REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 22, Number 101, January 1968, page 212.

26[2.10].—JAMES L. KINSEY, *Tables for Gaussian Quadrature of*

$$\int_0^\infty 2x^2 \exp(-x^2) f(x)\, dx,$$

6 pages of tables and 4 pages of explanatory text, reproduced on the microfiche card attached to this issue.

The abscissas and weights for n-point Gaussian quadrature of the integral in the title are tabulated to 20S for $n = 2(1)18$.

A second table gives, to the same precision, the coefficients α_n and β_n in the recurrence formula $P_{n+1}(x) = (x + \alpha_n)P_n(x) + \beta P_{n-1}(x)$ for the polynomials orthogonal on $(0, \infty)$ with respect to the Maxwellian weighting factor $2x^2 \exp(-x^2)$, as well as the normalization integral γ_n for $P_n(x)$. The corresponding error coefficients $d_n = \gamma_n/(2n)!$ are given to 6S in the first table.

Details of the underlying calculations on a CDC 3600 system at the University of Wisconsin are also furnished.

J. W. W.

27[3].—S. J. HAMMARLING, *Latent Roots and Latent Vectors*, The University of Toronto Press, 1970, xi + 172 pp., 26 cm. Price $13.50.

This book gives a descriptive account of methods for the numerical solution of eigenvalue problems. It begins with elementary properties of eigenvalues and their applications; this is not without some minor errors, notably stating a weak form of Gerschgorin's theorem (p. 9) and giving an oversimplified discussion of stability of the Crank-Nicolson method (Section 2.3). After this, the author proceeds to discuss in order the Danilevsky and Krylov methods, eigenvalues of tridiagonal matrices, the Givens and Householder methods, Lanczos' method, the power method, and finally QR.

Most of the discussion is dated, and can be found in either Wilkinson's or Householder's book; indeed the author appeals to one or the other continually for details and rigor. Besides this, there are some glaring omissions: the Givens and Householder methods are only described for symmetric matrices and, in fact, the Hessenberg form for nonsymmetric matrices is barely mentioned. Also, there is very little on inverse iteration for eigenvectors, and the author does not attach enough importance to the QR method (referred to as the method of Francis); only a cursory description is given and its use for symmetric tridiagonal matrices is not even mentioned.

Moreover, much is made of Danilevsky's method requiring only $O(n^3)$ operations to find the characteristic polynomial. However, the possibility of extreme loss of accuracy obtained the moments $I_{N,M}$ of the Nth degree polynomials

$$(4) \qquad I_{N,M} \equiv \int_0^\infty 2x^2 \exp(-x^2)\, x^M P_N(x)\, dx$$

by recursion from the moments of the polynomials of next lower degree:

$$(5) \qquad I_{N,M} = I_{N-1,M+1} + \alpha_{N-1} I_{N-1,M} + \beta_{N-1} I_{N-2,M}$$

and then used the orthogonality requirement that $I_{N,M}=0$ for $M<N$ to compute the next recursion coefficients

$$(6) \qquad \beta_N = -I_{N,N}/I_{N-1,N-1}$$

$$(7) \qquad \alpha_N = I_{N-1,N}/I_{N-1,N-1} - I_{N,N+1}/I_{N,N}$$

The starting values are known to high accuracy:

$$(8) \qquad I_{0,M} = \Gamma\left(\frac{M+3}{2}\right) \quad (M=0,1,\ldots)$$

$$(9) \qquad \alpha_0 = -\Gamma(2)/\Gamma(3/2) = -2\pi^{-1/2}$$

$$(10) \qquad \beta_0 = 0$$

From the set of recursion coefficients, the zeroes of the polynomials were determined by the Newton-Raphson method and the weights from

$$(11) \qquad W_{KN} = \gamma_{N-1}/[P'_N(x_{KN}) P_{N-1}(x_{KN})]$$

where x_{KN} is the Kth zero of the Nth degree polynomial and W_{KN} is the Kth weight in the Nth order Gaussian quadrature formula; γ_N is the normalization integral for the Nth degree polynomial:

$$(12) \qquad \gamma_N = \int_0^\infty 2x^2 \exp(-x^2)[P_N(x)]^2 dx = I_{N,N}$$

The computations were performed on the University of Wisconsin's CDC3600 computer in double precision (25S). The abscissas and weights so obtained were then truncated to 20S and tested by comparison of exact values of integrals of the form (8) with those obtained from the quadrature formulae. The Nth order formula should reproduce the values correctly for all values of M through M=2N-1. For values of N through 18, the maximum relative error in the computed values was 3.5×10^{-19}. For N=19 the relative error increased to approximately 5×10^{-15} and for N=20 no useful accuracy remained. This rapid deterioration of accuracy was not unexpected, since the program for obtaining the recursion coefficients and normalization integrals was also clearly beginning to produce unreasonable results at about N=20 to N=22.

In addition to the abscissae and weights, the tables contain the usual error coefficients[2] $\lambda_N = \gamma_N/(2N)!$

Acknowledgement

I am extremely grateful to Mr. Tony T. Warnock for his help in handling the computer programs.

References

1. N. M. Steen, G. D. Byrne, and E. M. Gelbard, "Gaussian Quadratures for the Integrals $\int_0^\infty \exp(-x^2) f(x)\,dx$ and $\int_0^b \exp(-x^2) f(x)\,dx$", Math. Comp., vol. 23, 1969, pp. 661-671.

2. A. H. Stroud and D. Secrest, Gaussian Quadrature Formulas, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966.

Footnotes:

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TABLE 1

ABSCISSAE AND WEIGHTS FOR GAUSSIAN QUADRATURES ON THE INTERVAL (0,INF.) WITH WEIGHT FUNCTION

$$W(X)=2X^{2}*EXP(-X^{2})$$

N= 2

ABSCISSAE	WEIGHTS
7.5398690798988710798-001	5.4768269345136692867-001
1.734095298879162795416000	3.3856423119993950849 6-001

D= 9.55497-003

N= 3

ABSCISSAE	WEIGHTS
5.5075549325050080740-001	2.0847703795130629099-001
1.28730902997026743506C00	5.2098955908719057350-001
2.22048494340982682966000	7.67693284615971149212-002

D=7.21402-005

N= 4

ABSCISSAE	WEIGHTS
4.2386281939005277604-001	1.5298149533475747617-001
1.01433210456675996866000	4.8708789892849061691-001
1.74243737514205029536C00	2.3259060710213911694-001
2.63981333163558626866C00	1.3966574086370904023-002

D=1.01181-006

N= 5

ABSCISSAE	WEIGHTS
3.3840959606959403634-001	8.4769421449679288069-002
8.2662954377359528989-001	9.8142187911134124639-001
1.43285437267355379016000	3.465945839799959872136-001
2.14940822242101348496000	7.14209556953457214920-002
3.01417246749010314976000	2.0612293974419431904-003

D=1.08255-009

N= 6

ABSCISSAE	WEIGHTS
2.7778943813951642991-001	4.9377146214145987517-002
6.8991114724401165395-001	2.7868356066153474690-001
1.26098733509792605616C00	3.81858757943074688428-001
1.81301245028710441986000	1.5861632251407936164-001
2.51041804359614412986000	1.7408679430942978029-002
3.35956946648594910376000	2.8244178898015544396-004

D=4.29331-011

N= 7

ABSCISSAE	WEIGHTS
2.3309732873350840319-001	3.0148945724911002126-002
5.8844294382868694882-001	1.9882890839832887474-001
1.03840973672703198746C00	3.6126525217149977796-001
1.56049433329589849016000	2.3736677251641494373-001
2.16450578236001099266000	5.4964311631551659494-002
2.84629099217405142976000	3.6108545690295973277-003
3.47142527951046078166000	3.5878440822465451289-005

D=6.65938-013

ABSCISSAE	WEIGHTS

N= 8

1.99000063179842944812-001	1.91902406720001361613-002
5.09552264502057941355-001	1.41490899526073231110-001
9.04160821020405600045-001	3.14733177400780613262-001
1.37261572397159844446000	2.05864544000774001006-001
1.90056957292497024423600	1.00628000005072020 26-001
2.49047984196743469696000	1.56704483062031524 33-002
3.15870067710524032106000	6.67790519404009572 65-004
3.96672040326539529096000	4.29953446531295491311-006

D=4.08901-015

N= 9

1.72399073533782300480-001	1.26028330197111327627-002
4.41544698421055734 01-001	1.01366989570851320 97-001
7.96071197394447203583-001	2.62021236239409666 43-001
1.21703340265700698096000	3.02256935517435983 03-001
1.69032196492399290016000	1.63942116972321409 97-001
2.21924460461709121706000	3.99020116059979097 26-002
2.75991170770902467463 6000	3.86117447340770907 85-003
3.49219405112479706466000	1.12926717955739809 06-004
4.24499495480604931413 6000	4.91064591950090830 20-007

D=2.19569-017

N=10

1.51114958421399311372-001	8.65107847674322296 32-003
3.90100817309415407 49-001	7.34992147582429040 96-002
7.07471989790740311026-001	2.12067720855633416750-001
1.08659540790387360 76000	2.94531111110499619231-001
1.51170734159572275695 6000	2.08773093457793629 48-001
1.99208300519540181786000	7.49977359856905717468-002
2.51200194419754621129 6000	1.24009197452014037 04-002
3.08420887661004958 4C6000	8.40006920329498757 42-004
3.72944198553736973456000	1.76227090507081102 0244-005
4.50887399669194024456000	5.41064945101991437 89-008

D=1.04699-019

N=11

1.33943391351460006457-001	6.06795963431212893 37-003
3.47476947760145070-001	5.40274366569951617737-002
6.33797601096 93029334-001	1.70896172556857683 49-001
9.78830029674629939801-001	2.72321117582851406271-001
1.37117961611851299533 6000	2.37937064949077460 73-001
1.80606010161650191 7426000	1.13429590549618810966-001
2.27522759823459558946000	2.83974963750169407 07-002
2.79945334964963 4115056000	3.42131790181120706 19-003
3.39716177583496255576000	1.70232092199335338 07-004
4.09925617251913024 5136000	2.99870470663009689 10-006
4.76036950387049684116000	5.77060953314420063195-009

D=4.48779-022

ABSCISSAE	WEIGHTS
	N=12
1.197C93109600404?326-001	4.301359875170433867?-0??
3.119441499719511946?-C01	4.070137992789370?490-0??
5.717036646673130995?-001	1.369095?2?1760291969-0?1
8.0717725170603645?27C-001	2.0314152?7195?1?65?6-0??
1.2402124627070C379?06000	2.901410971?2?1101669-0?1
1.64AC02122029903050?C00	1.59C9020440762964?9?-0?1
2.C0317060405004CC99556000	5.11979?5541215?37162-0??
2.5532029706077902094?C00	9.34?659?559619?1C?16-0??
3.0611?14?4399505?596CC0	0.9?0593379140912?259-0?6
4.61029?502057014C460600	1.14?4577604450915?26-0?5
4.2445384?4?1642455946?000	?.649?5?0?014?2992?6?-0?7
5.CC1C6434505?52911?96CC0	6.00C6077??C39??4150-01?
	D=1.?4677-024
	N=13
1.C7917296?4?9?22604?-001	1.20?9140199995621?2?-0??
2.0190C96115300718569-C01	1.0494421929?7417677?-0??
5.1903239309655143133-001	1.09056927999921124?6-0?1
8.C0E53712390001?C069-001	2.17637?9910?240971?8-0?1
1.14109222150372004326C00	2.496312900519646?47-0?1
1.512C700373901162030C000	1.?9449?0?0?52?1?3?5?-0?1
1.915C41010262?5?73?CC000	7.842?4?14?53401271??-0??
2.349123969040330?526000	1.991592074693394?5?6-0??
2.0193629916275293026000	7.76?69496?105493??0-0??
3.31022145?0?6607??0?000	1.906356095?6??1.?42??-0??
?.0676910917??C7249200007	9.51927?2421014371?11-0??
4.4095272489346595600956000	4.09?7919457073930212-0??
5.2322900590C0090?0526000	6.08779954967414029?6-0?1
	D=6.22?2?-027
	N=14
5.??9032?5424515915?93-CC2	2.??714??0??75?0C62?1-0??
2.?E45?C08J7793600?74-001	2.5?54216?7C2?760?1??-0??
4.7374341361509531?3?-CC1	8.7464?113R6?5?03?9?2-0??
7.4079273049694226503?-CC1	1.0?309467661095477?5-0?1
1.0459754315812792134600?	2.39?0?3?92?90473C96?-0?1
1.3937511911642169959600?	2.00614915295875?1992-0?1
1.76093909971220?C0306000	1.0671930136?95470124-0?1
2.177725?128730?05?0C600?	3.5140661149493690934-0??
2.6C474541300057059?C600?	6.05297500005437304?4-0??
?.0609146008992?547143600?	7.4035?01?3467535??89-0??
3.5E4?69041472753?945?CC?	1.9974?96326569?1933?-0??
4.10712690?75624360431600?	9.1031?76046271049312-0??
4.717412306906094657?6?00	5.3127903?3308?5557?16-0??
5.4549C71430301?963216000	6.052?3?9?59041?0C9?35-0??
	D=2.?4967-029

ABSCISSAE	WEIGHTS
N=15	
0.0916223790169854049-002	1.8347274048083219540-003
2.3495375695833815198-001	1.8105159697910463826-002
4.3494001030028630188-001	7.0304170338834977938-002
6.8100762911485565970-001	1.5663694931172022802-001
9.6078099411177094592 2-001	2.2394992015937062228-001
1.2859179110322732997 6000	2.1289763994704591549-001
1.6404891293917409437 6000	1.3296800721108300096-001
2.0102870579165659749 6000	5.4139290277083004391-002
2.4215286294643107859 6000	1.3059920620744709361-002
2.8596496316503328326 000	2.1266106787141812369-003
3.3089086353491057974 6000	1.8238209093026939410-004
3.8055907064053089606 6000	7.8635141108961002529-006
4.3376677510272775087 6000	1.4320658652111756008-007
4.9468121672959257833 6000	7.9312861430947515639 7-010
5.6695596459445088 7936 000	5.9049875575884729770-013
D=6.23559-017	
N=16	
8.1749135193414844598-002	1.4101455596930956912-003
2.1547619940528798177-001	1.4214773808005121498-002
4.0099305699305941811-001	5.6083772193046391309-002
6.2989364444454044480-001	1.3320470633963613470-001
8.9761244209026432684-001	2.0515714454571444787-001
1.1581469277080309435532 6000	2.1590040572907578784-001
1.5271881964280101080 6000	1.5494312467704859617 0-001
1.8817456108521059619 56000	7.5262129059811067943-002
2.2601132977685034892 96000	2.4097471594768543960-002
2.6619803286113862988 856000	4.9064169308574471264-003
3.0881763990962040891 6000	6.0406192468110434155-004
3.5422560313761376048 6000	4.1842417842567501224-005
4.0253122860780318735 6000	1.4629273220987372932-006
4.5602030449324930159 46000	2.1612934801075214322 7-009
5.1568767807803222227 86000	9.6561620019786580122-011
5.8781145019113730434 956000	5.6802530694214876429-014
D=1.77228-034	

ABSCISSAE	WEIGHTS
	N=17
7.9304117924451691362-002	1.1046967821222958394-013
1.9885926857601749946-001	1.1282421599141750619-012
3.7034114409809030637-001	4.6298802606795129798-012
5.8411570177600639070-001	1.1303247163122811381-011
8.3492689445983137761-001	1.8532430781526910629-011
1.1185438365534170534 6000	2.1241916099563766612-011
1.4260754637448986269 6000	1.7147541464911217751-011
1.7600830241372595570 96000	9.6637736007848648613-012
2.1169954784459427805 6000	3.7306529782309409844-012
2.4947095703106338711 86000	9.5946700175727704622-013
2.8948442734220689632 26000	1.5825676271823858975-013
3.3179512001150552604 486000	1.5890305966257955407-014
3.7677461767076962660 6000	9.0205681269878148688-016
4.2500241703110928797 6000	2.5914072733359629271-017
4.7754862606038005873 6000	3.1436059690829828706-019
5.3655701161021125573 6000	1.1497406541661878150-011
6.0799902963726922046 000	5.3821471596271957418-015

D=4.71199-037

ABSCISSAE	WEIGHTS
	N=18
6.5695895767014879231-002	8.7602772604527522192-014
1.8429062215711309240-001	9.0469646011627800928-013
3.4383463985364761366-001	3.7917509167598433287-012
5.4393065679629435967-001	9.5881261798466375814-012
7.7811371945462120976 3-001	1.6575974380651288359-011
1.0415059577496801770 86000	2.0462501619319324190-011
1.3359708923167596074 96000	1.8230196891431059910-011
1.6509017771119192850 16000	1.1698976260967924447-011
1.9079572643939805570 826000	5.2740490043178127916-012
2.3452119567599552631 36000	1.6509746647341804020-012
2.7[illegible]2365234940111110626000	3.4691459432655851912-013
3.1[illegible]C0733785817977397 6000	4.7064884858101428239 7-014
3.9[illegible]0159052269452067 46000	3.9076840397639720346-015
3.9[illegible]C6619659808510526C 00	1.8407427813242796993-016
4.4[illegible]0C5919844196 6C526000	4.3948129350137118028-014
4.9[illegible]16294813937652636000	4.4241110946877124717-01
5.5[illegible]098137921 30411456000	1.3206570844560175615-012
6.27[illegible]1198450446533198 76000	5.0350221056710173304-016

D=1.18103-039

TABLE II

RECURSION COEFFICIENTS AND NORMALIZATION INTEGRALS FOR POLYNOMIALS

ORTHOGONAL ON $(0,\infty)$ WITH WEIGHT FUNCTION $2X^2\mathrm{EXP}(-X^2)$

N	α_N	β_N	γ_N
0	-1.1243791670955125739+000	0.0000000000000000000+000	8.8622692545275801365-001
1	-1.3594630397735372082+000	-2.2676045525483731395-001	2.0096122109362444658-001
2	-1.5705068598616252099+000	-4.2455596300894734455-001	3.5319244744612089444-002
3	-1.7610965662237742365+000	-6.0895224074683321367-001	5.1955369624148635412-002
4	-1.9370615696730712383+000	-7.8676824918204031261-001	4.0876835194797181727-002
5	-2.0790006812220112459+000	-9.6102422811365706916-001	3.9283628909055462040-002
6	-2.2500781129554739394+000	-1.1331719141125172570+000	4.4515104964150196023-002
7	-2.3921011490565887447+000	-1.3039765714862162088+000	5.8044653950514455873-002
8	-2.5264577581087443640+000	-1.4738755211959316101+000	8.5553542344946671012-002
9	-2.6542297407193833425+000	-1.6431356752962060427+000	1.4057607757502537784-001
10	-2.7762746355007191917+000	-1.8119280081945109467+000	2.5471384469976908861-001
11	-2.8932834328558843043+000	-1.9803703400909055602+000	5.0442774325371034130-001
12	-3.0058213537847309536+000	-2.1495406370697460624+000	1.0837835048464290235+000
13	-3.1143570357320916627+000	-2.3164973520473803515+000	2.5105816191693820429+000
14	-3.2192037662491002222+000	-2.4842829018903459131+000	6.2369949902775699999+000
15	-3.3209352809243874284+000	-2.6519291319606968960+000	1.6549068715610011121+001
16	-3.4195986661431126520+000	-2.8194597658135943063+000	4.6634058253357260692+001
17	-3.5155412688234980459+000	-2.9868843924219021453+000	1.3922053888648426265+002

The editorial committee would welcome readers' comments about this microfiche feature. Please send comments to Professor Eugene Isaacson, MATHEMATICS OF COMPUTATION, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012.

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