, and 10 other elements which do not commute with C. These 10 elements lie in two sets, each of 5 elements, on which conjugation by C is a cyclic permutation. If one of these elements is α, then $\{C^{-j}\alpha C^j \mid j = 0, \cdots , 4\}$ is one set and $\{C^{-j}\alpha^{-1}C^j \mid j = 0, \cdots , 4\}$ is the other. Thus, $x_2\theta = C$, C^{-1}, or the automorphism of A_5, defined by conjugation by a suitable power of C, carries $x_2\theta$ to α or α^{-1}. Hence, we may assume $x_2\theta = C$, C^{-1}, α or α^{-1}. No automorphism of A_5 that leaves C fixed carries α to α^{-1}, so that two homomorphisms θ_1, θ_2, thus normalized with $x_2\theta_1 \neq x_2\theta_2$, are inequivalent. If $n = 2$, then $\langle C, x_2\theta \rangle = A_5$ so that $x_2\theta = \alpha$ or α^{-1}. We shall fix the element α to be $(13254) = [35214]$.

Suppose $n = 3$ and $x_2\theta = C^{\pm 1}$. We can apply the above argument to $x_3\theta$ and get $x_3\theta = \alpha$ or α^{-1}. However, if $x_2\theta = \alpha^{\pm 1}$ then $\langle C, x_2\theta \rangle = A_5$ so that the only auto-morphism which leaves $x_1\theta$ and $x_2\theta$ fixed is the identity. This means that $x_3\theta$ can be any of the 12 elements conjugate to C and that two homomorphisms normalized so $x_1\theta_1 = x_1\theta_2 = C$, $x_2\theta_1 = x_2\theta_2 = \alpha^{\pm 1}$, but $x_3\theta_1 \neq x_3\theta_2$, are inequivalent. Hence, when $n = 3$ the original homomorphism θ is equivalent to a homomorphism defined by one of $2\cdot 2 + 2\cdot 12 = 28$ choices for the images of x_1, x_2, x_3.

Now let θ be a homomorphism of order 3. We may assume $x_1\theta = 123$. We have two cases, either 123 and $x_2\theta$ generate A_5, or they do not. If they do, then $x_2\theta$ cannot leave 4 or 5 fixed and therefore $x_2\theta = a45$ or $a54$ where $a = 1, 2$ or 3. When $x_2\theta = a54$, the automorphism defined by $(45) \in S_5$ leaves 123 fixed and transforms $a54$ to $a45$, so that θ may be replaced by an equivalent homomorphism (still called) θ where $x_2\theta = a45$. If α is a suitable power of 123, then $a45^\alpha = 145$ so that we may finally assume $x_2\theta = 145$. Note that $\langle 123, 145 \rangle = A_5$ and that this argument shows that if β is a 3-cycle moving both 4 and 5 then $\langle 123, \beta \rangle = A_5$.

When $n = 2$ we must have $\langle x_1\theta, x_2\theta \rangle = A_5$ so that the normalized homomorphism can only be $x_1\theta = 123$, $x_2\theta = 145$. However, when $n = 3$ and θ_1, θ_2 are two homo-morphisms thus normalized, then $\theta_1 \equiv \theta_2 \Leftrightarrow x_3\theta_1 = x_3\theta_2$. Therefore, our original θ is equivalent to a homomorphism defined by one of 20 choices for the images of x_1, x_2, x_3 when $\langle x_1\theta, x_2\theta \rangle = A_5$.

Next, suppose that θ is a homomorphism of order 3 such that $x_1\theta = 123$ and $\langle 123 \rangle < \langle x_1\theta, x_2\theta \rangle < A_5$. Then $x_2\theta$ must move at least one of 4, 5, but not both.

If $x_2\theta$ moves 5, the automorphism defined by (45) $\in S_5$ transforms θ to an equivalent homomorphism such that $x_2\theta$ moves 4, therefore, we can assume $x_2\theta$ moves 4. Now $x_2\theta = ab4$ where a and b are 1, 2, 3. We wish to normalize θ so that $a = 1$, in which event $x_2\theta = 124$ or 134. If $a \neq 1$ already, the appropriate automorphism is defined by $1ac \in S_5$ where $c = 2$ or 3 as $a = 3$ or 2. It is easy to check that if $x_1\theta_1 = x_2\theta_2 = 123$, $x_2\theta_1 = 124$, $x_2\theta_2 = 134$, then $\theta_1 \neq \theta_2$. Furthermore, $\langle 123, 124 \rangle = \langle 123, 134 \rangle =$ the subgroup of A_5 leaving 5 fixed. This subgroup is maximal so that if ζ is a 3-cycle moving 5 then $\langle x_1\theta, x_2\theta, \zeta \rangle = A_5$. When $n = 3$, $x_3\theta$ must be one of these 12 3-cycles $\zeta = ab5$. There is no freedom in the choice of θ in its equivalence class left to move $x_3\theta$, so that two homomorphisms θ_1, θ_2 normalized so that $x_1\theta_1 = x_2\theta_2 = 123$, $x_2\theta_1 = x_2\theta_2 = 124$ or 134, but $x_3\theta_1 \neq x_3\theta_2$, are inequivalent. Therefore, our original θ is equivalent to a homomorphism defined by one of $2 \cdot 12 = 24$ choices for the images of x_1, x_2, x_3 when $n = 3$, θ has order 3, and $\langle x_1\theta \rangle < \langle x_1\theta, x_2\theta \rangle < A_5$.

Finally for order 3 assume that $x_2\theta = 123$ or 132. Then $\langle 123, x_3\theta \rangle = A_5$ and the normalization argument above shows that we may replace θ by a new θ such that $x_3\theta = 145$. Our original θ is equivalent to a homomorphism defined by one of 2 choices for the images of x_1, x_2, x_3.

Last and least we consider homomorphisms of order 2. It is well known (and easy to check for A_5) that a nonabelian group generated by two elements of order 2 is dihedral, hence such homomorphisms do not exist for $n = 2$. We omit the case by case argument for $n = 3$ and simply tabulate the results in Table 1 at the end which summarizes the discussion of this section.

Now that we have a normal form for each equivalence class of homomorphisms we can test for the existence of each equivalence class of homomorphisms by testing the assignments $x_1\theta, \cdots, x_r\theta$ of its normal form to see if θ actually defines a homomorphism, as remarked in the introduction.

3. We now generalize in a naive manner some of the results of the last section to a classification of homomorphisms of knot groups on the linear projective groups $L_p \equiv LF(2, p) \equiv PSL(2, p)$ where p is a rational prime. For $L_5 = A_5$, we found that the easiest case to handle was the homomorphisms of order 5. Accordingly, we shall fix our attention on the homomorphisms of order p on L_p, and for brevity, we shall call such a homomorphism a "rep". We refer to Chapter XIV of Burnside [1] for the relevant background. The group L_2 is somewhat exceptional so we shall assume p is an odd prime in the classification arguments that follow.

The group L_p has order $p \cdot (p^2 - 1)/2$ and is simple except when $p = 3$ where $L_3 \approx A_4$. It contains $p^2 - 1$ elements of order p and these lie in two conjugacy classes. If $\alpha \in L_p$ has order p, then α is conjugate to α^n iff n is a square (mod p). If β also has order p, either $\beta = \alpha^n$ for some n or $\langle \alpha, \beta \rangle = L_p$. All elements of order p are either powers of α or lie in one of $p - 1$ orbits of p elements under conjugation by α. The only automorphisms of L_p that leave α fixed are conjugations by powers of α. For any pair β_1, β_2 of elements of order p there is an automorphism ζ of L_p such that $\beta_2 = \beta_1\zeta$.

Let (1) be a normal presentation of the knot group πK and assume the number of generators n is 2 or 3. To begin the classifications of the reps of πK on L_p, we fix $\alpha \in L_p$ of order p and alter each rep by an automorphism of L_p so that the image of x_1 is α. Next, consider the $(p - 1)/2$ orbits of elements of order p which are conjugate

to α. If we choose an element α_j in the jth of these for $j = 1, \cdots, (p - 1)/2$, then every element β of order p which is not a power of α has the form $\beta = \alpha^{-s}\alpha_j\alpha^s$ for some j and some s. This means that every rep θ can be altered so that $x_1\theta = \alpha$, $x_2\theta = \alpha^s$ for some s or $x_2\theta = \alpha_j$ for some j. When $n = 2$ the equivalence class of θ is completely determined by j.

When $n = 3$, we have two cases. If $x_2\theta = \dot{\alpha}_j$ for some j, then $\langle x_1\theta, x_2\theta \rangle = L_p$ so that the equivalence class of θ is completely determined by j and $x_3\theta$. However, if $x_2\theta = \alpha^s$ for some s, $(s/p) = 1$, then we can alter θ by an automorphism of L_p so that $x_3\theta = \alpha_j$ for some j. In this event, the equivalence class of θ is determined by s and j.

This shows that when $n = 2$ the equivalence classes of reps can be found by $(p - 1)/2$ experiments and when $n = 3$ by

$$\frac{p - 1}{2} \cdot \frac{p^2 - 1}{2} + \left(\frac{p - 1}{2}\right)^2 = (p + 2)\left(\frac{p - 1}{2}\right)^2$$

experiments. The method generalizes immediately to $n > 3$ generators and the general formula for the number of experiments is

$$\left(\frac{p - 1}{2}\right)^{n-1} \frac{(p + 1)^{n-1} - 1}{p}.$$

We can determine the existence of reps on the solvable groups L_2 and L_3 from the Alexander polynomial $\Delta_K(x)$ of K. Since L_2 is dihedral of order 6, the argument of Section 10 of [6] shows that πK has a rep on L_2 iff 3 divides $\Delta_K(-1)$. By a variation of this argument, we can show πK has a rep on L_3 iff 2 divides $\Delta_K(\omega) \Delta_K(\bar{\omega})$ where ω is a nonreal cube root of 1. (This strange number is the product of the "Torsionszahlen dritter Stufe" in Reidemeister [13] which are tabulated there on p. 25 for the classical knots. We note that his table is wrong for the knots 9_1 and 9_6, the entries in both cases under $h = 3$ should read 2, 2. The rest of his table is correct.)

As a first hint that there is a better way to find reps, we will show that some knots have a rep on L_p for every prime p. Let M be the group $SL(2,\mathbf{Z})$, then every L_p is a homomorphic image of M. Furthermore, the image of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ in L_p is an element of order p. This means that it is sufficient to find a homomorphism θ of πK on M such that $x_1\theta = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. The elements $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ are conjugate and generate M, in fact $BAB = C = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, $AB = \left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right) = D$, and we know from Section 1.4, Exercise 19, of [11], that $M = \langle C, D \rangle$. Using the presentations in Table 2, we find the following homomorphisms θ of πK on M such that $x_1\theta = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

K	$x_2\theta$	$x_3\theta$
$3_1, 9_1, 9_6, 9_{23}$ $(\nu = 2)$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	
$8_5, 8_{18}, 9_{16}$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$8_{19}, 8_{21}, 9_{24}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

(*continued*)

K	$x_2\theta$	$x_3\theta$
8_{20}	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$
$8_{18}, 9_{28}$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$
$8_{10}, 8_{15}, 8_{18}, 9_{40}$	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$
8_{19}	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

The above list is complete up to equivalence for the classical knots. Because a necessary condition for such a homomorphism θ on M is the existence of a rep on L_p for every p, the nonexistence of other θ follows from the above mentioned table in Reidemeister or from Table 3 in the microfiche section in this issue.

4. If (1) is a normal presentation of πK on a small number n of generators, the relators are frequently very long words in $x_1, \cdots, x_n$. For example, for some rational knots with 9 crossings, the relator is a word in x_1, x_2 of length > 100. This difficulty can be avoided by introducing new generators $x_{n+1}, \cdots, x_m$, and new relations expressing x_{n+i} as a word in $x_1, \cdots, x_{n+i-1}$, for $i = n + 1, \cdots, m$. The resulting presentation of πK now has the form:

generators $\quad x_1, \cdots, x_m$,

(3) relations $\quad x_{n+i} = $ word in $x_1, \cdots, x_{n+i-1}, i = 1, \cdots, m - n$,

relators $\quad r_1(x_1, \cdots, x_m), \cdots, r_{n-1}(x_1, \cdots, x_m)$.

This is still a normal presentation on n generators $x_1, \cdots, x_n$ and much more convenient in practice. Note that the subordinate generators $x_{n+1}, \cdots, x_m$ are arbitrary elements of πK and not necessarily over generators of πK.

If one is given a model knot k for K, a presentation (3) is defined implicitly by orienting k and selecting n arcs of the picture to correspond to $x_1, \cdots, x_n$. Next, the other arcs are numbered $n + 1, \cdots, m$ in such a way that at one end arc(j) is crossed by an arc(i) with $i < j$ and on the other side of arc(i) the continuation of arc(i) is arc(s) with $s < j, j = n + 1, \cdots, m$. (This rule is stated for the Wirtinger presentation, but for an over-presentation for a nonalternating knot the rule is only slightly more complex. See the example below.)

Presentation $\pi K = |x_1, x_2, x_3 : r_1, r_2|$

$$x_4 = x_3^{-1} x_2 x_3,$$

$$x_5 = x_3^{-1} x_2^{-1} x_1 x_2 x_3,$$

$$r_1 = x_5 x_1 x_2^{-1} x_1^{-1},$$

$$r_2 = x_4 x_1 x_2 x_3^{-1} x_2^{-1} x_1^{-1},$$

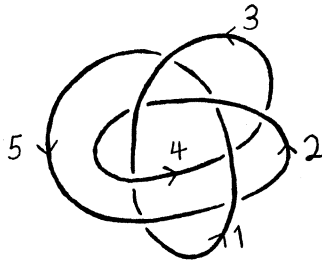longitude $\quad x_2 x_3 x_1 x_3 x_1 x_2 x_4 x_5 x_1^{-8}$.

FIGURE 1

Table 2 in the microfiche section of this issue is a list of normal presentations of the marked groups of the classical knots. These presentations have been checked in numerous ways and are almost certainly correct. Table 3 of homomorphisms of these knot groups on A_5 is based on Table 2.

5. Associated with a homomorphism θ of a knot group πK on a finite group G are certain topological spaces whose homotopy types are invariants of the knot type K. In particular, suppose G is a transitive group of permutations of the symbols $1, \cdots, s$, and let H be the subgroup of πK whose image in G keeps the symbol 1 fixed. Let $\mathfrak{U}$ be the covering space of $S^3 - k$ corresponding to the subgroup H, then $\mathfrak{U}$ is a noncompact s-sheeted covering manifold of $S^3 - k$. Furthermore, as explained in Section 8 of [6], $\mathfrak{U}$ has a certain natural closure $\mathfrak{M}$ which is a compact manifold known as the *s-sheeted cover of S^3 branched over k* (belonging to H). The spaces $\mathfrak{U}$, $\mathfrak{M}$ are the topological spaces in question, they depend (up to homeomorphism) only on πK, the permutation representation of the group G, and the weak equivalence class of θ. In particular, their integral homology groups $H_1\mathfrak{U} \equiv H_1(\mathfrak{U}; Z)$, $H_1\mathfrak{M} \equiv H_1(\mathfrak{M}; Z)$ are invariants of K. A general algorithm to compute integral presentation matrices for these homology groups is informally derived in Section 8 of [6] as a consequence of a presentation for the fundamental groups $\pi_1\mathfrak{U}$, $\pi_1\mathfrak{M}$.

We have translated this algorithm into a Fortran subroutine, Homology, which goes into action given a presentation (2) and the images of the generators $\{x_i\theta, i = 1, \cdots, n\}$ expressed as permutations. It returns the groups $H_1\mathfrak{U}$, $H_1\mathfrak{M}$ as Betti numbers and torsion coefficients. The subroutine has the same generality as the original algorithm (except for practical limitations) and so can be used to study more general groups than knot groups. A more complete description and a listing of Homology have been deposited in the UMT file; cf. the review in the review section of this issue.

To apply Homology to a homomorphism θ of πK on L_p, p prime, we need a permutation representation for L_p. The different permutation representations of L_p will give rise to different spaces $\mathfrak{U}$, $\mathfrak{M}$, but the smaller the number of sheets s, the smaller the presentation matrices for $H_1\mathfrak{U}$, $H_1\mathfrak{M}$. It is known ([1] or [9]) that L_p always has a permutation representation of degree $p + 1$ but none of lower degree, except for $p = 5, 7, 11$, where there are representations of degree p. So for all $p \neq 5$, $7, 11$, we restrict our attention to the representation of degree $p + 1$ but for $p = 5, 7$, 11 we shall use both representations. When πK, p and θ are understood, we shall write $\mathfrak{U}_{p+1}$, $\mathfrak{M}_{p+1}$, $\mathfrak{U}_p$, $\mathfrak{M}_p$ for the corresponding covering spaces.

For the exceptional primes 5, 7, 11, we took particular permutation groups of degree p for the groups L_p. For $p = 5$, we used A_5 as discussed in Section 2. For $p = 7$, we took the elements α, α_1, α_2, α_3 of Section 3 to be

$$\alpha = (1234567), \qquad \alpha_1 = (1675243), \qquad \alpha_2 = \alpha_1^2, \qquad \alpha_3 = \alpha_1^4.$$

This representation of L_7 is taken from p. 208, line 5 of [1]. We took the elements α, $\{\alpha_j, j = 1, \cdots, 5\}$ for L_{11} to be

$$\alpha = (123456789ab), \qquad \beta = (1a)(37)(56)(89)(2)(b), \qquad \alpha_j = \beta\alpha^{j'}\beta, \qquad j = 1, \cdots, 5.$$

This representation came from 7.8 of [3]. In addition, for $p = 5, 7$, we wrote out specific isomorphisms between the above permutation groups and permutation groups of degrees 6, 8 resp. We may someday do the same for $p = 11$.

For the nonexceptional primes p, we find it convenient to deal with the groups L_p directly as matrix groups and derive our permutations from the matrices. Our work in this direction is not complete except for the primes $p \leq 19$.

Given the permutation group G of degree s and the homomorphism θ of πK on G, suppose that $x_1\theta$ has c disjoint cycles. Then, as Professor Fox shows, the number of generators for $\pi_1\mathfrak{M}$ (and consequently for $H_1\mathfrak{M}$) is bounded above by $ns - nc - (s - 1)$; cf. [7]. When this bound is 0, $\mathfrak{M}$ is a homotopy sphere. When $\nu K = 2$ this is true for $\mathfrak{M}_3$ when θ is a rep on L_2 [7] and for $\mathfrak{M}_5$ when θ is a homomorphism of order 3 on A_5. Consequently, we do not compute $H_1\mathfrak{U}$, $H_1\mathfrak{M}$ in these cases. When the upper bound is 1, $\pi_1\mathfrak{M} = H_1\mathfrak{M}$ is cyclic and this happens for $G = L_2$, $\nu K = 3$, and for $G = L_3$, $\nu K = 2$. (The number c is always 2 for the permutation representation of L_p of degree $p + 1$.)

The space $\mathfrak{U}$ can be considered as the complementary space of the link $l = \mathfrak{M} - \mathfrak{U}$ in $\mathfrak{M}$. The link type of l in $\mathfrak{M}$ is an invariant of the knot type K and the group $H_1\mathfrak{U}$ is only the crudest of a set of link invariants of l analogous to the Alexander polynomials and the Hosokawa polynomial [8]. We have not pursued this matter beyond the calculation (by hand) of one specific example. The rational knot 5_1 has a homomorphism of degree 3 on A_5 (Table 3), consequently $\mathfrak{M}_5$ is a homotopy sphere and l is a link in this sphere. We found that the Alexander matrix of l (in one variable x) reduces to

$$\mathfrak{A}(x) = \begin{Vmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta\gamma \end{Vmatrix},$$

where $\alpha = 1 - x$, $\beta = 1 + x^5$ and $\gamma = 1 + x + x^2 + x^3 + x^4$. Consequently, the Hosokawa polynomial is $\nabla(x) = \beta^2\gamma$.

6. The algorithms of the preceding sections have been completely or partially coded as Fortran programs and put to work on up to 250 knots. In this and the next section we consider the results. They fill quite a few tables, some rather lengthy, so we will include just one in microfiche and merely discuss the conclusions to be drawn from the rest.

In this and the next section, we describe a homology group $H_1X \equiv H_1(X; Z)$ by the diagonal entries of a square presentation matrix for H_1X. If convenient, some of these entries may be 1. When the coefficient group is not Z, it will be $Z_n = Z/nZ$ for

some n. For example, $H_1X = 0, 1, 3$ means $H_1X = Z \oplus Z_3$, and if $H_1X = 3$ then $H_1(X; Z_2) = 1$.

We begin with the reps on L_2. After this paper was submitted for publication, the paper "Metacyclic invariants of knots and links" by R. H. Fox appeared in *Canad. J. Math.*, v. 22, 1970, pp. 193–201, and parts of that paper are very relevant to our work. In the first place, Fox points out that the longitudes of a knot group πK lie in the second commutator subgroup of πK, hence if $\theta : \pi K \to G$ where G is a group whose second commutator subgroup is trivial, the longitudes are in the kernel of θ. This applies when $G = L_2$ or L_3 and supersedes a more cumbersome argument of mine. Secondly, Fox includes a table (which agrees with mine) of the groups $H_1\mathfrak{M}_3$ associated with the reps of the groups of the classical knots on L_2. More interesting is the linking number v of the two components of $\mathfrak{M}_3 - \mathfrak{U}_3$ for each rep which Fox includes in his table. Although I do not know how to calculate v, I found on comparison with one of my tables that in certain cases there seems to be a strict relation between v and a number I can calculate. Namely, if θ is a rep of πK on $M = SL(2, Z)$ where $x_1\theta = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $\gamma \in \pi K$ is a longitude commuting with x_1, then $\gamma\theta$ necessarily has the form $\pm\left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right)$ for some $g \in Z$. In fact, $g \equiv 0 \pmod 6$ because the image of γ in L_2 or L_3 is E. Then, in the 15 cases where Fox's homomorphism is the mod 2 reduction of such a homomorphism on M, we have

$$g = \pm 3v.$$

The sign depends on the normalizations and we cannot settle it here. This relation is probably at the heart of the reason why in all known cases v is a rational fraction with an even numerator.

For $A_5 = L_5$, we tabulate the detailed results for the classical knots in Table 3 in microfiche. All permutations in that table are in the bracket [ ] notation. There are altogether 51 reps, 26 homomorphisms of order 3, and 9 of order 2, for a total of 86 homomorphisms. There are 30 knots with no homomorphism of which only $9_{32}, 9_{33}, 9_{34}$ are not rational. Five knots, viz. $9_{16}, 9_{22}, 9_{25}, 9_{30}, 9_{36}$, are proven nonrational by the number or types of their homomorphisms but not by any simpler method that I know. Most of the nonamphicheiral knots are proven to be such by Fox's argument [5].

The cases $p = 7$ and $p = 11$ have been run on the computer for the classical knots. We found 102 reps on L_7 for these knots and that 27 knots have no rep, of which 6 are not rational. The maximal number of reps for any knot is 13 for 9_{40} and these appear to lie in 6 weak equivalence classes. There are 100 reps on L_{11} and 29 knots with no reps, 8 of these are not rational. This time the maximal number of reps per knot is only 5, attained for $K = 9_{41}, 9_{47}$ and 9_{48}.

This is as far as we have gone for the block of classical knots and as far as we have computed the homology. We have gone further to find at least one rep on some L_p for every classical knot. The largest prime needed for the first rep was 23 which was required for 7_5 and 8_{12}. All the nonrational knots, save 9_{33}, have a rep on L_p for $p \leq 11$, but for 9_{33} we need $p = 17$.

Out of the detailed results of the calculation of around 1000 homology groups $H_1\mathfrak{U}_h, H_1\mathfrak{M}_h$, we have found three general conjectures that appear certain, and two more which are very likely. The discussion splits naturally into the cases $h = p + 1$, $h = p$, and we exclude $p = 2$ because 2 is quite exceptional in this context. As

notation, write $B(\mathfrak{U}_h)$, $B(\mathfrak{M}_h)$ for the Betti numbers of $\mathfrak{U}_h$, $\mathfrak{M}_h$, respectively. We begin with $h = p + 1$.

Conjecture A. If θ is a rep and $p \geqq 3$,

$$B(\mathfrak{M}_{p+1}) + 2 \geqq B(\mathfrak{U}_{p+1}) \geqq B(\mathfrak{M}_{p+1}) + 1,$$

and when $B(\mathfrak{M}_{p+1}) = 0$, $B(\mathfrak{U}_{p+1}) = 2$. Furthermore $H_1\mathfrak{M}_{p+1}$ has a torsion generator of even order.

The usual value for $B(\mathfrak{M}_{p+1})$ is zero. We know that $H_1\mathfrak{M}_{p+1}$ is a factor group of $H_1\mathfrak{U}_{p+1}$, but apart from that and Conjecture A there seems to be no further universal assertion to be made about the comparison of $H_1\mathfrak{U}_{p+1}$ with $H_1\mathfrak{M}_{p+1}$. However, when θ happens to be the reduction of a homomorphism on M the situation is much clearer. We have:

Conjecture B. For each prime $p \geqq 2$, there are integers b_p and d_p, where d_p is a divisor of $p + 1$, which have the following properties. Let $\theta : \pi K \to L_p$ be the reduction modulo p of a homomorphism $\phi : \pi K \to SL(2, \mathbf{Z})$ such that $x_1\phi = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let the torsion subgroups of the corresponding homology groups $H_1\mathfrak{U}_{p+1}$, $H_1\mathfrak{M}_{p+1}$ have orders u, m, respectively. Then, $B(\mathfrak{M}_{p+1}) \geqq b_p$ and m/u is an integer which divides d_p. If $B(\mathfrak{M}_{p+1}) = b_p$, then $m/u = d_p$.

We have verified this when $p \leqq 19$ for the homomorphisms on M listed in Section 3. The numbers b_p, d_p for $p \leqq 19$ are

p	2	3	5	7	11	13	17	19
b_p	0	0	0	0	2	0	2	2
d_p	1	2	2	4	2	14	6	10

It should be emphasized that Conjecture B only concerns the orders u, m of the torsion subgroups, not the relations between the actual torsion coefficients. The tables show that the passage from $\mathfrak{U}$ to $\mathfrak{M}$ will do the wildest things to these torsion numbers, subject to the known or conjectured restrictions. We have also considered the question of patterns in the groups $H_1\mathfrak{U}_{p+1}$, $H_1\mathfrak{M}_{p+1}$ for a fixed K and varying p (corresponding to a homomorphism on M), but the results are too vague to be commented on here.

There is a conjecture similar to conjecture A for the groups $H_1\mathfrak{U}_6$, $H_1\mathfrak{M}_6$ corresponding to a homomorphism of order 2 or 3 on $L_5 = A_5$. For order 3, the conjecture is the same as Conjecture A, and for order 2 the only change is

$$B(\mathfrak{M}_6) + 1 \leqq B(\mathfrak{U}_6) \leqq B(\mathfrak{M}_6) + 4.$$

In all known cases, $B(\mathfrak{M}_6) = 0$, $B(\mathfrak{U}_6) = 4$, but this is *not* a conjecture.

Now for the groups $H_1\mathfrak{U}_p$, $H_1\mathfrak{M}_p$ when $p = 5, 7$ or 11. We only discuss the cases where θ is a rep because the other cases are very different.

Conjecture C. If $\theta : \pi K \to L_p$ is a rep and $p = 5, 7$ or 11 then $B(\mathfrak{U}_p) = B(\mathfrak{M}_p) + 1$. If $H_1\mathfrak{U}_p = 0, a_1, \cdots, a_n$ with $1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n$ then for some $j \in \{1, \cdots, n\}$, $H_1\mathfrak{M}_p = a_1, \cdots, pa_j, \cdots, a_n$.

In Conjecture C the torsion number a_j may or may not be already divisible by p. The hypothesis $B(\mathfrak{U}_p) = 1$ is essential.

The two minor conjectures are that if $H_1\mathfrak{U}_p = 0, a_1, \cdots, a_n$ as in Conjecture C, then 3 divides a_n when $p = 5$ and 2 divides a_n when $p = 7$. The evidence for these is quite strong, but because there is no corresponding rule for $p = 11$, we hesitate to

be too dogmatic about them. As an example, the knot 7_4 has a unique rep on L_{11}, and $H_1 \mathfrak{U}_{11} = 0$, $H_1 \mathfrak{M}_{11} = 11$.

To conclude this section, consider the three knots of Fig. 2, of interest because they have Alexander polynomial $\Delta(x) \equiv 1$. The knot k_1 is

$k_1 \qquad\qquad k_2 \qquad\qquad k_3$
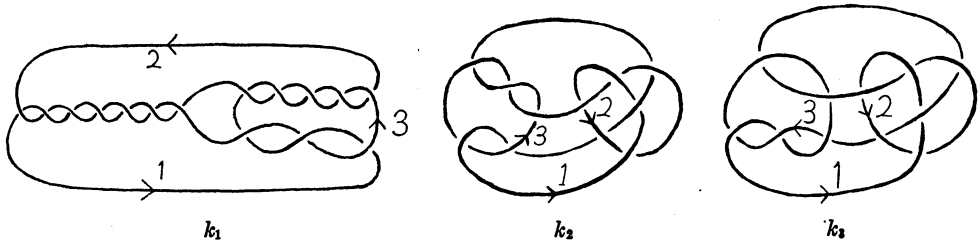
FIGURE 2

Seifert's classical example of such a nontrivial knot. The knot k_2 was constructed by Kinoshita and Terasaka [10] and further studied in [12]. The knot k_3 was discovered by J. Conway [2] in his classification of the nonalternating 11 crossing knots. He has shown that k_2 and k_3 are the only 11-crossing knots with $\Delta(x) \equiv 1$. (In Conway's notation, $k_2 = .-(2, 3).2$ and $k_3 = .-(3, 2).2$.) We first try the homomorphisms on A_5.

$K_1 \quad x_1\theta = (12345), \quad x_2\theta = (15432), \quad x_3\theta = (14523).$

$\qquad H_1\mathfrak{U}_5 = 0, 2, 6. \quad H_1\mathfrak{M}_5 = 2, 30. \quad H_1\mathfrak{U}_6 = 0, 0, 2, 4. \quad H_1\mathfrak{M}_6 = 2, 2, 16.$

$K_2 \quad x_1\theta = (123), \quad x_2\theta = (145), \quad x_3\theta = (123), \quad \text{(longitude)} \; \theta = E.$

$K_3 \quad x_1\theta = (123), \quad x_2\theta = (145), \quad x_3\theta = (142), \quad \text{(longitude)} \; \theta = E.$

For both K_2 and K_3:

$\qquad H_1\mathfrak{U}_5 = 0, 0, 0. \quad H_1\mathfrak{M}_1 = 7. \quad H_1\mathfrak{U}_6 = 0, 0, 12. \quad H_1\mathfrak{M}_6 = 180.$

This shows that all three knots are nontrivial and that K_1 is different from the other two. We have not yet shown $K_2 \neq K_3$. It is possible that the deeper Alexander invariants of the link $\mathfrak{M} - \mathfrak{U}$ in $\mathfrak{M}$ can do that, but I bet not.

We now try reps on L_7 for K_2 and K_3. For convenience, write $C = (1234567)$. Each rep satisfies $x_1\theta = C$ and each knot has two reps.

$K_2 \begin{cases} x_2\theta_1 = (1675243), \quad x_3\theta_1 = (1452736), \quad \text{(longitude)} \; \theta_1 = C. \\ H_1\mathfrak{U}_7 = 0, 4, 28. \quad H_1\mathfrak{M}_7 = 28, 28. \quad H_1\mathfrak{U}_8 = 0, 0, 3. \quad H_1\mathfrak{M}_8 = 1920. \\ x_2\theta_2 = C, \quad\qquad x_3\theta_2 = (1675243), \quad \text{(longitude)} \; \theta_2 = C^2. \\ H_1\mathfrak{U}_7 = 0, 2, 238. \quad H_1\mathfrak{M}_7 = 14, 238. \quad H_1\mathfrak{U}_8 = 0, 0, 6. \quad H_1\mathfrak{M}_8 = 3, 228. \end{cases}$

$K_3 \begin{cases} x_2\theta_1 = (1675243), \qquad x_3\theta_1 = (1723654), \quad \text{(longitude)} \; \theta_1 = C. \\ H_1\mathfrak{U}_7 = 0, 2, 2, 4, 12. \quad H_1\mathfrak{M}_7 = 2, 2, 4, 84. \quad H_1\mathfrak{U}_8 = 0, 0, 4, 24. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad H_1\mathfrak{M}_8 = 4, 8, 24. \\ x_2\theta_2 = C, \qquad\qquad x_3\theta_2 = (1264735), \quad \text{(longitude)} \; \theta_1 = C^2. \\ H_1\mathfrak{U}_7 = 0, 2, 2, 8, 40. \quad H_1\mathfrak{M}_7 = 2, 2, 8, 280. \quad H_1\mathfrak{U}_8 = 0, 0, 2, 36. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad H_1\mathfrak{M}_8 = 2, 6, 324. \end{cases}$

This shows that $K_2 \neq K_3$ and that neither knot is amphicheiral. The results for L_{11} are very similar: two reps per knot but the groups $H_1\mathfrak{U}_{11}$, $H_1\mathfrak{M}_{11}$ prove the knots are different.

We tried to prove that $K_2 \neq K_3$ by counting the reps on L_p for $p = 13, 17, 19, 23, 29$ and 31 to see if there was ever a difference between the corresponding numbers for K_2, K_3. (This experiment took more than $3\frac{1}{4}$ hours on the Southampton computer.) The results suggest the πK_2 and πK_3 have the same number of reps on L_p for every prime p and that these knots cannot be distinguished this way. In [12], Magnus and Peluso proved that πK_2 has reps on L_p for an infinity of p, but because they only netted a small proportion of the true set of reps their results cannot help us. We hope to return to this question in a later paper.

7. One of the best ways to understand the behavior of our knot invariants is to calculate them for a family of knots where the presentations of the knot groups depend on integral parameters. As a final collection of examples, we consider the rather complicated family of Kinoshita-Terasaka (KT) knots. The KT knot $\kappa(p, n)$ is drawn in Fig. 3 in the case
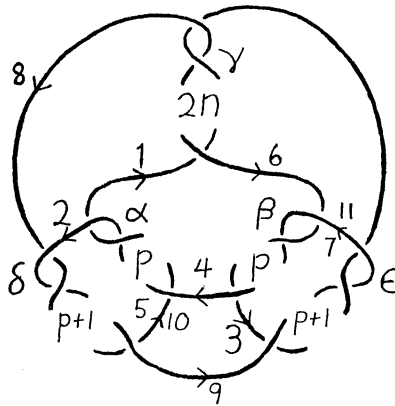
FIGURE 3. $\kappa(p, n)$

where p and n are positive integers. When $n < 0$, the sense of rotation in the integral tangle (winding) γ is reversed. In [10] Kinoshita and Terasaka proved that $\kappa(p, n)$ is a nontrivial knot with Alexander polynomial $\Delta(x) \equiv 1$ if $p \geqq 2$ and $n \neq 0$.* Obviously, $\kappa(p, n) = 0$ if $p = 0$ or 1 or if $n = 0$. The knot K_2 of the last section is $\kappa(2, 1)$.

We may also define $\kappa(p, n)$ for $p < 0$ by reversing the sense of rotation in the tangles α, β, δ, ϵ in Fig. 3. (Beware that δ and ϵ now have $|p| - 1 = |(p + 1)|$ crossings.) If we turn the knot over and reflect in a mirror, we see $\kappa(p, n) = \kappa(p, -n)$. Furthermore, we can turn the knot over and apply the *Edmunds Flip* to show $\kappa(p, n) = \kappa(-p - 1, n)$ for all p, n. This allows us to assume p is even and $n > 0$ from now on.

Let $\pi(p, n)$ be the group of $\kappa(p, n)$. Since we have taken p even, we may write $p = 2p_0$, $p_+ = p_0 + 1$. Then $\pi(p, n)$ has a normal presentation (4) of the form $|x_1, x_2, x_3 : r_1, r_2|$ where

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* Our $\kappa(p, n)$ is written $\kappa(p, 2n)$ in [10].

$$\alpha = x_2 \, x_1 \qquad x_4 = \alpha^{p_0} x_1 \alpha^{-p_0} \qquad x_5 = \alpha^{p_0} x_2 \alpha^{-p_0}$$

$$\beta = x_4^{-1} x_3^{-1} \qquad x_6 = \beta^{p_0} x_4 \beta^{-p_0} \qquad x_7 = \beta^{p_0} x_3 \beta^{-p_0}$$

$$(4) \qquad \gamma = x_6 \, x_1^{-1} \qquad x_8 = \gamma^{n} x_1 \gamma^{-n}$$

$$\delta = x_2 \, x_8 \qquad x_9 = \delta^{p_0} x_2 \delta^{-p_0} \qquad x_{10} = \delta^{p+} x_8 \delta^{-p+}$$

$$\epsilon = x_9^{-1} x_3^{-1} \qquad x_{11} = \epsilon^{p_0} x_9 \epsilon^{-p_0}$$

$$r_1 = x_5 \, x_{10}^{-1} \qquad r_2 = x_7 \, x_{11}^{-1}$$

Suppose for some fixed (p, n) that $\kappa(p, n)$ has a homomorphism θ of order s on A_5. Say that the elements $\alpha\theta$, $\beta\theta$, $\gamma\theta$, $\delta\theta$, $\epsilon\theta$ of A_5 have orders a, b, g, d, e, respectively. Let q be the l.c.m. of a, b, d, e, and let $x_i\theta = \zeta_i$, $i = 1, 2, 3$. From (4) it is easy to see that the assignment $x_i\phi = \zeta_i$, $i = 1, 2, 3$, defines a homomorphism ϕ of $\kappa(p', n')$ of order s on A_5 whenever $p \equiv p'$ (mod $2q$) and $n \equiv n'$ (mod g). Because $\kappa(p, 0) = 0$ we must have $g \geqq 2$, and because $\kappa(0, n) = \kappa(-2, n) = 0$ we must have $2q \geqq 6$ (in fact $2q$ is 30 or 60 in every case). We shall say that the homomorphisms θ and ϕ are in the same *clan* of homomorphisms Θ.

Because the numbers g and q for a clan of homomorphisms must be divisors of 30, we know in advance that infinitely many KT knots have no homomorphisms on A_5. Clearly, the same difficulty will occur whatever finite (nonabelian) group G we may use in place of A_5. Therefore, if we wish to resolve all the different types among the KT knots, we must either use an infinite group G or an infinite set of finite groups $\{G_\nu, \nu = 1, 2, \cdots\}$.

We can show however, that infinitely many of the knots $\kappa(p, n)$ are different by considering the homology groups $H_1\mathfrak{U}_h$, $H_1\mathfrak{M}_h$, $h = 5, 6$, associated with a clan of homomorphisms Θ. We examined a number of these clans and found by looking at 10 to 40 cases that these homology groups can usually be predicted by simple formulas. We must admit we have not tried to prove these formulas—the groups are complicated and the presentation matrices are rather large. However the method of proof (a giant calculation!) is clear and the results stated below should be convincing as they are. (Sceptics might try a simpler case of a clan Θ where the knots are alternating torus knots.) We present our formulas for five cases which illustrate most of what we should expect.

I. $x_1\theta = (123)$, $x_2\theta = (123)$, $x_3\theta = (145)$ when $p \equiv 2$ (mod 30), $n \equiv 1$ (mod 3). Since $\kappa(p, n) = \kappa(p, -n)$, this clan also allows $n \equiv 2$ (mod 3), but using the standard normalization of θ in its equivalence class we then write it $x_1\theta = (123)$, $x_2\theta = (145)$, $x_3\theta = (235)$. When $p = 2$ and $n = 3s + 1$:

$$H_1\mathfrak{U}_5 = 0, 0, 0. \quad H_1\mathfrak{M}_5 = |7 + 24s|. \quad H_1\mathfrak{U}_6 = 0, 0, 12. \quad H_1\mathfrak{M}_6 = |180 + 648s|.$$

II. $x_1\theta = (12345)$, $x_2\theta = (13254)$, $x_3\theta = (15324)$ when $p \equiv 2$ (mod 30), $n \equiv 2$ (mod 5). When $n \equiv 3$ (mod 5), we normalize θ to $x_2\theta = (14523)$, $x_3\theta = (15324)$. (This clan is one of three clans of reps which exist for the same set of (p, n).) When $p = 2$ write $n = 5s + 2$, $\tau = 2$ if $11 \nmid s$, $\tau = 22$ if $11 \mid s$,

$$H_1\mathfrak{U}_5 = 0, 2, 18. \quad H_1\mathfrak{M}_5 = 2, 90. \quad H_1\mathfrak{U}_6 = 0, 0, \tau. \quad H_1\mathfrak{M}_6 = |1100 + 2616s|.$$

III. $x_1\theta = (12)(34)$, $x_2\theta = (12)(35)$, $x_3\theta = (13)(25)$ when $p \equiv 4$ (mod 30), $n \equiv 1$ (mod 3). For $n \equiv 2$ (mod 3) we write $x_2\theta = (13)(24)$, $x_3\theta = (12)(35)$. When

$p = 4$, $n = 3s + 1$ and $\tau = |78 + 240s|$ we get:

$$H_1\mathfrak{U}_5 = 0, 0, 0, \tau. \qquad H_1\mathfrak{M}_5 = \tau. \qquad H_1\mathfrak{U}_6 = 0, 0, 0, 0, 2. \qquad H_1\mathfrak{M}_6 = 4.$$

IV. $x_1\theta = (12)(34)$, $x_2\theta = (13)(24)$, $x_3\theta = (14)(25)$ when $p \equiv 2 \pmod{30}$, $n \equiv 2 \pmod 5$. For $n \equiv 3 \pmod 5$ we write $x_3\theta = (13)(25)$, $x_3\theta = (15)(34)$. When $p = 2$ the groups are:

$$H_1\mathfrak{U}_5 = 0, 0, 0, 3. \qquad H_1\mathfrak{M}_5 = 3. \qquad H_1\mathfrak{U}_6 = 0, 0, 0, 0. \qquad H_1\mathfrak{M}_6 = 2.$$

V. $x_1\theta = (123)$, $x_2\theta = (134)$, $x_3\theta = (135)$ when $p \equiv 4 \pmod{60}$, $n \equiv 1 \pmod 2$. (This clan is one of four such which lie in pairs, the corresponding homology groups for the homomorphisms in a pair being the same.) When $p = 4$ write $n = 2s + 1$, $\rho = 18|s + 1|$, and let $\tau = 9$ if $3|s$, $\tau = 3$ if $3 \nmid s$. Then:

$$H_1\mathfrak{U}_5 = 0, 0, 0, \tau. \qquad H_1\mathfrak{M}_5 = 2, \rho.$$

$H_1\mathfrak{U}_6 = 0, 0, a(s), b(s)$ where $a(s) = 4\tau(s)$, $b(s) = 1$ when s is even, $b(s) = 4$ if $s \equiv 1 \pmod 4$, and $b(s) = 2$ if $s \equiv 3 \pmod 4$. $H_1\mathfrak{M}_6 = c(s), d(s), t(s)$ where $c(s) = 3$ if s is even, $c(s) = 2$, otherwise, and $d(s) = 1$ if s is even, $d(s) = 3 \cdot b(s)$ otherwise. The final summand $t(s)$ varies so wildly that I cannot guess a formula for it.

Table 1. The images $x_1\theta$, $x_2\theta$, $x_3\theta$ of the generators of a normal presentation on three generators for the equivalence classes of homomorphisms on A_5. See Section 1 for the [ ] notation.

I. *Order* 5. $x_1\theta = [23451]$. 28 combinations

$\qquad x_2\theta = [35214]$ or $[43152]$. $x_3\theta = [23451]$, $[51234]$, or one of

$\qquad\qquad [35214]$, $[41532]$, $[54213]$, $[31524]$, $[54132]$,

$\qquad\qquad [43152]$, $[25413]$, $[43521]$, $[24153]$, $[35421]$.

$\qquad x_2\theta = [23451]$ or $[51234]$. $x_3\theta = [35214]$ or $[43152]$.

II. *Order* 3. $x_1\theta = [23145]$. 46 combinations

$\qquad x_2\theta = [42351]$. $x_3\theta = $ any 3-cycle.

$\qquad x_2\theta = [24315]$ or $[32415]$. $x_3\theta = $ any 3-cycle which moves 5.

$\qquad x_3\theta = [23145]$ or $[31245]$. $x_3\theta = [42351]$.

III. *Order* 2. $x_1\theta = [21435]$. 19 combinations

$\qquad x_2\theta = [21543]$. $x_3\theta = $ one of $[35142]$, $[32154]$, $[34125]$, $[42513]$,

$\qquad\qquad\qquad [45312]$, $[52431]$.

$\qquad x_2\theta = [35142]$. $x_3\theta = $ one of $[21543]$, $[21354]$, $[34125]$, $[32154]$,

$\qquad\qquad\qquad [43215]$, $[45312]$, $[53241]$, $[52431]$,

$\qquad\qquad\qquad [14523]$, $[15432]$.

$\qquad x_2\theta = [34125]$. $x_3\theta = $ one of $[21543]$, $[35142]$, $[45312]$.

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PRESENTATIONS OF KNOT GROUPS

KNOT 3, 1   NUM 2   NWORD 5   TORSION 3
   X3   1 1 2 1 1-1
   RELATOR   1 1 2 1 3-1 2-1
   LONGITUDE   3-1 1-1 2-1 1 3

KNOT 4, 1   NUM 2   NWORD 6   TORSION 5
   X3   1 1 2 1 1-1   X4   2-1 3 1 2 1
   RELATOR   4 1 2 1 4-1 1-1
   LONGITUDE   4 1 1-1 2 1 3-1

KNOT 5, 1   NUM 2   NWORD 7   TORSION 5
   X3   2 1 1 1 2-1   X4   3 1 2 1 3-1   X5   4 1 3 1 4-1
   RELATOR   5 1 4 1 5-1 1-1
   LONGITUDE   2-1 4-1 1-2 3-1 5-1 1 5

KNOT 5, 2   NUM 2   NWORD 7   TORSION 7
   X3   1-1 2 1 1 1   X4   3 1 1 1 3-1   X5   4-1 3 1 4 1
   RELATOR   1 1 5 1 2-1 5-1
   LONGITUDE   5 1 1 1 4 1 2 1 3 1 1-5

KNOT 6, 1   NUM 2   NWORD 8   TORSION 9
   X3   1-1 2 1 1 1   X4   3 1 1 1 3-1   X5   4-1 3 1 4 1
   X6   3 1 5 1 2-1
   RELATOR   6 1 1 1 6-1 2-1
   LONGITUDE   6-1 1 1 4 1 2-1 5 1 3 1 1-2

KNOT 6, 2   NUM 2   NWORD 8   TORSION 11
   X3   2-1 1 1 1 2 1   X4   3-1 2 1 3 1   X5   4 1 3 1 4 1
   X6   5-1 1-1 5 1
   RELATOR   1 1 6 1 1-1 4-1
   LONGITUDE   5 1 1-1 3-1 6 1 4-1 2-1 1 6

KNOT 6, 3   NUM 2   NWORD 8   TORSION 13
   X3   2 3 1 1 2-1   X4   3 1 2 1 3-1   X5   4-1 2 1 4 1
   X6   5 1 1 1 5-1
   RELATOR   4 1 1 1 6-1 1-1
   LONGITUDE   5-1 1-1 3 1 4 1 6-1 2 1

KNOT 7, 1   NUM 2   NWORD 9   TORSION 7
   X3   2 1 1 1 2-1   X4   3 1 2 1 3-1   X5   4 1 3 1 4-1
   X6   5 1 4 1 5-1   X7   6 1 5 1 6-1
   RELATOR   2 1 1 1 7-1 1-1
   LONGITUDE   2-1 4-1 6-1 1-1 3-1 5-1 7-1 1 7

KNOT 7, 2   NUM 2   NWORD 9   TORSION 11
   X3   1-1 2 1 1-1   X4   2-1 1 1 3 1   X5   4 1 3 1 4-1
   X6   5-1 4 1 5 1   X7   6 1 5 1 6-1
   RELATOR   7 1 1 1 7-1 2-1
   LONGITUDE   3 1 5 1 2 1 6 1 4 1 1 1 7 1 1-7

KNOT 7, 3   NUM 2   NWORD 9   TORSION 13
   X3   1-1 2 1 1 1   X4   3 1 1 1 3-1   X5   4-1 3 1 4 1
   X6   2 1 4 1 2-1   X7   6 1 2 1 6-1
   RELATOR   5 1 7 1 4-1 7-1
   LONGITUDE   3-1 2-1 7-1 4-1 1-1 6-1 5-1 1 7

KNOT 7, 4   KNOT 2   NWIRD 9   TORSION 15
X3   8 1 1 9-1   X4   1 1 9 1 1-1   X5   4-1 1 1 4 1
X6   3-4 5 1 2 1   X7   5-1 9 1 5 1
RELATOR   6 1 7 1 4-1 7-1
LONGITUDE   8-1 4-1 5-1 1-1 7-1 3>1 4-1 1 1 7

KNOT 7, 4   NWR 2   NWIRD 9   TORSION 17
X3   1 1 2 1 1-1   X4   9-1 1 1 9 1   X5   4-1 9 1 4 1
X6   5 1 1 1 5-1   X7   4-1 5 1 6 1
RELATOR   7-1 4 1 7 1 3-1
LONGITUDE   5-1 3-1 6-1 4-1 1-1 7-1 9-1 1 7

KNOT 7, 6   NWR 2   NWIRD 9   TORSION 19
X3   8 1 1 9-1   X4   3 1 1 1 3-1   X5   4-1 1 1 4 1
X6   1 1 9 1 1-1   X7   5-1 9 1 5 1
RELATOR   6 1 5 1 4-1 7-1
LONGITUDE   5-1 7 1 1 1 5 1 6 1 4-1 9 1 1-1

KNOT 7, 7   NWR 2   NWIRD 9   TORSION 21
X3   8 1 9 1 9-1   X4   1-1 9 1 3 1   X5   4 1 1 1 4-1
X6   1-1 3 1 1 1   X7   5 1 4 1 4-1
RELATOR   7 1 9-1 7-1 5 1
LONGITUDE   4-1 7 1 3 1 4-1 5 1 1-1 9 1 1-1

KNOT 8, 1   NWR 9   NWIRD 18   TORSION 17
X3   9-1 1 1 9 1   X4   3 1 9 1 3-1   X5   4-1 1 1 4 1
X6   5 1 4 1 5-1   X7   6-1 5 1 9 1   X8   1-1 9 1 1 1
RELATOR   7 1 8 1 7-1 1-1
LONGITUDE   9 1 4 1 6 1 8-1 5 1 3 1 1 1 7-1 1-4

KNOT 8, 2   NWR 9   NWIRD 18   TORSION 17
X3   9-1 1 1 9 1   X4   1-1 9 1 3 1   X5   4-1 1 1 4 1
X6   5-1 4 1 5 1   X7   6-1 5 1 6 1   X8   7 1 9 1 7-1
RELATOR   1 1 8 1 6-1 8-1
LONGITUDE   9 1 4 1 6 1 1-1 7 1 9 1 5 1 8-1 1-4

KNOT 8, 3   NWR 9   NWIRD 18   TORSION 17
X3   1-1 9 1 1 1   X4   3 1 1 1 3-1   X5   4-1 1 1 4 1
X6   5 1 4 1 5-1   X7   6 1 1 1 6-1   X8   7-1 6 1 7 1
RELATOR   9 1 5 1 9-1 8-1
LONGITUDE   5-1 5-1 7 1 9 1 4-1 1-1 8 1 6 1

KNOT 8, 4   NWR 9   NWIRD 18   TORSION 19
X3   8-1 1 1 9 1   X4   3 1 9 1 3-1   X5   4-1 1 1 4 1
X6   5 1 4 1 5-1   X7   6 1 9 1 6-1   X8   7 1 5 1 7-1
RELATOR   8 1 7 1 8-1 1-1
LONGITUDE   9 1 4 1 7-1 1-1 5 1 3 1 6-1 8-1

KNOT 8, 5   NWR 3   NWIRD 11   TORSION 21
X4   9-1 1 1 9 1   X5   4-1 9 1 4 1   X6   5-1 4 1 5 1
X7   1 1 3 1 1-1   X8   6-1 7 1 6 1
RELATOR   8 1 3 1 9-1 3-1   RELATOR   7 1 5 1 7-1 1-1
LONGITUDE   9 1 5 1 8 1 1-1 6 1 3 1 4 1 7-1 1-4

KNOT 8, 6   NWR 2   NWIRD 10   TORSION 23
X3   5-1 9 1 1 1   X4   9 1 1 1 9-1   X5   3-1 1 1 3 1
X6   5 1 9 1 5-1   X7   6-1 5 1 6 1   X8   7 1 6 1 7-1
RELATOR   4 1 7 1 4-1 8-1
LONGITUDE   9-1 8 1 1-1 5-1 7-1 4 1 6-1 3-1 1 4

KNOT 8, 7   NGB 8   NWORD 10   TORSION 23
X3   8-1 1 1 8 1   X4   3-1 1 1 3 1   X5   4-1 3 1 1 6 1
X6   5-4 4 1 9 1   X7   4 1 6 1 1-1   X8   7 1 9 1 7-1
RELATOR   1 1 8 1 7-1 8-1
LONGITUDE   9 1 4 1 7-1 6 1 3 1 9 7 1-1 8-1 1-2

KNOT 8, 8   NTR 9   NWORD 10   TORSION 25
X3   1-1 9 1 1 1   X4   3-1 1 1 3 1   X5   4 1 1 1 4-1
X6   9-1 4 1 9 1   X7   4 1 5 1 6-1   X8   7 1 3 1 7-1
RELATOR   9 1 8 1 6-1 8-1
LONGITUDE   3 1 9 1 8-1 1 1 7-1 9-1 6 1 9 1 1-2

KNOT 8, 9   NTR 9   NWORD 10   TORSION 25
X3   9-1 1 1 9 1   X4   3-1 9 1 3 1   X5   9-1 3 1 4 1
X6   1 1 9 1 1-1   X7   5 1 9 1 5-1   X8   6-1 1 1 7 1
RELATOR   9 1 6 1 3-1 6-1
LONGITUDE   9 1 4 1 1-1 8-1 9 1 3 1 6-1 7-1

KNOT 8, 10   NTR 3   NWORD 11   TORSION 27
X4   1 1 9 1 1-1   X5   4 1 1 1 4-1   X6   5 1 4 1 9-1
X7   9-1 3 1 9 1   X8   6 1 7 1 6-1
RELATOR   1 1 9 1 6-1 8-1   RELATOR   3 1 9 1 3-1 8-1
LONGITUDE   9 1 9 1 1 1 3-1 6 1 9-1 7-1 4 1 1-9

KNOT 8, 11   NTR 9   NWORD 10   TORSION 27
X3   1-1 9 1 1 1   X4   3-1 1 1 3-1   X5   4-1 3 1 4 1
X6   9-1 1 1 9 1   X7   6-1 9 1 6 1   X8   7 1 4 1 7-1
RELATOR   9 1 2 1 7-1 9-1
LONGITUDE   3-1 7-9 9 1 6-1 4-1 1-1 8 1 9-1 1 4

KNOT 8, 12   NTR 9   NWORD 10   TORSION 29
X3   9 1 4 1 9-1   X4   3-1 9 1 3 1   X5   4-1 9-1 4 1
X6   9 1 4 1 9-1   X7   9 1 9 1 1-1   X8   7-9 1 1 7 1
RELATOR   9 1 6 1 1-1 3-1
LONGITUDE   7 1 4-1 1 1 4-1 3 1 9-1 9-1 9 1

KNOT 8, 13   NTR 9   NWORD 10   TORSION 29
X3   1-1 9 1 1 1   X4   9-9 1 1 9 1   X5   9 1 9 1 4-1
X6   3-1 4 9 3 1   X7   5-1 9 1 5 1   X8   7 1 6 1 7-1
RELATOR   1 1 9 1 7-1 8-1
LONGITUDE   9 1 3 1 7-1 6-1 9 1 9 9 1 3 1 8-1 1-2

KNOT 8, 14   NTR 9   NWORD 10   TORSION 31
X3   1 1 9 1 1-1   X4   9-1 3 1 9 1   X5   3-1 1 1 3 1
X6   9 1 9 1 9-1   X7   4 1 9 1 4-1   X8   7-1 9 1 7 1
RELATOR   1 1 9 9 6-1 8-1
LONGITUDE   9 1 9 1 1-1 9 1 7 1 6 1 4 1 3-1 1-4

KNOT 8, 15   NTR 3   NWORD 11   TORSION 33
X4   2 1 1 1 9-1   X5   4 1 9 1 4-1   X6   9-7 9 1 9 1
X7   6 1 3 1 6-1   X8   3 1 5 1 3-1
RELATOR   3 1 1 1 9-1 1-1   RELATOR   6 1 7 1 4-1 7-1
LONGITUDE   9-1 7-1 5-1 4-1 3-1 1-1 9-1 9-1 1 9

KNOT 8, 16   NTR 3   NWORD 11   TORSION 35
X4   9-1 1 1 9 1   X5   1 1 3 1 9-1   X6   5 1 9 1 9-1
X7   3 1 9 1 3-1   X8   6 1 5 1 6-1
RELATOR   4 1 3 1 4-1 6-1   RELATOR   1 1 7 1 8-1 7-1
LONGITUDE   9 1 9-1 3 1 9-1 4 1 1-1 6-1 7-1 1 9

KNOT 8.17   NOR 3   NWIRD 11   TORSION 37
X4   3-1 1 1 1 3 1   X5   3-1 2 1 1 1 1   X6   9 1 3 1 9-1
X7   6 1 4 1 6-1   X8   4-1 9-1 6 1
RELATUR   7 1 8 1 5-1 3-1   RELATUR   5 1 3 1 5-1 4-1
LONGITUDE   3 1 4-1 8 1 1-1 4 1 5 1 9-1 7-1

KNOT 8.18   NOR 3   NWIRD 11   TORSION 3 15
X4   1 1 3 1 1-1   X5   4 1 9 1 4-1   X6   5 1 1 1 5-1
X7   9 1 1 1 9-1   X8   3-1 5 1 1 1
RELATUR   7 1 6 1 3-1 6-1   RELATUR   6 1 8 1 9-1 4-1
LONGITUDE   5-1 9 1 4-1 3 1 7-1 1 1 6-1 9 1

KNOT 8.19   NOR 3   NWIRD 9   TORSION 3
X4   3-1 9 1 3 1   X5   3-1 9-1 1 1 9 1 1 1
RELATUR   3 1 1 1 9-1 1-1
RELATUR   4-1 1 1 9 1 3-1 9-1 1-1
LONGITUDE   9 1 3 1 1 1 9 1 1 1 9 1 4 1 5 1 1-9

KNOT 8.20   NOR 3   NWIRD 9   TORSION 9
X4   9 1 9 1 9-1   X5   1-1 9 1 1 1   X6   3-1 9 1 1 1
RELATUR   5 1 4 1 5-1 6-1
RELATUR   1 1 6 1 4-1 9-1 4 1 6-1
LONGITUDE   6 1 4-1 1 1 9 1 3-1 1-1 9-1 5-1 1 9

KNOT 8.21   NOR 3   NWIRD 9   TORSION 19
X4   9-1 9 1 9 1   X5   4-1 9 1 9 1   X6   1-1 5 1 1 1
RELATUR   1 1 9 1 5-1 4-1 5 1 9-1
RELATUR   6 1 9 1 6-1 3-1
LONGITUDE   9 1 5-1 9-1 4-1 1 1 3-1 1-1 4-1 3 4

KNOT 9.1   NOR 9   NWIRD 11   TORSION 9
X3   9-1 1 1 9 1   X4   3-1 9 1 3 1 X5   4-1 1 1 4 1
X6   5-1 4 1 5 1   X7   6-1 5 1 6 1   X8   7-1 9 1 1 1
X9   9-7 1 8 1
RELATUR   1 1 9 1 1-1 9-1
LONGITUDE   9 9 4 1 6 1 9 1 8 1 7 1 5 1 1 1 9 1 1-9

KNOT 9.2   NOR 9   NWIRD 11   TORSION 15
X3   1 1 9 1 1-1   X4   3-1 1 1 9 1   X5   4 1 9 1 4-1
X6   5-1 4 1 5 1   X7   6 1 5 1 6-1   X8   7-1 6 1 9 1
X9   9 1 7 1 9-1
RELATUR   9 1 9 1 9-1 9-1
LONGITUDE   9 1 5 1 7 1 9 1 9 1 6 1 4 1 1 1 9 1 1-9

KNOT 9.3   NOR 9   NWIRD 11   TORSION 19
X3   9 1 1 1 9-1   X4   3 1 9 1 3-1   X5   4 1 3 1 4-1
X6   5 1 4 1 5-1   X7   6 1 5 1 6-1   X8   1 1 7 1 1-3
X9   7 1 6 1 7-1
RELATUR   9 1 9 1 9-1 1-1
LONGITUDE   9-1 4-1 6-1 1-1 9-1 3-1 5-1 7-1 9-1 1 9

KNOT 9.4   NOR 9   NWIRD 11   TORSION 21
X3   1-1 9 1 1 1   X4   3 1 1 1 3-1   X5   4-1 3 1 4 1
X6   5 1 4 1 5-1   X7   6-1 5 1 6 1   X8   7-1 1 1 7 1
X9   9 1 6 1 9-1
RELATUR   7 1 9 1 9-1 9-1
LONGITUDE   3-1 5-1 9-1 9-1 6-1 4-1 1-1 9-1 7-1 1 9

KNOT 9.5   NOR 9   NWIRD 11   TORSION 23
X3   9-1 1 1 9 1   X4   3 1 9 1 3-1   X5   4-1 3 1 4 1
X6   5 1 4 1 5-1   X7   6-1 5 1 6 1   X8   7-1 9 1 7 1
X9   8 1 6 1 8-1
RELATUR   7 1 7 1 9-1 1-1
LONGITUDE   9-1 9-1 3-1 5-1 8-1 1-1 6-1 4-1 9-1 1 9

KNIT 9, 6   NUM 2, NWORD 11   TORSION 27
  X3   1-1 2 1 1 1   X4   3-1 1 1 3 1   X5   4-1 3 1 4 1
  X6   5-1 4 1 5 1   X7   6-1 5 1 6 1   X8   8-1 6 1 8 1
  X9   7-1 8 1 9 1
  RELATOR   7 1 9 1 1-4 9-1
  LONGITUDE   9 1 5 1 8 1 7 1 8 1 1 1 4 1 6 1 9 1 1-9

KNIT 9, 7   NUM 2   NWORD 11   TORSION 30
  X3   2 1 3 1 9-1   X4   1-1 2 1 1 1   X5   4 1 2 1 4-1
  X6   5-1 4 1 5 1   X7   6 1 5 1 6-1   X8   7-1 6 1 7 1
  X9   3 1 2 1 3-1
  RELATOR   8 1 9 1 3-1 9-1
  LONGITUDE   8-1 9-1 7-1 5-1 8-1 4-1 6-1 3-1 8-1 1 9

KNIT 9, 8   NUM 2   NWORD 11   TORSION 33
  X3   2 1 1 1 8-1   X4   1-1 3 1 1 1   X5   4 1 1 1 4-1
  X6   5-1 4 1 5 1   X7   6 1 5 1 6-1   X8   3-1 9 1 3 1
  X9   8 1 6 1 8-1
  RELATOR   7 1 9 1 9-1 9-1
  LONGITUDE   8-1 1 1 5 1 8-1 7-1 9-1 9-1 6 1 6 1 1 1

KNIT 9, 9   NUM 2   NWORD 11   TORSION 31
  X3   1-1 2 1 1 1   X4   3-1 1 1 3 1   X5   4-1 3 1 4 1
  X6   5-1 4 1 5 1   X7   9-1 5 1 9 1   X8   6-1 7 1 6 1
  X9   8-1 6 1 8 1
  RELATOR   9 1 1 1 9-1 9-1
  LONGITUDE   9 1 5 1 8 1 7 1 7 1 1 1 4 1 9 1 6 1 9 1 1-9

KNIT 9, 10   NUM 2   NWORD 11   TORSION 33
  X3   9-1 1 1 8 1   X4   1-1 2 1 1 1   X5   4-1 3 1 4 1
  X6   5-1 4 1 5 1   X7   7 1 9 1 3-1   X8   7-1 6 1 7 1
  X9   8-1 5 1 8 1
  RELATOR   9 1 1 1 9-1 8-1
  LONGITUDE   9 1 4 1 4 1 8 1 3 1 1 1 5 1 7 1 9 1 9 1 1-9

KNIT 9, 11   NUM 2   NWORD 11   TORSION 33
  X4   1-1 2 1 1 1   X4   3-1 1 1 3 1   X5   4-1 3 1 4 1
  X6   5-1 4 1 5 1   X7   8-1 5 1 8 1   X8   6-1 9 1 6 1
  X9   7 1 8 1 7-1
  RELATOR   9 1 1 1 9-1 7-1
  LONGITUDE   9 1 5 1 8-1 7 1 6-1 1 1 4 1 9 1 9 1 1-5

KNIT 9, 12   NUM 2   NWORD 11   TORSION 33
  X3   1 1 2 1 1-1   X4   3-1 1 1 3 1   X5   4 1 3 1 4-1
  X6   5-1 4 1 5 1   X7   6-1 1 1 6 1   X8   7 1 6 1 7-1
  X9   9-1 8 1 9 1
  RELATOR   9 1 9 1 7-1 9-1
  LONGITUDE   9 1 5 1 7-1 9 1 8 1 4 1 1 1 9 1 4-1 1-5

KNIT 9, 13   NUM 2   NWORD 11   TORSION 37
  X3   9-1 1 1 9 1   X4   3-1 2 1 3 1   X5   4-3 3 1 4 1
  X6   1-1 4 1 1 1   X7   5-1 6 1 5 1   X8   7 1 8 1 7-1
  X9   8-1 7 1 8 1
  RELATOR   9 1 8 1 9-1 5-1
  LONGITUDE   9 1 4 1 9 1 7 1 3 1 1 1 5 1 8 1 6 1 1-9

KNIT 9, 14   NUM 2   NWORD 11   TORSION 37
  X3   1-1 2 1 1 1   X4   3 1 1 1 3-1   X5   4-1 3 1 4 1
  X6   5 1 4 1 5-1   X7   6-1 8 1 6 1   X8   9 1 5 1 9-1
  X9   4-1 6 1 9 1
  RELATOR   1 1 9 1 7-1 9-1
  LONGITUDE   3-1 5-1 9 1 7-1 9 1 4-1 3-1 6 1 9-1 1 3

KNOT 9.15   NOB 8   NWORD 11   TORSION 39
   X3   8-1 L 1 8 1   X4   3-1 8 1 5 1   X5   1-1 3 1 1 1
   X6   6-1 8 1 4-1   X7   6-1 4-1 6 1   X8   7-1 1 1 7 1
   X9   8 1 7 1 6-1
   RELATOR   9 1 6 1 9-1 5-1
   LONGITUDE   8 1 1 1 9 1 4 1 9 1 6-1 8-1 5 1 7-1 1-5

KNOT 9.16   NOB 3   NWORD 12   TORSION 39
   X4   8 1 1 1 8-1   X5   4 1 8 1 4-1   X6   5 1 4 1 3-1
   X7   6-1 8 1 6 1   X8   7 1 6 1 3-1   X9   7 1 4 1 7-1
   RELATOR   9 1 5 1 9-1 1-1   RELATOR   7 1 8 1 1-1 9-1
   LONGITUDE   9 1 4 1 6 1 8 1 1 1 1-7 1 1 1 5 1 2 1 1-9

KNOT 9.17   NOB 8   NWORD 11   TORSION 39
   X3   8-1 1 1 8 1   X4   3 1 8 1 7-1   X5   6 1 2 1 6-1
   X6   5 1 3 1 5-1   X7   6 1 5 1 6-1   X8   7 1 6 1 7-1
   X9   1-1 8 1 1 1
   RELATOR   7 1 9 1 1-1 9-1
   LONGITUDE   8 1 1 5-1 7-1 1 1 9-1 1 1 4-1 6-1 9 1 1 1

KNOT 9.18   NOB 8   NWORD 11   TORSION 41
   X3   1 1 8 1 1-1   X4   3-1 1 1 7 1   X5   4 1 1 1 4-1
   X6   5 1 1 1 5-1   X7   6 1 5 1 6-1   X8   7-1 6 1 7 1
   X9   8-1 8 1 8 1
   RELATOR   9 1 6 1 8 1 2-1
   LONGITUDE   5-1 9-1 1-1 4-1 6-1 9-1 2-1 7-1 3-1 1 1

KNOT 9.19   NOB 8   NWORD 11   TORSION 41
   X3   1 1 8 1 1-1   X4   3 1 2 1 3-1   X5   4-1 1 1 4 1
   X6   5 1 4 1 5-1   X7   6 1 1 1 6-1   X8   7-1 5 1 7 1
   X9   8-1 6 1 8 1
   RELATOR   9 1 1 1 9-1 9-1
   LONGITUDE   8-1 9-1 5 1 1 1 1-1 4 1 7 1 9-1 6 1 1 1

KNOT 9.20   NOB 8   NWORD 11   TORSION 41
   X3   8 1 1 1 8-1   X4   3-1 1 1 3 1   X5   4-1 2 1 4 1
   X6   5 1 4 1 5-1   X7   6 1 3 1 6-1   X8   7 1 5 1 7-1
   X9   8 1 7 1 8-1
   RELATOR   1 1 9 1 1-1 9-1
   LONGITUDE   8-7 6-7 8-1 1-1 4 1 7-1 9-1 5 1 7-1 1 5

KNOT 9.21   NOB 8   NWORD 11   TORSION 47
   X3   8-1 1 1 8 1   X4   3 1 2 1 3-1   X5   4-1 3 1 4 1
   X6   5-1 8 1 5 1   X7   6 1 4 1 6-1   X8   7 1 5 1 7-1
   X9   1-1 8 1 1 1
   RELATOR   9 1 1 1 9-1 9-1
   LONGITUDE   8 1 4 1 7-1 1 1 9-1 6-1 3 1 5 1 9 1 1-5

KNOT 9.22   NOB 3   NWORD 12   TORSION 47
   X4   8-1 1 1 8 1   X5   4 1 8 1 4-1   X6   5 1 2 1 5-1
   X7   6 1 4 1 6-1   X8   3 1 7 1 3-1   X9   1-1 8 1 1 1
   RELATOR   5 1 8 1 3-1 9-1   RELATOR   9 1 1 1 9-1 6-1
   LONGITUDE   8 1 6-1 3-1 1 1 7-1 8-1 4 1 5-1 9 1 1 1

KNOT 9.23   NOB 8   NWORD 11   TORSION 45
   X3   8-1 1 1 8 1   X4   1 1 8 1 1-1   X5   3 1 1 1 3-1
   X6   5-1 3 1 5 1   X7   6 1 4 1 6-1   X8   7-1 6 1 7 1
   X9   4 1 5 1 4-1
   RELATOR   9 1 7 1 9-1 8-1
   LONGITUDE   3-1 4-1 8-1 1-1 6-1 9-1 7-1 5-1 2-1 1 1

KNIT 9.94 NIR 3 NWIRD 12 TORSION 45
X4 4 1 2 1 1-1 X5 2-1 4 1 2 1 XR 9-1 1 1 4 1
X7 3 1 5.1 3-1 XR 5 1-7 1 5-1 X9 R-1 L 1 R 1
RELATIR 9 1 4 1 9-1 R-1 RELATIR 9 1 6 1 3-1 6-1
LONGITUDE R 1 6 1 7-1 1-1 2 1 3-1 5-1 9 1 4-1 1 1

KNIT 9.95 NIR 3 NWIRD 12 TORSION 47
X4 41-1 R 1 1 1 X5 4-1 R 1 4 1 X6 5 1 4 1 5-1
X7 9-1-1 1 9 1 XR 3-1 5 1 3 1 X9 R 1 7 1 R-1
RELATIR 4 1 9 1 4-1 3-1 RELATIR 7 1 3 1 7-1 R-2
LONGITUDE 2 1 R-1 6-1 4-1 3-1 4-1 1 1 5-1 9-1 1 5

KNIT 9.96 NIR 2 NWIRD 11 TORSION 47
X3 1 1 2 1 1-1 X4 3 1 1 1 3-1 X5 2 1 4 1 9-1
X6 4 1 3 1 4-1 X7 6 1 2 1 6-1 XR 5 1 7 1 5-1
X9 9 1 1 1 7-1
RELATIR 9 1 5 1 9-1 R-1
LONGITUDE 7-1 R 1 4 1 1 1 6-1 5-1 9 1 2 1 3 1 1-2

KNIT 9.97 NIR 2 NWIRD 11 TORSION 49
X3 1-1 9 1 1 1 X4 3-1 1 1 3 1 X5 4 1 1 1 4-1
X6 2 1 4 1 9-1 X7 5-1 R 1 5 1 XR 6-1 7 1 6 1
X9 7-1 5 1 7 1
RELATIR 9 1 R 1 9-1 6-1
LONGITUDE 3 1 2-1 9 1 6-1 5-1 1 1 R 1 7-1 4 1 1-1

KNIT 9.98 NIR 3 NWIRD 12 TORSION 51
X4 2 1 1 1 2-1 X5 4 1 2 1 6-1 X6 5-1 2 1 5 1
X7 1-1 5 1 1 1 XR 3 1 4 1 3-1 X9 6-1 3 1 6 1
RELATIR 7 1 9 1 7-1 R-1 RELATIR 6 1 R 1-3-1 R-1
LONGITUDE 9-1 3-1 7 1 6-1 R-1 5-L 4-1 1 1 9 1 1 3

KNIT 9.99 NIR 3 NWIRD 12 TORSION 51
X4 2-1 1 1 9 1 X5 1 1 3 1 1-1 X6 3-1 2 1 3 1
X7 4 1 6 1 4-1 XR 5 1 6 1 5-1 X9 R-1 7 1 R 1
RELATIR 9 1 4 1 9-1 4-1 RELATIR 6 1 3 1 6-1 9-1
LONGITUDE 5 1 7-1 3 1 4-1 R 1 6 1 1-1 9-1 2-1 3 1

KNIT 9.30 NIR 3 NWIRD 12 TORSION 53
X4 1 1 9 1 1-1 X5 4-1 1 1 4 1 X6 5-1 1 1 5 1
X7 9 1 5 1 9-1 XR 6-1 3 1 6 1 X9 3-1 7 1 3 1
RELATIR 7 1 9 1 7-1 9-1 RELATIR R 1 9 1 R-1 6-1
LONGITUDE 5 1 R 1 1-1 9-1 6 1 7-1 3-1 9 1 4-1 1 1

KNIT 9.31 NIR 2 NWIRD 11 TORSION 55
X3 2-1 1 1 2 1 X4 1 1 2 1 1-1 X5 4 1 3 1 4-1
X6 3 1 1 1 3-1 X7 5-1 6 1 5 1 XR 6-1 4 1 6 1
X9 R-1 9 1 R 1
RELATIR 4 1 7 1 4-1 7-1
LONGITUDE 3-1 5 1 9-1 1-1 6 1 7-1 R-1 4 1 2-1 1 3

KNIT 9.32 NIR 3 NWIRD 12 TORSION 59
X4 1-1 2 1 1 1 X5 2 1 4 1 9-1 X6 4 1 1 1 4-1
X7 3-1 1 1 3 1 XR 6 1 3 1 6-1 X9 5 1 3 1 5-1
RELATIR 7 1 R 1 7-1 5-1 RELATIR R 1 6 1 R-1 9-1
LONGITUDE 3 1 9-1 1 1 2-1 7 1 6 1 5-1 R 1 4 1 1-3

```
KNOT  9,33   NOR 3   NWORD 12   TORSION 61
   X4    1 1 2 1 1-1    X5    2 1 1 1 2-1   X6    5 1 1 1 5-1
   X7    3 1 4 1 3-1    X8    4-1 7 1 4 1   X9    8-1 2 1 8 1
   RELATOR  6 1 5 1 8-1 3-1    RELATOR  7 1 9 1 7-1 5-1
   LONGITUDE  5-1 7-1 4 1 6-1 9 1 1 1 8 1 7-1 2 1 1-1

KNOT  9,34   NOR 3   NWORD 12   TORSION 69
 1 X4    2 1 1 1 2-1    X5    1-1 3 1 1 1   X8    3-1 2 1 1 1
   X7    5 1 4 1 5-1    X8    4 1 1 1 8-1   X9    4 1 7 1 4-1
   RELATOR  7 1 9 1 7-1 8-1    RELATOR  2 1 3 1 9-1 4-1
   LONGITUDE  6-1 7 1 3 1 5 1 4-1 8 1 1 1-1 9-1 2 1 1 1

KNOT  9,35   NOR 3   NWORD 12   TORSION 3 9
   X4    2-1 1 1 8 1    X5    4 1 2 1 4-1   X6    5-1 4 1 5 1
   X7    1-1 3 1 1 1    X8    7 1 1 1 5-1   X9    3-1 2 1 1 1
   RELATOR  9 1 6 1 8-1 8-1    RELATOR  7 1 9 1 5-1 9-1
   LONGITUDE  7-1 4-1 3-1 4-1 9-1 1-1 9-1 5-1 2-1 1 9

KNOT  9,36   NOR 3   NWORD 12   TORSION 17
   X4    1 1 8 1 3-1    X5    2 1 1 1 2-1   X6    4 1 2 1 4-1
   X7    8-1 4 1 6 1    X8    7 1 7 1 7-1   X9    8 1 7 1 8-1
   RELATOR  6 1 3 1 8-1 3-1    RELATOR  5 1 9 1 5-1 1-1
   LONGITUDE  2-1 3-1 4 1 1-1 6 1 9-1 5-1 7-1 9-1 1 5

KNOT  9,37   NOR 3   NWORD 12   TORSION 8 15
   X4    2 1 1 1 2-1    X5    1 1 2 1 1-1   X6    5 1 2 1 5-1
   X7    3 1 1 1 3-1    X8    6 1 7 1 6-1   X9    4-1 8 1 4 1
   RELATOR  5 1 9 1 9-1 4-1    RELATOR  9 1 7 1 6-1 7-1
   LONGITUDE  3-1 4-1 4 1 8-1 1 6 5-1 7-1 9 1 2 1 1 1

KNOT  9,38   NOR 3   NWORD 12   TORSION 57
   X4    1 1 2 1 1-1    X5    2-1 1 1 2 1   X6    3 1 8 1 3-1
   X7    6 1 1 1 6-1    X8    7-1 3 1 7 1   X9    4 1 1 1 4-1
   RELATOR  5 1 4 1 5-1 9-1    RELATOR  9 1 9 1 8-1 5-1
   LONGITUDE  4-1 9-1 1-1 3-1 5-1 7-1 4-1 8-1 2-1 1 9

KNOT  9,39   NOR 3   NWORD 12   TORSION 55
   X4    1 1 2 1 1-1    X5    2 1 1 1 2-1   X6    3 1 1 1 3-1
   X7    4-1 3 1 6 1    X8    7 1 4 1 7-1   X9    4-1 6 1 4 1
   RELATOR  5 1 8 1 5-1 9-1    RELATOR  8 1 5 1 8-1 3-1
   LONGITUDE  3-1 4 1 5 1 7 1 1 1 9 1 6-1 8 1 2 1 1-5

KNOT  9,40   NOR 3   NWORD 12   TORSION 3 15
   X4    1-1 2 1 1 1    X5    3-1 1 1 1 1   X6    2-1 3 1 2 1
   X7    5-1 2 1 5 1    X8    6-1 1 1 6 1   X9    4-1 3 1 4 1
   RELATOR  7 1 8 1 7-1 4-1    RELATOR  8 1 2 1 8-1 5-1
   LONGITUDE  3 1 8 1 4-1 2 1 9 1 5-1 1 1 7 1 6-1 1-3

KNOT  9,41   NOR 3   NWORD 12   TORSION 7 7
   X4    1-1 8 1 1 1    X5    3 1 1 1 3-1   X6    2 1 3 1 2-1
   X7    4 1 3 1 4-1    X8    6-1 2 1 6 1   X9    5 1 4 1 5-1
   RELATOR  7 1 1 1 7-1 8-9    RELATOR  7 1 8 1 9-1 8-1
   LONGITUDE  7-1 6-1 1 1 5-1 8-1 4 1 2-1 9-1 3 1 1 1
```

KNIT 9,42   NWG 3   NWORD 18   TORSION 7
  X4   2-1 1 1 1 2 1   XS   1 1 2 1 1-1
  X6   4-1 2-1 3 1 2 1 4 1   X7   3 1 6 1 3-1
  RELATION   7 1 6 1 4-1 6-1
  RELATION   3 1 7 1 5-1 7-1
  LONGITUDE   2 1 6-1 3 1 4-1 2-1 7 1 1 1 3-1 5 1 1-1

KNIT 9,43   NWG 3   NWORD 18   TORSION 13
  X4   1 1 2 1 1-1   XS   3 1 1 1 3-1   X6   2 1 5 1 2-1
  X7   3-1 2 1 3 1
  RELATION   2 1 3 1 3-1 5-1 2-1 7 1
  RELATION   4 1 1 1 5-1 4-1
  LONGITUDE   3-1 2-1 4 1 7-1 5-1 2-1 5-1 1-1 6 1 1 5

KNIT 9,44   NWG 3   NWORD 18   TORSION 17
  X4   1 1 2 1 1-1   XS   4 1 2 1 4-1   X6   2 1 1 1 2-1
  X7   3 1 5 1 3-1
  RELATION   3 1 6 1 7-1 6-1
  RELATION   3 1 1 1 7-1 6-1 7 1 1-1
  LONGITUDE   2-1 7 1 1-1 6 1 3 1 4 1 1-1 5 1 3-1 1-1

KNIT 9,45   NWG 3   NWORD 18   TORSION 23
  X4   1 1 2 1 1-1   XS   3-1 4 1 3 1
  X6   2-1 5-1 3 1 5 1 2 1   X7   4 1 6 1 4-1
  RELATION   6 1 2 1 6-1 1-1
  RELATION   5 1 1 1 5-1 7-1
  LONGITUDE   6 1 1-1 3 1 7 1 4 1 5 1 2 1 4-1 5 1 1-5

KNIT 9,46   NWG 3   NWORD 18   TORSION 3 3
  X4   2-1 1 1 2-1   XS   4-1 2 1 4 1
  X6   3-1 5 1 3 1   X7   2 1 1-1 4 1 1 1 2-1
  RELATION   7 1 3 1 7-1 2-1
  RELATION   1 1 6 1 3-1 6-1
  LONGITUDE   6 1 7-1 4 1 3 1 1 1 2-1 3-1 5 1 2 1 6-3

KNIT 9,47   NWG 3   NWORD 18   TORSION 3 9
  X4   3-1 2 1 3 1   XS   4-1 3 1 4 1   X6   5 1 3 1 5-1
  X7   1-1 6 1 1 1
  RELATION   2 1 5 1 2-1 7-1
  RELATION   2 1 6 1 4 1 6-1 2-1 1-1
  LONGITUDE   2 1 6 1 3-1 3 1 7 1 5-1 1 1 2 1 4-1 1-3

KNIT 9,48   NWG 3   NWORD 18   TORSION 3 9
  X4   2 1 1 1 2-1   XS   1 1 2 1 1-1
  X6   4-1 3 1 4 1   X7   3-1 5 1 2 1 5-1 3 1
  RELATION   7 1 3 1 7-1 1-1
  RELATION   5 1 3 1 6-1 3-1
  LONGITUDE   7 1 4 1 3-1 1 1 5-1 3 1 1 1 6 1 2 1 1-5

KNIT 9,49   NWG 3   NWORD 18   TORSION 5 5
  X4   2 1 1 1 2-1   XS   1 1 2 1 1-1   X6   4-1 3 1 4 1
  X7   3-1 2 1 3 1
  RELATOR   7 1 5 1 7-1 6-1
  RELATOR   7 1 1 1 6 1 4-1 6-1 1-1
  LONGITUDE   7 1 5 1 4 1 7 1 1 1 3 1 1 1 6 1 2 1 1-9
....

Table Three

The homomorphisms on A_3 of the knot groups in table two, together with the corresponding homology groups

$$H_1 M_5, \quad H_1 \tilde{M}_5, \quad H_1 M_6, \quad H_1 \tilde{M}_6.$$

Each homomorphism is defined by the permutations which are the images of the "independent" over generators. The image of x_1 is inferred from the order of the homomorphism.

ORDER 5 X1=23451   ORDER 3 X1=23145   ORDER 2 X1=21435

KNOT 3, 1
1. ORDER 5 / X2 35214   LONGITUDE   23451
  5 SHEETS BETTI 1   TORSION   3
  BRANCHED BETTI 0   TORSION   15
  6 SHEETS BETTI 2   TORSION   2
  BRANCHED BETTI 0   TORSION   4

KNOT 4, 1   NONE

KNOT 5, 1
1. ORDER 5   X2 43152   LONGITUDE 49345
  5 SHEETS BETTI 3
  BRANCHED BETTI 2
  6 SHEETS BETTI 2   TORSION   2
  BRANCHED BETTI 0   TORSION   12
2. ORDER 3 X2 42391   LONGITUDE 23145
  6 SHEETS BETTI 2   TORSION   2
  BRANCHED BETTI 0   TORSION   12

KNOT 5, 2
1. ORDER 5   X2 35214   LONGITUDE 34512
  5 SHEETS BETTI 1   TORSION   3   3
  BRANCHED BETTI 0   TORSION   3   15
  6 SHEETS BETTI 3   TORSION   2
  BRANCHED BETTI 1   TORSION   2   2

KNOT 6, 1   NONE

KNOT 6, 2   NONE

KNOT 6, 3
1. ORDER 3. X2 42391   LONGITUDE 12349
  6 SHEETS BETTI 2   TORSION   2   12
  BRANCHED BETTI 1   TORSION   2   2

KNOT 7, 1   NONE

KNOT 7, 2   NONE

KNOT 7, 3   NONE

KNOT 7, 4
1. ORDER 3   X2 42391   LONGITUDE 23145
  6 SHEETS BETTI 2   TORSION   2   4
  BRANCHED BETTI 0   TORSION   2   2

KNOT 7, 5   NONE

KNOT 7, 6   NONE

KNOT 7, 7   NONE

KNOT 8, 1
1. ORDER 5   X2 35214   LONGITUDE 45123
  5 SHEETS BETTI 1   TORSION   3   3
  BRANCHED BETTI 0   TORSION   3   15
  6 SHEETS BETTI 3   TORSION   4
  BRANCHED BETTI 1   TORSION   8   4

```
KNOT  6.  2   NONF

KNOT  6.  3
1. ORDER  5   X2  35214   LONGITUDE  45123
5 SHEETS   BETTI   1   TORSION        91
BRANCHED   BETTI   0   TORSION       105
6 SHEETS   BETTI   2   TORSION        9          A
BRANCHED   BETTI   0   TORSION        9          9
2. ORDER  5   X2  43152   LONGITUDE  34512
5 SHEETS   BETTI   1   TORSION        91
BRANCHED   BETTI   0   TORSION       105
6 SHEETS   BETTI   2   TORSION        2          A
BRANCHED   BETTI   0   TORSION        9          9

KNOT  6.  4
1. ORDER  5   X2  43152   LONGITUDE  12345
5 SHEETS   BETTI   2   TORSION        2A
BRANCHED   BETTI   1   TORSION        9A
6 SHEETS   BETTI   9   TORSION        9
BRANCHED   BETTI   0   TORSION        12

KNOT  6.  5
1. ORDER  5   X2  35214   X3  23451   LONGITU
5 SHEETS   BETTI   1   TORSION        6        4A
BRANCHED   BETTI   0   TORSION        6       940
6 SHEETS   BETTI   9   TORSION        9          9
BRANCHED   BETTI   0   TORSION        9          9
2. ORDER  5   X2  43152   X3  43521   LONGITU
5 SHEETS   BETTI   1   TORSION        9
BRANCHED   BETTI   0   TORSION        4A
6 SHEETS   BETTI   9   TORSION        9          4
BRANCHED   BETTI   0   TORSION        9          9

KNOT  N.  6
1. ORDER  5   X2  43152   LONGITUDE  51234
5 SHEETS   BETTI   1   TORSION        9         91
BRANCHED   BETTI   0   TORSION        9        165
6 SHEETS   BETTI   2   TORSION        4
BRANCHED   BETTI   0   TORSION        4.0
2. ORDER  3   X2  42351   LONGITUDE  12345
6 SHEETS   BETTI   2   TORSION        9          2A
BRANCHED   BETTI   1   TORSION        2           A

KNOT  6.  7
1. ORDER  5   X2  35214   LONGITUDE  23451
5 SHEETS   BETTI   1   TORSION        3          3
BRANCHED   BETTI   0   TORSION        3          3
6 SHEETS   BETTI   3   TORSION        9
BRANCHED   BETTI   0   TORSION        9

KNOT  6.  8
1. ORDER  5   X2  35214   LONGITUDE  23451
5 SHEETS   BETTI   2   TORSION        4
BRANCHED   BETTI   1   TORSION        26
6 SHEETS   BETTI   2   TORSION        2          A
BRANCHED   BETTI   0   TORSION        2          2
```

KNOT R. 9 NONE

KNOT R.10
1º ORDER 5   X2 34819   X3 54132   LONGITUDE 18345
5 SHEETS BETTI 1   TORSION   45
BRANCHED BETTI 0   TORSION   883
6 SHEETS BETTI 2   TORSION   8   8   4
BRANCHED BETTI 0   TORSION   8   4   4
2º ORDER 5-4 X2 43152   X3 41532   LONGITUDE 41234
5 SHEETS BETTI 1   TORSION   8   96
BRANCHED BETTI 0   TORSION   2   480
6 SHEETS BETTI P   TORSION   2
BRANCHED BETTI 0   TORSION   4

KNOT R.11
1º ORDER 3   X2 42351   LONGITUDE 18345
6 SHEETS BETTI 0   TORSION   8
BRANCHED BETTI 0   TORSION   38

KNOT R.12 NONE

KNOT R.13 NONE

KNOT R.14 NONE

KNOT R.15
1º ORDER 5   X2 34819   X3 54132   LONGITUDE 34819
5 SHEETS BETTI 1   TORSION   8   848
BRANCHED BETTI 0   TORSION   8   1710
6 SHEETS BETTI P   TORSION   8   4   24
BRANCHED BETTI 0   TORSION   4   4   24
2º ORDER 5   X2 43152   X3 25413   LONGITUDE 34819
5 SHEETS BETTI L   TORSION   24   24
BRANCHED BETTI 0   TORSION   24   185
6 SHEETS BETTI 3   TORSION   4
BRANCHED BETTI 1   TORSION   8   4

KNOT R.16
1º ORDER 3   X2 42351   X3 48542   LONGITUDE 18345
5 SHEETS BETTI 3
BRANCHED BETTI 0
6 SHEETS BETTI P   TORSION   4
BRANCHED BETTI 0   TORSION   168

KNOT R.17
1º ORDER 3   X2 32819   X3 13384   LONGITUDE 18345
5 SHEETS BETTI 3
BRANCHED BETTI 0
6 SHEETS BETTI P   TORSION   8   8   6
BRANCHED BETTI 1   TORSION   8   8

```
KNOT  8.18
1. ORDER 5   X2  35814   X3  23451   LONGITUDE  23451
5 SHEETS  BETTI  1   TORSION        PA      PA
BRANCHED  BETTI  0   TORSION        PA      120
6 SHEETS  BETTI  2   TORSION         5      30
BRANCHED  BETTI  0   TORSION         5      60
2. ORDER 5   X2  35814   X3  31584   LONGITUDE   1234
5 SHEETS  BETTI  1   TORSION        PA      24
BRANCHED  BETTI  0   TORSION        PA      120
6 SHEETS  BETTI  2   TORSION         5      30
BRANCHED  BETTI  0   TORSION         5      60
3. ORDER 5   X2  35814   X3  54132   LONGITUDE  51234
5 SHEETS  BETTI  1   TORSION        PA      24
BRANCHED  BETTI  0   TORSION        PA      120
6 SHEETS  BETTI  2   TORSION         5      30
BRANCHED  BETTI  0   TORSION         5      60
4. ORDER 5   X2  23451   X3  35814   LONGITUDE  23451
5 SHEETS  BETTI  1   TORSION        PA      PA
BRANCHED  BETTI  0   TORSION        PA      120
6 SHEETS  BETTI  2   TORSION         5      30
BRANCHED  BETTI  0   TORSION         5      60


KNOT  8.19
1. ORDER 5   X2  35814   X3  41532   LONGITUDE  65123
5 SHEETS  BETTI  1   TORSION         2      6
BRANCHED  BETTI  0   TORSION         2      30
6 SHEETS  BETTI  2   TORSION         2      
BRANCHED  BETTI  0   TORSION         2      2      8


KNOT  8.20
1. ORDER 5   X2  35814   X3  35814   LONGITUDE  12345
5 SHEETS  BETTI  1   TORSION        45
BRANCHED  BETTI  0   TORSION        225
6 SHEETS  BETTI  2   TORSION         2      2      6
BRANCHED  BETTI  0   TORSION         2      4      4
2. ORDER 3   X2  24315   X3  12534   LONGITUDE  51234
5 SHEETS  BETTI  3
BRANCHED  BETTI  0
6 SHEETS  BETTI  2   TORSION         2      4
BRANCHED  BETTI  0   TORSION         2      4      12


KNOT  8.21
1. ORDER 5   X2  23451   X3  35814   LONGITUDE  34512
5 SHEETS  BETTI  1   TORSION         2      6
BRANCHED  BETTI  0   TORSION         2      30
6 SHEETS  BETTI  2   TORSION         2      4      4
BRANCHED  BETTI  0   TORSION         4      4      6
2. ORDER 3   X2  42351   X3  32541   LONGITUDE  31245
5 SHEETS  BETTI  3
BRANCHED  BETTI  0
6 SHEETS  BETTI  2   TORSION         2
BRANCHED  BETTI  0   TORSION        60
3. ORDER 3   X2  42351   X3  24315   LONGITUDE  31245
5 SHEETS  BETTI  3
BRANCHED  BETTI  0
6 SHEETS  BETTI  2   TORSION         2
BRANCHED  BETTI  0   TORSION        60


KNOT  9.1
1. ORDER 5   X2  35814   LONGITUDE  34512
5 SHEETS  BETTI  1   TORSION         3      9
BRANCHED  BETTI  0   TORSION         3      45
6 SHEETS  BETTI  2   TORSION         6
BRANCHED  BETTI  0   TORSION        18
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KNOT   9, 2  NONE

KNOT   9, 3  NONE

KNOT   9, 4  NONE

KNOT   9, 5
1. ORDER 5   X2  35214   LONGITUDE  34512
5 SHEETS  BETTI  1   TORSION      91
BRANCHED  BETTI  0   TORSION     105
6 SHEETS  BETTI  2   TORSION       2      8
BRANCHED  BETTI  0   TORSION       2      2     16

KNOT   9, 6
1. ORDER 5   X2  35214   LONGITUDE  34512
5 SHEETS  BETTI  1   TORSION      63
BRANCHED  BETTI  0   TORSION     315
6 SHEETS  BETTI  2   TORSION       2
BRANCHED  BETTI  0   TORSION       4

KNOT   9, 7  NONE

KNOT   9, 8
1. ORDER 3   X2  42351   LONGITUDE  23145
6 SHEETS  BETTI  2   TORSION      10
BRANCHED  BETTI  0   TORSION      60

KNOT   9, 9  NONE

KNOT   9,10  NONE

KNOT   9,11  NONE

KNOT   9,12  NONE

KNOT   9,13
1. ORDER 5   X2  43152   LONGITUDE  23451
5 SHEETS  BETTI  1   TORSION       3    117
BRANCHED  BETTI  0   TORSION       3    585
6 SHEETS  BETTI  2   TORSION       6
BRANCHED  BETTI  0   TORSION      60

KNOT   9,14
1. ORDER 5   X2  35214   LONGITUDE  51934
5 SHEETS  BETTI  1   TORSION       3     39
BRANCHED  BETTI  0   TORSION       3    195
6 SHEETS  BETTI  2   TORSION       2     48
BRANCHED  BETTI  0   TORSION       2      8    96

KNOT   9,15  NONE

KNOT   9,16
1. ORDER 5   X2  35214   X3  23451   LONGITUDE  45123
5 SHEETS  BETTI  1   TORSION       8     24
BRANCHED  BETTI  0   TORSION       8    120
6 SHEETS  BETTI  2   TORSION       2      4      8
BRANCHED  BETTI  0   TORSION       4      4      8
1. ORDER 3   X2  42351   X3  42351   LONGITUDE  12345
5 SHEETS  BETTI  3
BRANCHED  BETTI  0
6 SHEETS  BETTI  2   TORSION       2     24
BRANCHED  BETTI  0   TORSION       2      8    144
1. ORDER 3   X2  24315   X3  12453   LONGITUDE  12345
5 SHEETS  BETTI  3
BRANCHED  BETTI  0
6 SHEETS  BETTI  2   TORSION       2     24
BRANCHED  BETTI  0   TORSION       2      8    144
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KNOT 9.17
10 ORDER 3   XP 48351   LONGITUDE 31245
6 SHEETS BETTI 2 TORSION   P   4
BRANCHED BETTI 0 TORSION   P   P   4

KNOT 9.18   NONE

KNOT 9.19   NONE

KNOT 9.20
10 ORDER 5   XP 43152   LONGITUDE 45123
5 SHEETS BETTI 1 TORSION   R   P4
BRANCHED BETTI 0 TORSION   P   150
6 SHEETS BETTI 2 TORSION   P   4
BRANCHED BETTI 0 TORSION   P   P   14

KNOT 9.21   NONE

KNOT 9.22
10 ORDER 3   XP 24315   X3 51342   LONGITUDE 25145
5 SHEETS BETTI 3 TORSION   3
BRANCHED BETTI 0 TORSION   3
6 SHEETS BETTI 2 TORSION   4   19
BRANCHED BETTI 0 TORSION   P   19   4
20 ORDER 2   XP 35142   X3 13241   LONGITUDE 21245
5 SHEETS BETTI 3 TORSION   3
BRANCHED BETTI 0 TORSION   3
6 SHEETS BETTI 4
BRANCHED BETTI 0 TORSION   P

KNOT 9.23
10 ORDER 4   XP 35214   LONGITUDE 45123
5 SHEETS BETTI 2 TORSION   P   P   4
BRANCHED BETTI 1 TORSION   P   2   40
6 SHEETS BETTI 2 TORSION   4   14
BRANCHED BETTI 0 TORSION   7   24

KNOT 9.24
10 ORDER 5   XP 23451   X3 35214   LONGITUDE 12345
5 SHEETS BETTI 1 TORSION   33   145
BRANCHED BETTI 0 TORSION   33   435
6 SHEETS BETTI 2 TORSION   P   2   13
BRANCHED BETTI 0 TORSION   P   4   14
20 ORDER 3   XP 48351   X3 12534   LONGITUDE 31245
5 SHEETS BETTI 3
BRANCHED BETTI 0
6 SHEETS BETTI P TORSION   2
BRANCHED BETTI 0 TORSION   133

KNOT 9.25
10 ORDER 3   XP 24315   X3 51342   LONGITUDE 45123
5 SHEETS BETTI 3 TORSION   3
BRANCHED BETTI 0 TORSION   3
6 SHEETS BETTI P TORSION   42
BRANCHED BETTI 0 TORSION   3   4A
20 ORDER 2   XP 35142   X3 43215   LONGITUDE 12345
5 SHEETS BETTI 3 TORSION   3
BRANCHED BETTI 0 TORSION   3
6 SHEETS BETTI 4
BRANCHED BETTI 0 TORSION   P

KNOT 9.26
10 ORDER 5   XP 43152   LONGITUDE 51234
5 SHEETS BETTI 1 TORSION   1071
BRANCHED BETTI 0 TORSION   5355
6 SHEETS BETTI 2 TORSION   10
BRANCHED BETTI 0 TORSION   220

KNOT 9.27 NONE

KNOT 9.28
1. ORDER 5   X2 35214   X3 31524   LONGITUDE 34512
5 SHEETS   BETTI 1   TORSION   3   .6   186
BRANCHED   BETTI 0   TORSION   3   6   630
6 SHEETS   BETTI 2   TORSION   2   6   12
BRANCHED   BETTI 0   TORSION   2   6   24

KNOT 9.29
1. ORDER 5   X2 43152   X3 54213   LONGITUDE 51234
5 SHEETS   BETTI 1   TORSION   33   39
BRANCHED   BETTI 0   TORSION   03   165
6 SHEETS   BETTI 3   TORSION   16
BRANCHED   BETTI 1   TORSION   2   16

KNOT 9.30
1. ORDER 2   X2 35142   X8 23241   LONGITUDE 21435
5 SHEETS   BETTI 3   TORSION   3
BRANCHED   BETTI 0   TORSION   3
6 SHEETS   BETTI 4
BRANCHED   BETTI 0   TORSION   2

KNOT 9.31
1. ORDER 5   X2 35214   LONGITUDE 45123
5 SHEETS   BETTI 1   TORSION   3   12   96
BRANCHED   BETTI 0   TORSION   3   12   ???
6 SHEETS   BETTI 2   TORSION   2   12
BRANCHED   BETTI 1   TORSION   2   6
2. ORDER 5   X2 43152   LONGITUDE 45123
5 SHEETS   BETTI 1   TORSION   3   12   96
BRANCHED   BETTI 0   TORSION   3   12   AND
6 SHEETS   BETTI 2   TORSION   2   12
BRANCHED   BETTI 1   TORSION   2   6

KNOT 9.32 NONE

KNOT 9.33 NONE

KNOT 9.34 NONE

KNOT 9.35
1. ORDER 5   X2 43152   X3 43521   LONGITUDE 12349
5 SHEETS   BETTI 2   TORSION   2   6   6
BRANCHED   BETTI 1   TORSION   2   6
6 SHEETS   BETTI 2   TORSION   3   12
BRANCHED   BETTI 0   TORSION   3   36
2. ORDER 5   X2 43152   X3 84153   LONGITUDE 12349
5 SHEETS   BETTI 2   TORSION   2   6   6
BRANCHED   BETTI 1   TORSION   2   6   6
6 SHEETS   BETTI 2   TORSION   3   12
BRANCHED   BETTI 0   TORSION   3   36
3. ORDER 3   X2 24315   X3 25341   LONGITUDE 12349
5 SHEETS   BETTI 3   TORSION   2   6
BRANCHED   BETTI 0   TORSION   2   6
6 SHEETS   BETTI 2   TORSION   3   18
BRANCHED   BETTI 0   TORSION   3   36

KNOT 9.36
1. ORDER 2   X2 35142   X3 52431   LONGITUDE 21435
5 SHEETS   BETTI 2   TORSION   3
BRANCHED   BETTI 0   TORSION   3
6 SHEETS   BETTI 4
BRANCHED   BETTI 0   TORSION   2

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KNOT  9.37
1. ORDER  5   X2  43152   X3  24153   LONGITUDE  51.36
5 SHEETS   BETTI  1   TORSION          3      549
BRANCHED   BETTI  0   TORSION          3      2745
6 SHEETS   BETTI  2   TORSION         20
BRANCHED   BETTI  0   TORSION                 68
2. ORDER  5   X2  43152   X3  35421   LONGITUDE  51.36
5 SHEETS   BETTI  1   TORSION          3      549
BRANCHED   BETTI  0   TORSION          3      2745
6 SHEETS   BETTI  2   TORSION         20
BRANCHED   BETTI  0   TORSION                 68
3. ORDER  3   X2  42351   X3  18453   LONGITUDE  12.45
5 SHEETS   BETTI  3
BRANCHED   BETTI  0
6 SHEETS   BETTI  2   TORSION          2      18
BRANCHED   BETTI  1   TORSION          2
4. ORDER  3   X2  24315   X3  35143   LONGITUDE  12.45
5 SHEETS   BETTI  3   TORSION          6
BRANCHED   BETTI  0   TORSION          2      6
6 SHEETS   BETTI  2   TORSION        180
BRANCHED   BETTI  0   TORSION        370    186
5. ORDER  3   X2  32415   X3  42351   LONGITUDE  12.45
5 SHEETS   BETTI  3
BRANCHED   BETTI  0
6 SHEETS   BETTI  2   TORSION          2      12
BRANCHED   BETTI  1   TORSION          2      2


KNOT  9.38
1. ORDER  5   X2  35214   X3  35421   LONGITUDE  37.41
5 SHEETS   BETTI  1   TORSION          3              10
BRANCHED   BETTI  0   TORSION          3              40
6 SHEETS   BETTI  3   TORSION          2
BRANCHED   BETTI  1   TORSION          2


KNOT  9.39
1. ORDER  5   X2  35214   X3  35214   LONGITUDE  37.41
5 SHEETS   BETTI  1   TORSION        609
BRANCHED   BETTI  0   TORSION       3845
6 SHEETS   BETTI  2   TORSION          2
BRANCHED   BETTI  0   TORSION          2              14


KNOT  9.40
1. ORDER  5   X2  35214   X3  54139   LONGITUDE  24.13
5 SHEETS   BETTI  1   TORSION          3      3      40
BRANCHED   BETTI  0   TORSION          3      3      40
6 SHEETS   BETTI  3   TORSION          6      14
BRANCHED   BETTI  1   TORSION          6      14
2. ORDER  2   X2  35142   X3  14593   LONGITUDE  21.43
5 SHEETS   BETTI  3   TORSION          2      6
BRANCHED   BETTI  0   TORSION          2      6
6 SHEETS   BETTI  4
BRANCHED   BETTI  0   TORSION          2


KNOT  9.41
1. ORDER  5   X2  43152   X3  25413   LONGITUDE  45.13
5 SHEETS   BETTI  1   TORSION          3      9
BRANCHED   BETTI  0   TORSION          3      45
6 SHEETS   BETTI  2   TORSION          2      2       2
BRANCHED   BETTI  1   TORSION          2      2
2. ORDER  5   X2  43152   X3  43521   LONGITUDE  45.41
5 SHEETS   BETTI  1   TORSION          2      2       6
BRANCHED   BETTI  0   TORSION          2      2      30
6 SHEETS   BETTI  0   TORSION          2
BRANCHED   BETTI  0   TORSION         12
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KNOT  9.42
1. ORDER 3   X2  48391   X3  14835      LONGITUDE  28145
5 SHEETS  BETTI  3    TORSION         3
BRANCHED  BETTI  0    TORSION         3
6 SHEETS  BETTI  3    TORSION         2
BRANCHED  BETTI  1    TORSION         2
2. ORDER 2   X2  21543   X3  35142      LONGITUDE  21435
5 SHEETS  BETTI  3    TORSION         3
BRANCHED  BETTI  0    TORSION         3
6 SHEETS  BETTI  4
BRANCHED  BETTI  0    TORSION         2

KNOT  9.43
1. ORDER 2   X2  35142   X3  34125      LONGITUDE  21435
5 SHEETS  BETTI  3    TORSION         3
BRANCHED  BETTI  0    TORSION         3
6 SHEETS  BETTI  4
BRANCHED  BETTI  0    TORSION         2

KNOT  9.44
1. ORDER 2   X2  35142   X3  43215      LONGITUDE  21435
5 SHEETS  BETTI  3    TORSION         3
BRANCHED  BETTI  0    TORSION         3
6 SHEETS  BETTI  4
BRANCHED  BETTI  0    TORSION         2

KNOT  9.45
1. ORDER 2   X2  35142   X3  10523      LONGITUDE  21435
5 SHEETS  BETTI  3    TORSION         3
BRANCHED  BETTI  0    TORSION         3
6 SHEETS  BETTI  4
BRANCHED  BETTI  0    TORSION         2

KNOT  9.46
1. ORDER 5   X2  43152   X3  31524      LONGITUDE  18345
5 SHEETS  BETTI  0    TORSION        12
BRANCHED  BETTI  1    TORSION        12
6 SHEETS  BETTI  2    TORSION        12
BRANCHED  BETTI  0    TORSION        36
2. ORDER 4   X2  431     X3  54213      LONGITUDE  18345
5 SHEETS  BETTI  2    TORSION        12
BRANCHED  BETTI  1    TORSION        12
6 SHEETS  BETTI  2    TORSION        12
BRANCHED  BETTI  0    TORSION        36
3. ORDER 3   X2  42351   X3  52143      LONGITUDE  31845
5 SHEETS  BETTI  3
BRANCHED  BETTI  0
6 SHEETS  BETTI  2    TORSION        14
BRANCHED  BETTI  0    TORSION        44
4. ORDER 3   X2  24315   X3  51342      LONGITUDE  18345
5 SHEETS  BETTI  3    TORSION         2
BRANCHED  BETTI  0    TORSION         2
6 SHEETS  BETTI  2    TORSION        12
BRANCHED  BETTI  0    TORSION        36

KNOT  9.47
1. ORDER 5   X2  35214   X3  25413      LONGITUDE  34512
5 SHEETS  BETTI  1    TORSION         3     12     36
BRANCHED  BETTI  0    TORSION         3     12    180
6 SHEETS  BETTI  3    TORSION         6
BRANCHED  BETTI  1    TORSION         2      6
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KNOT 9.48
1. ORDER 5   X2   43152 .X3 31524   LONGITUDE .1.04
5 SHEETS BETTI 1 .TORSION   9   27
BRANCHED BETTI 0 TORSION   9   135
4 SHEETS BETTI P TORSION   A
BRANCHED BETTI 0 TORSION   12
2. ORDER 5   X2   43152 X3 54132   LONGITUDE .1.04
5 SHEETS BETTI 1 TORSION   9   27
BRANCHED BETTI 0 TORSION   9   135
4 SHEETS BETTI P TORSION   A
BRANCHED BETTI 0 TORSION   12
3. ORDER 3   X2   P4315 X3 32541   LONGITUDE .31.45
5 SHEETS BETTI 3 TORSION   P   P
BRANCHED BETTI 0 TORSION   P
6 SHEETS BETTI P TORSION
BRANCHED BETTI 0 TORSION   60

KNOT 9.49
1. ORDER 5   X2   48152 X3 43581   LONGITUDE .1.04
5 SHEETS BETTI 1 TORSION   9   27
BRANCHED BETTI 0 TORSION   9   135
4 SHEETS BETTI 3 TORSION   6
BRANCHED BETTI 3 TORSION
2. ORDER 3   X2   84314 X3 48351   LONGITUDE .1.04
5 SHEETS BETTI 3
BRANCHED BETTI 0 TORSION   5
6 SHEETS BETTI P TORSION   P   4
BRANCHED BETTI 0 TORSION   2   68
3. ORDER 3   X2   38415 X3 48351   LONGITUDE .1.04
5 SHEETS BETTI 3
BRANCHED BETTI 0 TORSION   P
6 SHEETS BETTI 8 TORSION   P   4
BRANCHED BETTI 0 TORSION   2   P
4. ORDER 3   X2   38415 X3 58314   LONGITUDE .1.04
5 SHEETS BETTI 3
BRANCHED BETTI 0 TORSION
6 SHEETS BETTI 2 TORSION   P
BRANCHED BETTI 0 TORSION   2   4
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Tables for Gaussian Quadrature of $\int_0^\infty 2x^2 \exp(-x^2) f(x)dx$[a]

by

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Abstract

Weights and abscissae for N-point Gaussian Quadrature of integrals of the form $\int_0^\infty 2x^2 \exp(-x^2) f(x)dx$ are given to 20 places for N = 2 through N = 18.

Steen, Byrne, and Gelbard recently published tables of weights
and abscissas for the evaluation by Gaussian quadrature of integrals
of the form:

$$(1) \qquad \int_0^\infty f(x)\exp(-x^2)dx$$

which arise in nuclear reactor design calculations. Similar integrals,
except with a **Maxwellian** weighting function $w(x)\langle 2x^2\exp(-x^2)$ rather
than their Boltzmann weight, occur frequently in atomic and molecular
collision problems. Although these integrals could be regarded as
special cases of (1), better convergence can be obtained from quadrature
formulae explicitly taking into account the additional factor $2x^2$ as
part of the weight function. Table I of this note gives abscissas and
weights for N point Gaussian quadratures of the form:

$$(2) \qquad \int_0^\infty 2x^2\exp(-x^2)f(x)dx$$

These were obtained in the usual way[2] from the (non-classical)
polynomials $P_N(x)$ orthogonal on $(0,\infty)$ with respect to $w(x)$,
uniquely defined by taking $P_0(x)=1$ and the following convention
for recursion:

$$(3) \qquad P_{N+1}(x) = (x+\alpha_N)P_N(x) + \beta_N P_{N-1}(x)$$

Of several numerical techniques investigated for determining
these polynomials the one suffering least from loss of numerical