The Dirichlet Problem for a Class of Elliptic Difference Equations*

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Abstract. Under suitable assumptions on the order of nonlinearity we prove existence and uniqueness theorems for difference Dirichlet problems of divergence type. We also show that the discrete solutions converge to a solution of the continuous problem. We do not assume that our equation comes from a variational problem. Some of our results are constructive or allow for the application of constructive methods.

Introduction. Let Ω be a plane bounded region such that the boundary of Ω, $\partial\Omega$, is of class C^1.

If P is a point in the plane, then we denote it by (x_1, x_2). Place a square grid on the plane of grid width h. All points of the form (mh, nh), with m and n integers, are called mesh points. Let $P_0 = (x_{01}, x_{02})$ be a mesh point. Then a neighborhood of P_0 is the set of points $\mathfrak{N}(P_0) = \{(x_{01}, x_{02}), (x_{01} + h, x_{02}), (x_{01} + h, x_{02} + h), (x_{01}, x_{02} + h), (x_{01} - h, x_{02} + h), (x_{01} - h, x_{02}), (x_{01} - h, x_{02} - h), (x_{01}, x_{02} - h), (x_{01} + h, x_{02} - h)\}$. We define Ω_h as that set of mesh points P such that $\mathfrak{N}(P) \subset \bar{\Omega}$, and we define the boundary of Ω_h as those mesh points P in $\bar{\Omega}$ such that at least one element of $\mathfrak{N}(P)$ is in the exterior of $\bar{\Omega} = \Omega + \partial\Omega$.

Let $V(P)$ be any function which is everywhere finite for $P \in \Omega_h + \partial\Omega_h = \bar{\Omega}_h$; such a function will be called a mesh function. Let $P = (x_1, x_2)$ be a mesh point. Then, for any mesh function we define forward difference quotients by

$$V_{x_1}(P) = \{V(x_1 + h, x_2) - V(P)\}/h, \qquad V_{x_2}(P) = \{V(x_1, x_2 + h) - V(P)\}/h$$

and backward difference quotients by

$$V_{\bar{x}_1}(P) = \{V(P) - V(x_1 - h, x_2)\}/h, \qquad V_{\bar{x}_2}(P) = \{V(P) - V(x_1, x_2 - h)\}/h.$$

The vector $(V_{x_1}(P), V_{x_2}(P))$ is denoted by $\nabla_h V(P)$ and the vector $(V_{\bar{x}_1}(P), V_{\bar{x}_2}(P))$ is denoted by $\bar{\nabla}_h V(P)$.

We denote by Ω_h' (and Ω_h'') the set of points $P \in \bar{\Omega}_h$ such that, for any mesh function $V(P)$ defined on $\bar{\Omega}_h$, the vector $\nabla_h V(P)$ (and $\bar{\nabla}_h V(P)$) is defined using only mesh points in $\bar{\Omega}_h$. If D_h is any set of mesh points in the plane, then we define $m_h(D_h)$ to be h^2 times the number of points in D_h. A set of mesh points will be called connected iff one can go from any mesh point in the set to any other mesh point in the set along line segments of length h connecting only elements of the set. We assume all mesh sets are connected.

A function $u(P) \in C_c^p(\Omega)$ iff the support of $u(P)$ is a compact subset of Ω and all pth order partial derivatives of $u(P)$ are continuous over Ω. The set $\mathcal{L}_m(\Omega)$ denotes all

functions whose absolute value is Lebesgue integrable to the mth power. The space $\mathcal{H}C^p_{m,0}(\Omega)$ is the completion of $C^p_c(\Omega)$ with respect to the metric of $\mathcal{L}_m(\Omega)$ applied to all partial derivatives up to order p; for a more detailed discussion see Morrey [12, pp. 62–90].

In this paper, we shall study selfadjoint uniformly elliptic differential problems of the form

$$(*)\qquad (a_i(P,\ u(P),\ \nabla u(P)))_{x_i} = f(P,\ u(P),\ \nabla u(P)),\qquad P \in \Omega,$$

$$u(P) = q(P),\qquad P \in \partial\Omega,$$

where the repeated indices are summed and $\nabla u(P) = (u_{x_1}(P),\ u_{x_2}(P))$ with $u_{x_i}(P) = \partial u(P)/\partial x_i$. We consider the difference approximation associated with this equation for given mesh size h to be given as

$$(**)\qquad (a_i(P,\ U,\ \nabla_h U))_{\bar{x}_i} = f(P,\ U,\ \nabla_h U),\qquad P \in \Omega_h,$$

$$U(P) = Q(P),\qquad P \in \partial\Omega_h.$$

The mesh function $Q(P)$ is related to $q(P)$ in that we assume $q(P)$ has a nice extension to Ω; for our purposes it is sufficient to assume that it may be extended to Ω as an element of $C^2(\bar{\Omega})$, call it $\tilde{q}(P)$, and we define $Q(P) = \tilde{q}(P)$ for $P \in \partial\Omega_h$.

The results of this paper will also hold for the difference problem

$$(**)'\qquad \begin{aligned} (a_i(P,\ U,\ \nabla_h U))_{\bar{x}_i} &+ (a_i(P,\ U,\ \overline{\nabla}_h U))_{x_i} \\ &= f(P,\ U,\ \nabla_h U) + f(P,\ U,\ \overline{\nabla}_h U),\qquad P \in \Omega_h, \end{aligned}$$

$$U(P) = Q(P),\qquad P \in \partial\Omega_h.$$

Let $\mathcal{C}_0(\Omega_h)$ be the set of all mesh functions defined on Ω_h and such that they vanish on $\partial\Omega_h$. Any solution $U(P)$ of $(**)$ will be such that, for every $\zeta \in \mathcal{C}_0(\Omega_h)$,

$$(***)\qquad h^2 \sum_{\Omega_h'} \{a_i(P,\ U,\ \nabla_h U)\zeta_{x_i} + f(P,\ U,\ \nabla_h U)\zeta\} = 0.$$

To save space, we shall often drop the index set of the summation. If we used the approximation in $(**)'$, we would add to $(***)$ a summation over Ω_h''. It is this last equation we shall study when we prove results of existence, uniqueness and convergence.

It is clear that to prove conditions for existence, uniqueness and convergence, we must make some assumptions which describe in a gross sense the types of nonlinearity we are considering in $(*)$. We list these classical assumptions as follows (for a more detailed analysis of the genesis of these conditions see [7] and [12]):

Condition (A). There exists a nonnegative constant C_1 such that, for mesh functions $V(P)$ and $W(P)$,

$$\sum_{i=1}^{2} a_i(P,\ V(P),\ \nabla_h W(P))p_i(P) \geqq C_1|\nabla_h W(P)|^m,$$

where $p_i(P) = W_{x_i}(P)$ for $i = 1, 2$ and $P \in \Omega_h$.

Condition (B). There exist nonnegative constants C_2 and C_3 such that for any nonzero vector $\xi = (\xi_1, \xi_2)$, for any $P \in \Omega_h$ and for any mesh functions $V(P)$ and $W(P)$, we have

$$C_3\{|\nabla_h W(P)|^2 + 1\}^{(m-2)/2} |\xi|^2 \geq \frac{\partial a_i(P,\ V(P),\ \nabla_h W(P))}{\partial p_i} \xi_i \xi_j$$

$$\geq C_2\{|\nabla_h W(P)|^2 + 1\}^{(m-2)/2} |\xi|^2,$$

where $a_{i,p_i} = a_{j,p_i}$, $|\xi|^2 = \xi_1^2 + \xi_2^2$ and the repeated indices indicate a summation over all i and j. This condition includes the statement that the problem in (1) is elliptic.

Condition (C). There exists a nonnegative constant C_4 such that

(α) $|f(P,\ V(P),\ \nabla_h W(P))| \leq C_4\{1 + |\nabla_h W|^2\}^{(m-1)/2}$, or

(β) $|f(P,\ V(P),\ \nabla_h W(P))| \leq C_4\{1 + |\nabla_h W|^2\}^{(m-1-\kappa)/2}$, with $\kappa > 0$.

Condition (D). There exist nonnegative constants C_5, C_6 such that for mesh functions $V(P)$, $W(P)$ we have the relations

$$\sum_i \{|\partial a_i(P,\ V,\ \nabla_h W)/\partial x_i| + |a_i(P,\ V,\ \nabla_h W)|\} \leq C_5(1 + |\nabla_h W|^2)^{(m-1)/2}$$

and

$$\sum_i |\partial a_i(P,\ V,\ \nabla_h W)/\partial V| \leq C_6(1 + |\nabla_h W|^2)^{(m-2)/2}.$$

Condition (E). There exists a nonnegative constant C_7 such that one of the following is valid:

(α)
$$|\partial f(P,\ V(P),\ \nabla_h W(P))/\partial V|$$
$$+ \sum_i \{|\partial f(P,\ V(P),\ \nabla_h W(P))/\partial W_{x_i}| + |\partial f(P,\ V(P),\ \nabla_h W(P))/\partial x_i|\}$$
$$\leq C_7(1 + |\nabla_h W|^2)^{(m-2)/2},$$

(β)
$$|\partial f(P,\ V(P),\ \nabla_h W(P))/\partial V|$$
$$+ \sum_i \{|\partial f(P,\ V(P),\ \nabla_h W(P))/\partial W_{x_i}| + |\partial f(P,\ V(P),\ \nabla_h W(P))/\partial x_i|\}$$
$$\leq C_7(1 + |\nabla_h W|^2)^{(m-2-\kappa)/2}$$

where $\kappa > 0$, or

(γ) $\quad |\partial f(P,\ V(P),\ \nabla_h W(P))/\partial V| + \sum_i |\partial f(P,\ V(P),\ \nabla_h W(P))/\partial W_{x_i}| = 0$

and

$$h^2 \sum_{\Omega_h} |f(P)| \leq C_7, \quad \text{for } m > 2,$$

or

$$h^2 \sum_{\Omega_h} |f(P)|^m \leq C_7, \quad \text{for } m \leq 2.$$

We are assuming that $m > 1$ in all of these conditions and that all the constants C_1, C_2, etc. are independent of h. By being independent of h we mean that if we imagine that h goes to zero then these numbers are to remain finite in the limit.

We shall use the notation, in order to save space in the sequel, m' and $\bar{m}$ where

$$m' = m - 1 \quad \text{and} \quad \bar{m} = m - 2.$$

We shall also consider problems where we replace the expression $1 + |\nabla_h W|^2$ by the expression $1 + |V|^2 + |\nabla_h W|^2$ in the appropriate conditions above. When we do this we shall denote the conditions corresponding to (A), (B), etc. by (A'), (B'), etc.

Ladyženskaya and Ural'ceva [8, p. 230] have considered the case that the in-homogeneous term in the differential equation associated with (1) satisfies the condition that $|f(P, u, \nabla u)| \leq C_4(1 + |\nabla u|^2)^{m/2}$. Their development is relative to the max norm, ours is not, and their analysis rests heavily on the Dirichlet Growth Theorem of Morrey [12, p. 79] and on the extension of results of De Giorgi [12, p. 194]. We can prove some of these preliminary results but when we attempt to apply them to a solution of (**), by a summation by parts, we cannot use these results because on the $\partial A_{h,k,\rho}$, $A_{h,k,\rho} = \{P: P \in \bar{\Omega}_h, U(P) > k, |P - P_0| < \rho\}$, we do not have that $U(P) = k$.

There is one more condition that we will add in order to prove the existence of a solution to (***) over general domains.

Condition (F). For any mesh function $\xi(P) \in \mathcal{C}_0(\Omega_h)$ and for every mesh function $W(P) \in \mathcal{C}_0(\Omega_h)$, there exists a function $F(P, \xi(P), \nabla_h W(P))$ such that

$$a_i(P, \xi(P), \nabla_h W(P)) = \partial F(P, \xi(P), \nabla_h W(P))/\partial W_{x_i}(P)$$

and

$$a_i(P, \xi(P), \overline{\nabla}_h W(P)) = \partial F(P, \xi(P), \overline{\nabla}_h W(P))/\partial W_{\bar{x}_i}(P).$$

We remark that this Condition (F) does *not* say that our equation in (**) is the Euler equation of a variational problem. Hence, our condition in (F) is more general than a requirement of Frehse [3, p. 316] who assumes that his equations come from a variational problem. We will also consider a sufficient condition for removing Condition (F).

We also consider the special case that Ω is a rectangular region with sides parallel to the coordinate axes. In this case, we assume that the lengths of two adjacent sides are commensurable and that $\partial \Omega_h \subset \partial \Omega$. We denote the sides of Ω by s_i, $i = 1, \cdots, 4$, with s_1 on the side perpendicular to the x_1-axis and the ordering increasing in the counterclockwise direction. The rectangular region offers simplicities which are not present in any other region.

In this paper, we show, under certain constraints on the constants C_i of our conditions, that a solution to (**) exists; the problem in (**) is equivalent to that in (***). The proof of the existence of a solution is given in Theorem 1 and it makes essential use of the Brouwer Fixed Point Theorem. To apply that theorem, we must prove, in part, that a certain function, called ϕ, defined on a ball $\mathcal{S}_1$ in $H^1_{m,0}(\Omega_h)$, maps $\mathcal{S}_1$ into itself. By requiring $C_1^m > C_4^m C_8 2^{m'm'}$ or $C_1^m > C_4^m 3^{m'm'} C_8(C_8^{m'/m} + 1)^m$, along with other conditions in the various parts of Theorem 1, we are able to show that a $J_1 > 0$ exists so that for $\mathcal{S}_1$ of radius $\geq J_1$ the mapping property is satisfied. We also prove that this ϕ is Hölder-continuous on $\mathcal{S}_1$ for $m > 2$ and ϕ is Lipschitz-continuous on $\mathcal{S}_1$ for $m \in [4/3, 2)$. These continuity properties allow, under certain assumptions, the application of constructive methods; e.g. the Banach Fixed Point Theorem or some of the approximation methods in [16]. We also obtain certain uniqueness results from these methods. The special case $m = 2$ is presented in Theorem 2.

For the proof of Theorem 1 we assumed that Condition (F) holds. In Theorem 3, we present a sufficient condition—again based on the size of the constants C_i—for removing that condition.

In Theorem 4, we obtain "interior estimates" on the norm of second-order difference quotients for solutions to (***); these estimates hold up to the boundary if Ω is a rectangular region and if the expressions a_i satisfy a condition given in Corollary 2.

In our proof of convergence, Theorem 5, we make essential use of Theorem 4 and the fact that we have two independent variables. In this respect, our proof of convergence is different from that of [3] and [5], although a more general proof of convergence is given in [3]. We show that for certain mesh sizes h_n tending to zero, those associated solutions U_n of (***) may be extended to $\mathfrak{U}_n \in \mathcal{3C}^1_{m,0}(\Omega)$ such that a subsequence converges weakly to an element $\mathfrak{U}_0 \in \mathcal{3C}^1_{m,0}(\Omega)$ and strongly to $\mathfrak{U}_0$ over D' where $\bar{D}' \subset \Omega$ and the $\partial\Omega$, $\partial D'$ are in C^1. From this and the fact that $\mathfrak{U}_n = U_n$ over D'_{h_n}, we conclude that $\mathfrak{U}_0$ is a weak solution to (*). This result is strengthened in the case Ω is a rectangle; that result is given in Corollary 3.

All of the results mentioned above were explicitly derived under the assumption that $Q(P) \equiv 0$ for $P \in \partial\Omega_h$. In part (II) of the paper, we mention how this assumption may be removed and the resulting effects on our computations become obvious.

Our assumption that the number n of independent variables is two was only important in Theorem 5. All results, with the exception of this, go through for $n > 2$ with slight modification. The proof of Theorem 5 would require a constraint on the relationship between m and n.

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Existence, Uniqueness and Convergence. In this section, we shall consider uniformly elliptic difference problems of the form

$$(a_1(P,\ U(P),\ \nabla_h U(P)))_{\bar{x}_1} + (a_2(P,\ U(P),\ \nabla_h U(P)))_{\bar{x}_2}$$

$$(1) \qquad\qquad = f(P,\ U(P),\ \nabla_h U(P)), \qquad P \in \Omega_h,$$

$$U(P) = Q(P), \qquad P \in \partial\Omega_h.$$

A more general type of problem has been treated in [9] where Ω was a rectangular region; the geometry of Ω was essential to the methods developed in that paper. Here, we wish to specialize the equation to be of divergence form but to leave the geometry of Ω as broad as possible.

In the development of this section, we shall refer to Condition (A), Condition (A'), etc., and by this we shall mean those conditions given in the Introduction which describe the types of nonlinearity we are considering.

We now divide this section into two parts.

(I). *The Case that* $Q(P) \equiv 0$ *for* $P \in \partial\Omega_h$. Let $A_0(\Omega_h) \equiv \mathfrak{C}_0 = \{\xi(P)\colon \xi(P)$ is a finite mesh function defined on $\bar{\Omega}_h$ and $\xi(P) = 0$ for $P \in \partial\Omega_h\}$. Then, we define a *weak solution* to (1) by following Ladyženskaya [7, p. 91], as a solution $U(P)$ to the problem

$$(2) \qquad h^2 \sum_{P \in \Omega_h} \left\{ \sum_{i=1}^{2} (a_i(P,\ U(P),\ \nabla_h U(P)))_{\bar{x}_i} + f(P,\ U(P),\ \nabla_h U(P)) \right\} \zeta(P) = 0,$$

where $U(P) \in \mathfrak{C}_0$ and (2) is to hold for *all* $\zeta(P) \in \mathfrak{C}_0$. The Theorem of Gauss, see von Koppenfels [6, p. 10] or Cryer [2, p. 160], allows us to write (2) as: (repeated indices indicate summation over i)

(3) $h^2 \sum_{P \in \Omega_h'} \{a_i(P,\ U(P),\ \nabla_h U(P))\zeta_{x_i}(P) - f(P,\ U(P),\ \nabla_h U(P))\zeta(P)\} = 0;$

here Ω_h' is the subset of $\bar{\Omega}_h$ described in the Introduction.

Let $H^1_{m,0}(\Omega_h') = \{V(P): V(P) \in \mathcal{C}_0$ and there exists a constant K_1 such that

(4) $\|V(P)\|^1_{m,0} \equiv \left(h^2 \sum_{P \in \Omega_h'} |V(P)|^m\right)^{1/m} + \left(h^2 \sum_{P \in \Omega_h'} |\nabla_h V(P)|^m\right)^{1/m} \leqq K_1$

and K_1 may be bounded, independent of mesh size $h\}$. If we have two different mesh widths h_1 and h_2, then the domain of definition of the function $V(P) \in H^1_{m,0}(\Omega_{h_1}')$ is different from the domain of definition of the function $V(P) \in H^1_{m,0}(\Omega_{h_2}')$, because $\Omega_{h_1}' \neq \Omega_{h_2}'$. We shall not dwell on this idea until we consider the question of convergence. Suffice it to say at this point that we are taking h to be arbitrarily small but fixed.

For any mesh function $W(P) \in \mathcal{C}_0$, we define the l_m-norm as $\|W(P)\|^0_m = (h^2 \sum_{P \in \Omega_h} |W(P)|^m)^{1/m}$.

In the sequel, we shall drop the prime from Ω_h as it does not add to the exposition; our meanings will be clear from the context.

We shall now state some lemmas of the Sobolev type which are essential for the technical manipulations which will follow. Their proofs proceed exactly as in [10], [11], [12, p. 80], [7, p. 82] and [14, p. 10].

LEMMA 1. *If* $V(P) \in H^1_{m,0}(\Omega_h)$, $m \geqq 1$, *and* Ω_h *is bounded and connected, then* $h^2 \sum_{\Omega_h} |V(P)|^m \leqq C_8 h^2 \sum_{\Omega_h} |\nabla_h V(P)|^m$ *where* $C_8 \leqq 2 \max\{d^m_{x_1}, d^m_{x_2}\} \equiv 2d^m$ *with* d_{x_i} *the maximum width of* Ω_h *in the* x_i-*direction.*

LEMMA 2. *If* $\xi(P) \in H^1_{m,0}(\Omega_h)$ *with* $1 \leqq m < 2$, *then* $\xi(P) \in l_r(\Omega_h)$ *where* $r = 2m/(2 - m)$ *and* $\|\xi(P)\|^0_r \leqq (2m/(2 - m)) \|\nabla_h \xi(P)\|^0_m$.

LEMMA 3. *If* $\xi(P) \in H^1_{m,0}(\Omega_h)$, $m > 2$, *and* Ω *is strongly Lipschitz and bounded, then there exists a positive constant* C_9 *such that* $\max_{P \in \Omega_h} |\xi(P)| \leqq C_9 \|\nabla_h \xi(P)\|^0_m$ *where* $C_9 \leqq 2\sqrt{2}(C_8 + 3m(\Omega)d)/m(\Omega)$.

LEMMA 4. *If* $f(P)$ *is everywhere finite for* $P \in \Omega_h$ *and if* $h^2 \sum_{\Omega_h} f(P)\zeta(P) = 0$ *for all* $\zeta(P) \in \mathcal{C}_0$, *then, for all* $P \in \Omega_h$, $f(P) = 0$.

An immediate consequence of Lemma 4 is that a weak solution to the discrete problem in (3) exists iff a solution to the discrete problem in (1) exists for the same mesh width h. This lemma gives "numerical meaning" to the idea of a weak discrete solution. We introduce the weak solution idea *only* because we need some quantitative estimates in order to prove that solutions to (3), and hence (1), exist and to establish uniqueness and convergence criteria. This definition is made in "formal anology" with that used in partial differential equations.

We now turn our attention to the question of existence and uniqueness of a solution to (3). The case $m = 2$ will be treated separately. We do not list explicitly all possible cases which can occur from our assumptions but only a representative sample to indicate the methods of proof. Some of our constraints on the constants of the problem, as given in the next theorem, come about as we are not able to prove all the results of the De Giorgi-Nash-Moser type; some general reasons for this were given in the Introduction. After the proof of the following theorem, we will give an example to which it may be applied.

THEOREM 1. *We assume in all cases to be considered that Condition* (F) *holds.*

(i) *Let Conditions* (A), (B), *and* (D) *hold with* $m > 2$. *If Conditions* (Cα) *and* (Eα)

also hold with $C_1 > C_4 C_8^{1/m} 2^{m'\,2/m}$, *then* (3) *has a solution in* $H_{m,0}^1(\Omega_h)$. *If Conditions* (Cβ) *and* (Eβ) *also hold and* $\kappa > 0$, *then* (3) *has a solution in* $H_{m,0}^1(\Omega_h)$. *If Condition* (Eγ) *also holds, then* (3) *has a solution in* $H_{m,0}^1(\Omega_h)$.

(ii) *If* $m \in (4/3, 2)$, *if Conditions* (A), (B), (D), (Cα) *and* (Eα) *hold and if* $C_1 > C_4 C_8^{1/m} 2^{m'\,2/m}$, *then* (3) *has a solution in* $H_{m,0}^1(\Omega_h)$. *Similar results hold for the other cases.*

(iii) *Let Conditions* (A'), (B'), *and* (D') *hold with* $m > 2$. *If Conditions* (C'α) *and* (E'α) *also hold with* $C_1 > C_4 3^{m'\,2/m} C_8^{1/m} (C_8^{m'/m} + 1)$, *then* (3) *has a solution in* $H_{m,0}^1(\Omega_h)$. *Similar results hold for the other conditions.*

(iv) *If* $m \in (4/3, 2)$, *if Conditions* (A'), (B'), (D'), (C'α) *and* (E'α) *hold and if* $C_1 > C_4 3^{m'\,2/m} C_8^{1/m} (C_8^{m'/m} + 1)$, *then* (3) *has a solution in* $H_{m,0}^1(\Omega_h)$. *Similar results hold for the other conditions.*

Proof. Case (a): $2 < m$. We first assume that (Cα) and (Eα) hold with $C_1 > C_4 C_8^{1/m} 2^{m'\,2/m}$.

Let $S_1 = \{\xi(P): \xi(P) \in \mathcal{C}_0, h^2 \sum |\nabla_h \xi|^m \leq J_1^m\}$ for *some* positive number J_1 which is independent of h. Let us "formally define" the quantity $\phi(\xi(\cdot); P)$ as a solution to the problem

$$(5) \qquad h^2 \sum_{\Omega_h'} \{a_i(P, \xi(P), \nabla_h \phi) \zeta_{x_i} + f(P, \xi, \nabla_h \xi) \zeta\} = 0$$

for all $\zeta \in \mathcal{C}_0$ and with $\nabla_h \phi = (\phi_{x_1}(\xi(\cdot); P), \phi_{x_2}(\xi(\cdot); P))$.

It follows from Condition (F) that to each $\xi(P) \in S_1$ there exists at least one $\phi(\xi(\cdot); P)$ and it follows from Condition (B) that to each $\xi(P) \in S_1$ there is at most one $\phi(\xi(\cdot); P)$.

Now, we determine those conditions which allow us to conclude that $\phi: S_1 \to S_1$. In (5) let $\zeta = \phi$ and apply the conditions in (A) and (C) and Lemma 1 to get

$$(6) \qquad C_1 h^2 \sum |\nabla_h \phi|^m \leq h^2 \sum |f(P, \xi, \nabla_h \xi)| \cdot |\phi|$$

$$\leq C_4 C_8^{1/m} 2^{m'\,2/m} \{J_1^{m'} + (m_h(\Omega_h))^{m'/m}\} (h^2 \sum |\nabla_h \phi|^m)^{1/m}.$$

Now we want the right-hand side of (6) to be $\leq C_1 J_1^{m-1}$, i.e. we want J_1 so that

$$(7) \qquad J_1^{m-1} \geq C_4 C_8^{1/m} 2^{m'\,2/m} (m_h(\Omega_h))^{m'/m} / (C_1 - C_4 C_8^{1/m} 2^{m'\,2/m}).$$

Hence, with J_1 satisfying (7), we have that $\phi: S_1 \to S_1$.

We shall show that ϕ is a Hölder-continuous function of ξ in the topology on S_1 induced by its defining norm. Let $\xi_1, \xi_2 \in S_1$ with ϕ_1, ϕ_2 the associated solutions to (5). Then, if we set $\zeta = \phi_1 - \phi_2$, we get: (the summation is over $P \in \Omega_h'$)

$$h^2 \sum \{(a_i(P, \xi_1, \nabla_h \phi_1) - a_i(P, \xi_1, \nabla_h \phi_2) + a_i(P, \xi_1, \nabla_h \phi_2) - a_i(P, \xi_2, \nabla_h \phi_2))$$

$$\cdot (\phi_1 - \phi_2)_{x_i} + (f(P, \xi_1, \nabla_h \xi_1) - f(P, \xi_2, \nabla_h \xi_2))(\phi_1 - \phi_2)\} = 0.$$

The Mean Value Theorem and our assumptions give

$$h^2 \sum \int_0^1 a_{i,p_i}(P, \xi_1, (1-t)\nabla_h \phi_1 + t\nabla_h \phi_2)\, dt \cdot (\phi_1 - \phi_2)_{x_i}(\phi_1 - \phi_2)_{x_i}$$

$$\leq h^2 \sum \left\{ \left| \int_0^1 a_{i,u}(P, (1-t)\xi_1 + t\xi_2, \nabla_h \phi_2)\, dt \cdot (\xi_1 - \xi_2) \cdot (\phi_1 - \phi_2)_{x_i} \right| \right.$$

$$\left. + (|\nabla_h \xi_1|^2 + |\nabla_h \xi_2|^2 + 1)^{\overline{m}/2} C_7(|\xi_1 - \xi_2| + |\nabla_h(\xi_1 - \xi_2)|) |\phi_1 - \phi_2| \right\}.$$

Now, for $m \geq 2$, $\bar{m} = m - 2$, $m' = m - 1$,

$$(\phi_1 - \phi_2)_{x_i}(\phi_1 - \phi_2)_{x_j} \int_0^1 a_{i,p_j}(P, \xi_1, (1 - t)\nabla_h\phi_1 + t\nabla_h\phi_2) \, dt$$

$$\geq C_2 \int_0^1 \{|\nabla_h((1 - t)\phi_1 + t\phi_2)|^2 + 1\}^{\bar{m}/2} \, dt \, |\nabla_h(\phi_1 - \phi_2)|^2$$

$$\geq C_2' \, |\nabla_h(\phi_2 - \phi_1)|^m/m',$$

where $C_2' = C_2/2$ if $(m - 2)/2 \geq 1$ and $C_2' = 2^{1/m}C_2/2^2$ if $0 < (m - 2)/2 < 1$. Hence, for $m > 2$,

$$(C_2'/m')(||\nabla_h(\phi_2 - \phi_1)||_m^0)^m$$

$$\leq h^2 \sum \{C_6(1 + |\nabla_h\phi_2|^2)^{\bar{m}/2} \, |\xi_1 - \xi_2| \cdot |\nabla_h(\phi_2 - \phi_1)|$$

$$+ (1 + |\nabla_h\xi_1|^2 + |\nabla_h\xi_2|^2)^{\bar{m}/2}C_7(|\xi_1 - \xi_2| + |\nabla_h(\xi_1 - \xi_2)|) \, |\phi_1 - \phi_2|\}$$

$$\leq C_6(h^2 \sum (1 + |\nabla_h\phi_2|^2)^{m/2})^{\bar{m}/m}C_8^{1/m} \, ||\nabla_h(\xi_1 - \xi_2)||_m^0 \, ||\nabla_h(\phi_1 - \phi_2)||_m^0$$

$$+ (h^2 \sum (1 + |\nabla_h\xi_1|^2 + |\nabla_h\xi_2|^2)^{m/2})^{\bar{m}/m}C_8^{(m+1)/m}C_7 \, ||\nabla_h(\xi_1 - \xi_2)||_m^0$$

$$\cdot ||\nabla_h(\phi_1 - \phi_2)||_m^0 + C_8^{1/m}C_7 \, ||\nabla_h(\xi_1 - \xi_2)||_m^0 \, ||\nabla_h(\phi_1 - \phi_2)||_m^0.$$

Therefore, for $m > 2$,

$$(C_2'/m')(||\nabla_h(\phi_2 - \phi_1)||_m^0)^{m'}$$

$$\leq \{C_8^{1/m}C_6(2^{\bar{m}/2}(J_1^m + m_h(\Omega_h)))^{\bar{m}/m}$$

$$+ C_8^{(m+1)/m}C_7(3^{m'}(m_h(\Omega_h) + 2J_1^m))^{\bar{m}/m} + C_8^{1/m}C_7\} \, ||\nabla_h(\xi_1 - \xi_2)||_m^0.$$

This is the definition of a Hölder-continuous function of $\xi \in \mathcal{S}_1$ with respect to the norm on $\mathcal{S}_1$. The Hölder exponent is independent of the mesh size h if J_1 has this property.

The existence of *an* element $U(P) \in \mathcal{S}_1$ satisfying the equation in (3) is now an immediate consequence of the Brouwer Fixed Point Theorem.

Now, consider that the Conditions (Cβ) and (Eβ) hold. The only part of the above proof which is in need of analysis is the determination of J_1. Here we have, using Young's Inequality,

$$C_1h^2 \sum |\nabla_h\phi|^m \leq h^2 \sum |f(P, \xi, \nabla_h\xi)| \cdot |\phi| \leq h^2 \sum |\phi| \cdot (1 + |\nabla_h\xi|^2)^{(m-1-\kappa)/2}$$

$$\leq C_8^{1/m}C_7 \, ||\nabla_h\phi||_m^0 \, \{m_h(\Omega_h) + \epsilon^{m/(m-\kappa m/m')}(m - \kappa m/m')(||\nabla_h\xi||_m^0)^m/m$$

$$+ \kappa m_h(\Omega_h)/m'\epsilon^{m'/k}\}^{m'/m},$$

where $C_7' = C_7 \cdot 2^{m-1-\kappa-m'/m}$ if $m - \kappa m/m' \geq 1$ and $C_7' = C_7$ otherwise, and $\epsilon > 0$. Therefore, for $\bar{\epsilon} = \epsilon^{m/(m-\kappa m/m')}$, we choose $\bar{\epsilon}$ so that (for example), $C_8^{1/m}C_7'\bar{\epsilon}^{m'/m} (m - \kappa m/m') = C_1/2$. Then pick J_1 so large that

$$C_8^{1/m}C_7'(m_h(\Omega_h))^{m'/m} + \{\kappa m_h(\Omega_h)/m'\epsilon^{m'\kappa}\}^{m'/m} \leq C_1J_1^{m'}/2.$$

We now proceed as in the earlier situation, since the left-hand side of the above inequality is known.

Now consider the case that (Eγ) holds. Using Lemma 3 we have that

$$C_1 h^2 \sum |\nabla_h \phi|^m \leq \max_{P \in \Omega_h} |\phi| \cdot h^2 \sum |f| \leq C_9 (h^2 \sum |\nabla_h \phi|^m)^{1/m} \cdot C_7$$

and hence we choose J_1 so that $C_9 C_7 \leq C_1 J_1^{m'}$.

Case (b): $4/3 \leq m < 2$. Assume that $(E\alpha)$ and $(C\alpha)$ hold. We prove the continuity of $\phi(\xi(\cdot); P)$ over $\mathcal{S}_1$.

We obtain, using the Hölder Inequality,

$$h^2 \sum (\phi_1 - \phi_2)_{x_i} (\phi_1 - \phi_2)_{x_i} \int_0^1 a_{i,p_i}(P, \xi_1, (1 - t)\nabla_h \phi_1 + t\nabla_h \phi_2) \, dt$$

$$\geq C_2 h^2 \sum \left\{ \int_0^1 \{1 + |\nabla_h\{(1 - t)\phi_1 + t\phi_2\}|^2\}^{\overline{m}/2} \, dt \, |\nabla_h(\phi_1 - \phi_2)|^2 \right\}$$

$$\geq C_2 \left(h^2 \sum \left(\int_0^1 \{1 + |\nabla_h\{(1 - t)\phi_1 + t\phi_2\}|^2\}^{\overline{m}/2} \, dt \right)^{m/\overline{m}} \right)^{\overline{m}/m} (||\nabla_h(\phi_1 - \phi_2)||_m^0)^2.$$

Therefore,

$$C_2(||\nabla_h(\phi_1 - \phi_2)||_m^0)^2 \leq \left(h^2 \sum \left(\int_0^1 \{1 + |\nabla_h\{(1 - t)\phi_1 + t\phi_2\}|^2\}^{\overline{m}/2} \, dt \right)^{m/\overline{m}} \right)^{-\overline{m}/m}$$

$$\cdot \{(C_6 + C_7)h^2 \sum |\xi_1 - \xi_2| \cdot |\nabla_h(\phi_1 - \phi_2)| + C_7 h^2 \sum |\nabla_h(\xi_1 - \xi_2)| \cdot |\phi_1 - \phi_2|\}.$$

Now

$$\left(\int_0^1 \{1 + |\nabla_h\{(1 - t)\phi_1 + t\phi_2\}|^2\}^{\overline{m}/2} \, dt \right)^{m/\overline{m}} \leq (1 + |\nabla_h \phi_1|^2 + |\nabla_h \phi_2|^2)^{-m/2}$$

and hence, using the Hölder Inequality and Lemma 2,

$$C_2(||\nabla_h(\phi_1 - \phi_2)||_m^0)^2 \leq \{2^{(2+m)/2} J_1^m + m_h(\Omega_h)\}^{-\overline{m}/m}$$

$$\cdot \{(C_6 + C_7)h^2 \sum |\xi_1 - \xi_2| \cdot |\nabla_h(\phi_1 - \phi_2)| + C_7 h^2 \sum |\nabla_h(\xi_1 - \xi_2)| \cdot |\phi_1 - \phi_2|\}$$

$$\leq \{2^{(2+m)/2} J_1^m + m_h(\Omega_h)\}^{\overline{m}/m} \{(C_6 + 2C_7)(-2m/\overline{m})^{m'/m}\} \cdot ||\nabla_h(\phi_1 - \phi_2)||_m^0$$

$$\cdot ||\nabla_h(\xi_1 - \xi_2)||_m^0.$$

Note that this shows that the function $\phi(\xi(\cdot); P)$ is a Lipschitz function on $\mathcal{S}_1$.

The remainder of the proof proceeds exactly as in Case (a), even to the use of Lemma 2 in establishing the mapping property of ϕ, i.e. for the computation of J_1.

Now, we consider the mapping property when (A'), (B'), $(C'\alpha)$, (D') and $(E'\alpha)$ hold for $m > 4/3$. As in (6), we have

$$C_1 h^2 \sum |\nabla_h \phi|^m \leq h^2 \sum |f(P, \xi, \nabla_h \xi)| \cdot |\phi|$$

$$\leq C_4 C_8^{1/m} ||\nabla_h \phi||_m^0 (h^2 \sum (1 + \xi^2 + |\nabla_h \xi|^2)^{m/2})^{m'/m};$$

that is

$$C_1(||\nabla_h \phi||_m^0)^{m'} \leq C_4 C_8^{1/m} 3^{m'^2/m} \{(m_h(\Omega_h))^{m'/m} + (C_8^{m'/m} + 1)(||\nabla_h \xi||_m^0)^{m'}\}.$$

Hence, we must require that

$$C_1 > C_4 3^{m'^2/m} C_8^{1/m}(C_8^{m'/m} + 1)$$

in order to establish the mapping property.

The Hölder constant now comes from the relation

$$(C_2'/m')(||\nabla_h(\phi_2 - \phi_1)||_m^0)^m$$

$$\leq \{C_6(h^2 \sum (1 + |\xi_1|^2 + |\xi_2|^2 + |\nabla_h\phi_2|^2)^{m/2})^{\bar{m}/m}C_8^{1/m} + C_8^{1/m}C_7$$

$$+ (h^2 \sum (1 + |\xi_1|^2 + |\xi_2|^2 + |\nabla_h\xi_1|^2 + |\nabla_h\xi_2|^2)^{m/2})^{\bar{m}/m}C_8^{(m+1)/m}C_7\}$$

$$\cdot ||\nabla_h(\phi_1 - \phi_2)||_m^0 \cdot ||\nabla_h(\xi_1 - \xi_2)||_m^0.$$

Now, use the mapping property to complete the bound on the Hölder constant.

The case that $4/3 \leq m < 2$ yields a Lipschitz constant computed from the relation, assuming that the primed conditions hold,

$$||\nabla_h(\phi_1 - \phi_2)||_m^0$$

$$\leq \{(2^{(2+m)/2}J_1^m + m_h(\Omega_h))^{-\bar{m}/m}C_2\}\{(C_6 + 2C_7)(2m/\bar{m})^{m'/m}\} ||\nabla_h(\xi_1 - \xi_2)||_m^0.$$

Remark 1. If we assume that the inhomogeneous term $f(P, \xi, \nabla \xi)$ satisfies a condition of the type,

$$(8) \qquad \sum \{|f(P, \xi, \nabla_h\xi)| + \nu_i |\partial f/\partial\xi_{x_i}|\} + \nu_3 |\partial f/\partial\xi| \leq C_7,$$

where ν_1, ν_2 and ν_3 are positive constants, then our mapping property is always established for $m > 2$ when we pick $C_7 m_h(\Omega_h) \leq C_1 J_1^{m-2}$ and when we set $\nu_j = 0$ for $j = 1, 2, 3$, in the above. If $\nu_j \in \{0, 1\}$ and $\partial a_i/\partial\xi = 0$ over S_1, then the continuity condition takes the simpler form, setting

$$[\nu_1, \nu_2] = \max\{\nu_1, \nu_2\},$$

$$||\nabla_h(\phi_1 - \phi_2)||_m^m \leq C_7 h^2 \sum \{[\nu_1, \nu_2] |\nabla_h(\xi_1 - \xi_2)| + \nu_3 |\xi_1 - \xi_2|\} |\phi_1 - \phi_2|.$$

This case occurs if, for example, $f = \sin\{\xi_{x_1} + \xi_{x_2} + \xi\}$.

We will illustrate some of the ideas of the last result in the following example.

Example. Let $a_i(\nabla_h U) = (1 + |\nabla_h U|^2/\bar{m})^{\bar{m}/2}U_{x_i}$ for $m > 2$. Then, $a_i(\nabla_h\phi) = (\bar{m}/m)\partial(1 + |\nabla_h\phi|^2/\bar{m})^{m/2}/\partial\phi_{x_i}$ and hence Condition (F) is satisfied. Also, we have that $C_1 = \bar{m}^{-\bar{m}/2} = C_2$ if $m \geq 3$, $C_1 = C_2 = 1$ if $m \in (2, 3]$, $C_3 = 3/\bar{m}$ if $m \in (2, 3)$, $C_3 = 1 + 2\bar{m}^2$ if $m \geq 3$, $C_5 = \bar{m}^{1/2}$ if $m \geq 3$, and $C_5 = \bar{m}^{(1-m')/2}$ if $m \in (2, 3)$. Suppose we consider the problem of finding a mesh function $\phi(\xi(\cdot); P) \in \mathcal{a}_0(\Omega_h)$ such that for every $\zeta(P) \in \mathcal{a}_0(\Omega_h)$ we have

$$h^2 \sum \{(1 + |\nabla_h\phi|^2/\bar{m})^{\bar{m}/2}\phi_{x_i}\zeta_{x_i} + (1 + |\nabla_h\xi|^2 + \sin^2 \xi)^{m'/2}\zeta\} = 0$$

where Ω_h has not been specified yet. Note that $C_4 = 2$. If we can find a mesh function $\psi \in \mathcal{a}_0(\Omega_h)$ such that

$$I(\psi) = \min_{V \in \mathcal{a}_0} I(V),$$

where

$$I(V) = h^2 \sum \{(\bar{m}/m)(1 + |\nabla_h V|^2/\bar{m})^{m/2} + (1 + |\nabla_h\xi|^2 + \sin^2 \xi)^{m'/2} V\},$$

then this mesh function ψ is what we will call ϕ. If we require that $C_1^m > C_4^m C_2 2^{m'm'}$, i.e. $1 > 2^{m+m'm'+1} \max(d_{x_1}^2, d_{x_2}^2)$ for $m \in (2, 3)$ and $1 > \bar{m}^{\bar{m}m/2}2^{m+m'm'+1} \max(d_{x_1}^2, d_{x_2}^2)$ for $m \geq 3$, then a constant J_1 exists and the set S_1 is meaningful. In fact, we may take

$$J_1^{m'/m} = 2^{2m+m'm'}(\max(d_{x_1}^2, d_{x_2}^2))^m/(1 - 2^{m+m'm'+1}\max(d_{x_1}^2, d_{x_2}^2)), \qquad m \in (2, 3],$$

$$= 2^{2m+m'm'}(\max(d_{x_1}^2, d_{x_2}^2))^m/(1 - \bar{m}^{\bar{m}m/2}2^{m+m'm'+1}\max(d_{x_1}^2, d_{x_2}^2)), \qquad m \geq 3.$$

Then, for $V \in \mathcal{C}_0(\Omega_h)$, $\xi \in \mathcal{S}_1$ and $\epsilon > 0$, we have that

$$I(V) \geq (1/m\bar{m}^{\bar{m}} - \epsilon/m) \|\nabla_h V\|_m^m - (2^{\bar{m}} + m2^{(\bar{m}m+1)/m'} J_1^m/\epsilon m')(\max(d_{x_1}^2, d_{x_2}^2))^{m/m'}.$$

Now, choose ϵ so that $1/m\bar{m}^{\bar{m}} - \epsilon/m > 0$ and we have that $I(V)$ is bounded from below on $\mathcal{C}_0(\Omega_h)$. Since $I(V)$ is clearly continuous, we have that an ψ exists. Now call $\phi(\xi(\cdot); P)$ this mesh function ψ.

As an almost immediate consequence of Theorem 1, we have the next result.

COROLLARY 1. *If* $4/3 \leq m < 2$ *and if*

$$\{2^{(2+m)/2} J_1^m + m_h(\Omega_h)\}^{(2-m)/m}\{(C_6 + 2C_7)(2m/(2 - m))^{(m-1)/m}\} \leq C_2,$$

whenever conditions (A), (B), (D), *and* (Cα), (Eα), *or* (Cβ), (Eβ), *or* (Eγ) *hold, or if*

$$\{2^{(2+m)/2} J_1^m + m_h(\Omega_h)\}^{-m/m}\{(C_6 + 2C_7)(-2m/\bar{m})^{m'/m}\} < C_2,$$

whenever Conditions (A'), (B'), (D'), *and* (C'α), (E'α), *or* (C'β), (E'β) *hold, then* $\phi(\xi(\cdot); P)$ *is a Lipschitz function on* $\mathcal{S}_1$ *and has one and only one fixed-point, i.e. one and only one solution to* (3) *exists.*

We shall now consider the case that $m = 2$ in establishing criteria for the constructive existence and uniqueness.

THEOREM 2. *If conditions* (A'), (B'), (C'α), (E'α) *and* (D') *are satisfied with* $m = 2$, *then* (3) *has a unique solution in* $\mathcal{C}_0(\Omega_h)$ *if there exists a positive constant* μ *such that*

$$1 - \mu C_2 + \mu\{(C_6 + C_7)C_8^{1/2} + C_7C_8\} < 1;$$

in this case the solution is completely constructible. Note that this condition is satisfied if (Eγ) *holds with* $\partial a_i/\partial V \equiv 0$.

Proof. Let $\mathcal{S}_2 = \{\xi(P): \xi(P) \in \mathcal{C}_0(\Omega_h), \|\nabla_h \xi\|_m \leq J_1\}$ and define the function $\phi(\xi(\cdot); P)$ as *the* solution to the problem, for μ a parameter,

$$h^2 \sum \{\phi_{x_i}\zeta_{x_i} - \xi_{x_i}\zeta_{x_i} + \mu a_i(P, \xi, \nabla_h\xi)\zeta_{x_i} + \mu f(P, \xi, \nabla_h\xi)\zeta\} = 0.$$

Remark 2. We observe that in all cases considered on m, if $\partial a_i/\partial V = 0$ and f depends on P alone, then a solution to (3) exists and it must be unique.

It is natural at this point, especially after our last result, to determine if we may remove the very confining Condition (F). We shall state and prove a theorem on a sufficient condition for the removal of this condition for the case $m > 2$. The case $m \in [4/3, 2)$ is treated in a similar manner and the modifications necessary will be read off from what we give.

THEOREM 3. *If the hypotheses of Theorem* 1(iii) *hold, with the exception of Condition* (F), *and if a positive constant* μ *exists such that* $\mu C_2 - 2^{m'}m' \geq 0$ *and*

$$\mu(C_5 + C_4C_8^{1/m})3^{m'm'/m}(1 + C_8)^{m'/m} < 1,$$

then (3) *has a solution in* $H_{m,0}^1(\Omega_h)$.

Proof. We need only prove that $\phi(\xi(\cdot); P)$ exists for all $\xi(P) \in \mathcal{S}_1$ as this was the only place Condition (F) was used. Let $\psi(v(\cdot); P)$, for $v(P) \in \mathcal{S}_1$ and μ a parameter to be determined, be a solution to

$$h^2 \sum \{\{(1 + |\nabla_h \psi|^2)^{\overline{m}/2} \psi_{x_i} + \mu a_i(P, \xi, 0)$$

$$+ \mu \tilde{a}_{i,\nu_i} \psi_{x_i} - (1 + |\nabla_h \nu|^2)^{\overline{m}/2} \psi_{x_i}\} \zeta_{x_i} + \mu f(P, \xi, \nabla_h \xi)\zeta\} = 0,$$

where

$$\tilde{a}_{i,\nu_i} = \int_0^1 a_{i,\nu_i}(P, \xi, t\nabla_h \nu) \, dt.$$

Clearly, $\psi(\nu(\cdot); P)$ exists as

$$I(\psi) = \min_{V \in S_1} I(V)$$

where

$$I(V) = h^2 \sum \{(1 + |\nabla_h V|^2)^{m/2}/m + \mu a_i(P, \xi, 0) V_{x_i} + \mu \tilde{a}_{i,\nu_i} V_{x_i} V_{x_i}$$

$$- (1 + |\nabla_h \nu|^2)^{\overline{m}/2} V_{x_i}^2 + \mu f(P, \xi, \nabla_h \xi) V\}.$$

Now, we show that $\psi : S_1 \to S_1$. As in Theorem 1, we have that

$$||\nabla_h \psi||_m^{m'} \leq \mu(C_5 + C_4 C_8^{1/m})3^{m'm'/m}\{(m_h(\Omega_h))^{m'/m} + (1 + C_8)^{m'/m} J_1^{m'}\}.$$

Now, pick $J_1^{m'} = \max\{A_1, A_2\}$ where

$$A_1 = \mu(C_5 + C_4 C_8^{1/m})3^{m'm'/m} B \div (1 - \mu(C_5 + C_4 C_8^{1/m})3^{m'm'/m}(1 + C_8)^{m'/m}),$$

$$A_2 = C_4 C_8^{1/m} 2^{m'm'/m} B \div (C_1 - C_4 C_8^{1/m} 2^{m'm'/m})$$

and

$$B = (m_h(\Omega_h))^{m'/m};$$

note that A_2 comes from (7) when the primed conditions are used. Now we have that $\psi : S_1 \to S_1$.

Now, observe that $||\nu_n - \nu||_m^1 \to 0$ iff $\nu_n - \nu_0 \to 0$ for each $P \in \Omega_h$. Hence ψ is a continuous, in $H_{m,0}^1(\Omega_h)$ norm, function of $\nu(P) \in S_1$. By Brouwer's Fixed Point Theorem, at least one fixed point of ψ exists on S_1; we call one of these $\phi(\xi(\cdot); P)$.

Now, we shall obtain "interior estimates" for powers of second-order difference quotients, i.e. we shall show that for any subregion D of Ω such that $\bar{D} \subset \Omega$, we have that $|\nabla_h U_{x_s}|$, $s = 1$ or 2, is in $l_p(D_h)$ for some $p > 0$ and for all h. These estimates will be used when we prove the convergence of the solutions of the difference equation to a solution of the differential equation. In a remark at the end of the proof of the next theorem, we will explain the generality of parts (d) and (e) in the statement of the next theorem.

THEOREM 4. (a) *If $U(P)$ is a solution to (3), if $h^2 \sum |\nabla_h U|^m$ is bounded—independent of h—over its domain of definition, if $m > 2$, and if conditions (B), (Cα) and (D) hold, then there exists a positive constant J_4 which is independent of h such that*

$$h^2 \sum \eta^2 (1 + |\nabla_h U|^2)^{(m-2)/2} |\nabla_h U_{x_s}|^2 \leq J_4,$$

where $\eta(P)$ is a smooth mesh function with support over compact mesh regions of Ω_h, and J_4 is given in (10), and $U_{x_s} = U_{x_1}$ or U_{x_2}.

(b) *If $U(P)$ is a solution of (3) with $m = 2$ and if the corresponding hypotheses of (2) are satisfied, then $h^2 \sum \eta^2 |\nabla_h U_{x_s}|^2 \leq J_5$ with J_6 given in (11).*

(c) If $U(P)$ is a solution to (3) with $m \in [4/3, 2]$ and the appropriate hypotheses hold as in (a), then $h^2 \sum \eta^m |\nabla_h U_{x_s}|^m \leq J_6^{m/2}$ where J_6 is given in (12).

(d) If $m \geq 2$ and $\xi(P) \in S_3$ where, for some positive number J_7, $S_3 = \{\xi(P): \xi(P) \in S_1$ and $\|\nabla_h \xi\|_{2(m-1-\kappa)}^0 \leq J_7$ for $\kappa \geq 0\}$, if Conditions (B), (C), and (D) are satisfied, and if $\phi(\xi(\cdot); P) \in H^1_{m,0}(\Omega_h)$ for each $\xi \in S_3$, then there exists a positive constant J_8 which depends on J_7 and other quantities such that

$$h^2 \sum \eta^2 (1 + |\nabla_h \phi|^2)^{(m-2)/2} |\nabla_h \phi_{x_s}|^2 \leq J_8.$$

The number J_8 is estimated in (13).

(e) If $m \in [4/3, 2]$, if $\xi(P) \in S_4$ where, for some positive number J_9,

$$S_4 = \{\xi(P): \xi(P) \in S_1 \text{ and } \max[\|\nabla_h \xi\|_{2(m-1-\kappa)}, \|\nabla_h \xi\|_{m/(m-1)}] \leq J_9, \kappa > 0\},$$

if Conditions (B), (C), and (D) are satisfied, and if $\phi(\xi(\cdot); P) \in H^1_m(\Omega_h)$, then a positive constant J_{10} exists such that

$$h^2 \sum \eta^m |\nabla_h \phi_{x_s}|^m \leq J_{10}.$$

Proof. (a) Let $\zeta = \mu_{x_s}$ for $s = 1$ or 2, and $\mu_{x_s}(P) \in \mathcal{C}_0(\Omega_h)$. Then, using Gauss' Theorem, we obtain

$$(9) \quad h^2 \sum a_i(P, U(P), \nabla_h U(P)) \zeta_{x_i} = -h^2 \sum (a_i(P, U, \nabla_h U))_{x_i} \mu_{x_i}$$
$$= -h^2 \sum \{\tilde{a}_{i,p_k} U_{x_k x_s} + \tilde{a}_{i,u} U_{x_s} + \tilde{a}_{i,x_s}\} \mu_{x_i},$$

where

$$\tilde{a}_{i,p_k} = \int_0^1 a_{i,p_k}(P, U, \nabla_h \tilde{U}) \, dt, \qquad \tilde{U} = (1-t)U(P - h_s) + tU(P),$$

$$\tilde{a}_{i,u} = \int_0^1 a_{i,u}(P, \tilde{U}, \nabla_h U(P - h_s)) \, dt, \qquad P - h_1 = (x_p - h, y_p),$$

$$P - h_2 = (x_p, y_p - h),$$

$$\tilde{a}_{i,x_s} = \int_0^1 a_{i,x_s}(\tilde{P}, U(P - h_s), \nabla_h U(P - h_s)) \, dt, \qquad \tilde{P} = (1-t)(P - h_s) + tP.$$

Substitution of (9) into (3) gives

$$h^2 \sum \{\{\tilde{a}_{i,p_k} U_{x_k x_s} + \tilde{a}_{i,u} U_{x_s} + \tilde{a}_{i,x_s}\} \mu_{x_i} - f(P, U, \nabla_h U) \mu_{x_s}\} = 0.$$

Let $\eta(P)$ be a nonnegative mesh function such that the closure, relative to our neighborhood system, of its support is contained in a mesh region D_h such that $\bar{D}_h \subset \Omega_h$. Let $\mu(P) = \eta^2(P) U_{x_s}(P)$. Using [8, p. 10] and Conditions (B), (D), (Cα), we obtain the estimate

$$C_2 h^2 \sum \eta^2 (1 + |\nabla_h U|^2)^{\bar{m}/2} |\nabla_h U_{x_s}|^2$$
$$(10) \quad \leq h^2 \sum \{(12C_3 |\nabla_h \eta| \, \eta + \eta^2 C_6)(1 + |\nabla_h U|^2)^{\bar{m}/2} |\nabla_h U_{x_s}| \cdot |U_{x_s}|$$
$$\cdot (\eta^2 C_5' + 4 |\nabla_h \eta| (\eta |U_{x_s}|^2 C_6 + \eta |U_{x_s}| C_5'))(1 + |\nabla_h U|^2)^{\bar{m}/2}$$
$$+ C_4 (1 + |\nabla_h U|^2)^{m'/2} (\eta^2 |\nabla_h U_{x_s}| + \eta 2 |\nabla_h \eta| \cdot |U_{x_s}|)\}.$$

Using the Cauchy Inequality, for $\epsilon_i > 0$ ($i = 1, \cdots, 4$) and so that $12C_3 \epsilon_1 + C_6 \epsilon_2 + C_5' \epsilon_3 + C_4 \epsilon_4 = C_2$ with $\epsilon_1 = C_2/48C_3$, $\epsilon_2 = C_2/4C_6$, $\epsilon_3 = C_2/4C_5'$, and $\epsilon_4 = C_2/4C_4$, then we obtain the bound

$$\tfrac{1}{2}C_2 h^2 \sum \eta^2 (1 + |\nabla_h U|^2)^{\overline{m}/2} |\nabla_h U_{x_s}|^2$$

$$\leqq 2^{m'} h^2 \sum \{2\eta^2 C_4^2 / C_2 + (288 C_3^2 + 2C_6^2 + 2C_5') |\nabla_h \eta|^2 / C_2$$

$$+ 2\eta |\nabla_h \eta| (C_6 + C_4)\}(1 + |\nabla_h U|^m)$$

$$+ \max\{1, 2^{\overline{m}}\} h^2 \sum 2 |\nabla_h \eta| \eta C_5' (1 + |\nabla_h U|^{m'}).$$

If we now choose $|\eta(P)|$ and $|\nabla_h \eta(P)|$ to be bounded over $\overline{\Omega}_h$, then we have that $\eta(1 + |\nabla_h U|^2)^{(m-2)/4} |\nabla_h U_{x_s}| \in l_2(\Omega_h)$ for $s = 1, 2$. Our constant J_3 is now estimated by the inequality

$$C_2^2 J_4 \leqq 4A\{2^{m'}(C_4^2 + (144 C_3^2 + C_6^2 + C_5'^2) + C_2(C_6 + C_4)\}(m_h(\Omega_h) + J_1^m)$$

$$+ 2AC_2 \max\{1, 2^{\overline{m}}\} C_5'(m_h(\Omega_h) + J_1^{m'}(m_h(\Omega_h))^{1/m}),$$

where $A = (\max\{\max|\eta|, \max|\nabla_h \eta|\})^2$.

(b) If $m = 2$, we have

$$C_2(\||\nabla_h U_{x_s}|\|_2^0)^2 \leqq 6C_3\{h^2 \sum \eta^2 \epsilon_1 |\nabla_h U_{x_s}|^2 + h^2 \sum |\nabla_h \eta|^2 |\nabla_h U|^2 / \epsilon_1\}$$

$$+ C_5' h^2 \sum \eta^2 + 4h^2 \sum (C_6 |\nabla_h \eta| \eta |\nabla_h U|^2 + C_5' \eta |\nabla_h U|)$$

$$+ 2C_4 h^2 \sum \eta |\nabla_h \eta| (1 + |\nabla_h U|^2)$$

$$+ C_4 \epsilon_2 h^2 \sum \eta^2 |\nabla_h U_{x_s}|^2 / 2 + C_4 h^2 \sum \eta^2 (1 + |\nabla_h U|^2) / 2\epsilon_2.$$

Now choose $\epsilon_1 = C_2 / 24 C_3$ and $\epsilon_2 = C_2 / 2C_4$ to get the estimate

$$\begin{aligned}
C_2 J_5 / 2 &\leqq 144(C_3^2 / C_2) h^2 \sum |\nabla_h \eta|^2 |\nabla_h U|^2 \\
&\quad + C_5' h^2 \sum \eta^2 + C_4^2 h^2 \sum \eta^2 (1 + |\nabla_h U|^2) / C_2 \\
&\quad + 4h^2 \sum (C_6 |\nabla_h \eta| \cdot |\nabla_h U| + C_5') \eta |\nabla_h U| \\
&\quad + 2C_4 h^2 \sum \eta |\nabla_h \eta| (1 + |\nabla_h U|^2).
\end{aligned}$$

(11)

(c) Now apply the Holder Inequality to (10) to get

$$C_2 (h^2 \sum (1 + |\nabla_h U|^2)^{m/2})^{\overline{m}/m} (h^2 \sum \eta^m |\nabla_h U_{x_s}|^m)^{2/m}$$

$$\leqq (h^2 \sum \{(12 C_3 |\nabla_h \eta| + \eta C_6)(1 + |\nabla_h U|^2)^{m'/2}\}^{m/m'})^{m'/m} \||\eta(\nabla_h U_{x_s})|\|_m^0$$

$$+ (h^2 \sum \{\eta(1 + |\nabla_h U|^2)^{m'/2}\}^{m/m'})^{m'/m} (h^2 \sum (\eta |\nabla_h U_{x_s}|)^m)^{1/m}$$

$$+ C_5' h^2 \sum \eta^2 + 4C_6 h^2 \sum |\nabla_h \eta| \eta (1 + |\nabla_h U|^2)^{m/2}$$

$$+ 4C_5' h^2 \sum \eta |\nabla_h \eta| (1 + |\nabla_h U|^2)^{m'/2}.$$

Now apply the Schwartz Inequality to the first two terms on the right side of the above to get, taking $\epsilon_1 = C_2 (h^2 \sum (1 + |\nabla_h U|^2)^{m/2})^{\overline{m}/m} / 2$ and $\epsilon_2 = \epsilon_1$,

$$(h^2 \sum \eta^m |\nabla_h U_{x_s}|^m)^{2/m} \leqq (2/C_2)(h^2 \sum (1 + |\nabla_h U|^2)^{m/2})^{-\overline{m}/m} C_5' h^2 \sum \eta^2$$

$$+ (h^2 \sum (1 + |\nabla_h U|^2)^{m/2})^{-\overline{m}/m}$$

(12)

$$\cdot \{(h^2 \sum \{(12 C_3 |\nabla_h \eta| + \eta C_6)(1 + |\nabla_h U|^2)^{m'/2}\}^{m/m'})^{2m'/m}$$

$$+ (h^2 \sum \{\eta(1 + |\nabla_h U|^2)^{m'/2}\}^{m/m'})^{2m'/m}\}/C_2$$

$$+ 4C_6 h^2 \sum |\nabla_h \eta| \eta (1 + |\nabla_h U|^2)^{m/2} + 4C_5' h^2 \sum \eta |\nabla_h \eta| (1 + |\nabla_h U|^2)^{m'/2}.$$

(d) Now let $\xi(P) \in \mathcal{S}_3$ and let $\phi(\xi(\cdot); P)$ be the solution to (5). We now derive an estimate on the norm of $\nabla_h \phi_{x_s}(\xi(\cdot); P)$ which is valid for all $\xi(P) \in \mathcal{S}_3$.

In this case, we have

$$h^2 \sum a_i(P, \xi, \nabla_h \phi) \zeta_{x_i} = -h^2 \sum (a_i(P, \xi, \nabla_h \phi))_{x_s} \mu_{x_i}$$

$$= -h^2 \sum \{\tilde{a}_{i,p_k} \phi_{x_k x_s} + \tilde{a}_{i,\xi} \xi_{x_s} + \tilde{a}_{i,x_s}\} \mu_{x_i},$$

where $\tilde{a}_{i,p_k}$, $\tilde{a}_{i,\xi}$ and $\tilde{a}_{i,x_s}$ is defined as in (9) but now the arguments of the associated integrands are $(P, \xi, \nabla_h \tilde{\phi})$ with $\tilde{\phi} = (1 - t)\phi(P - h_s) + t\phi(P)$, $(P, \xi, \nabla_h \phi(P - h_s))$ with $\tilde{\xi} = (1 - t)\xi(P - h_s) + t\xi(P)$, and $(\tilde{P}, \xi(P - h), \nabla_h \phi(P - h_s))$. Then, taking $\mu(P) = \eta^2(P)\phi_{x_s}$, assuming $m \geqq 2$ and $\kappa \geqq 0$ and setting $\bar{m} = m - 1 - \kappa$,

$$\begin{aligned}
&C_2 h^2 \sum \eta^2 (1 + |\nabla_h \phi|^2)^{(m-2)/2} |\nabla_h \phi_{x_s}|^2 \\
&\quad \leqq h^2 \sum (6C_3 \eta^2 \epsilon_1 + \eta^2 C_6 \epsilon_2/2 + \eta^2 C_5' \epsilon_3/2)(1 + |\nabla_h \phi|^2)^{\bar{m}/2} |\nabla_h \phi|^2 \\
&\qquad + C_4 (h^2 \sum (1 + |\nabla_h \xi|^2)^{\tilde{m}})^{1/2} (h^2 \sum \eta^4 |\nabla_h \phi_{x_s}|^2)^{1/2} \\
\text{(13)} &\qquad + h^2 \sum 6C_3 |\nabla_h \eta|^2 (1 + |\nabla_h \phi|^2)^{m/2} \epsilon_1 \\
&\qquad + C_6 (h^2 \sum |\nabla_h \xi|^m)^{2/m} (h^2 \sum \eta^{2m/\bar{m}}(1 + |\nabla_h \phi|^2)^{m/2})^{\bar{m}/m}/2\epsilon_2 \\
&\qquad + C_5' h^2 \sum \eta^2 (1 + |\nabla_h \phi|^2)^{\bar{m}/2}/2\epsilon_3 + 4h^2 \sum C_5' \eta (1 + |\nabla_h \phi|^2)^{m'/2} \\
&\qquad + 4C_6 (h^2 \sum \eta^{m/m'}(1 + |\nabla_h \phi|^2)^{m/2})^{m'/m}(h^2 \sum |\nabla_h \eta|^m |\nabla_h \xi|^m)^{1/m} \\
&\qquad + 2C_4 (h^2 \sum \eta^m |\nabla_h \phi|^m)^{1/m}(h^2 \sum |\nabla_h \eta|^{m/m'}(1 + |\nabla_h \xi|^2)^{\tilde{m}m/m'})^{m'/m}.
\end{aligned}$$

Now choose $\epsilon_1 = C_2/48C_3$, $\epsilon_2 = C_2/16C_6$, $\epsilon_3 = C_2/16C_5'$, $\epsilon_4 = C_2/16C_4$ and observe that for $p > 2$ we have $\eta^p \leqq \eta^2$ whenever $\eta \leqq 1$ to get an estimate on J_8 from (13).

(e) If $4/3 \leqq m < 2$, then we use (13) and the Schwartz Inequality to obtain an estimate.

Remark 3. Parts (d) and (e) of the last theorem are stated in their present generality so that we may show the exact dependence of the norms of $|\nabla_h \phi_{x_s}|$ on the properties of the coefficients.

In our next result we prove that if Ω is a rectangular region, then our "interior estimates" may hold for *all* of Ω_h.

COROLLARY 2. *If Ω is a rectangular domain with vertices $(0, 0)$, $(a, 0)$, (a, b), $(0, b)$, if a/b is rational and h divides a and b, if $a_1(P, U, \nabla_h U) = 0$ for $P \in s_2 + s_4$ and $a_2(P, U, \nabla_h U) = 0$ for $P \in s_1 + s_3$, if conditions (B), (C) and (D) are satisfied, and if $U(P)$ is a solution to (3) in $H_{m,0}^1(\Omega_h)$ with $m \geqq 2$, then there exists a positive constant J_{11} such that, for $s = 1$ and $s = 2$,*

$$h^2 \sum (1 + |\nabla_h U|^2)^{\bar{m}/2} |\nabla_h U_{\tilde{x}_s}|^2 \leqq J_{11}$$

where J_{11} depends on $\|\nabla_h U\|_m$ and the material constants of our conditions.

Proof. Let $\zeta = U_{x_1 \tilde{x}_1}$ where we have reflected U as an odd function. Then, integrating by parts in the x_1-direction, we get the identity

$$h^2 \sum_{\Omega_h} \{a_{i,\tilde{x}_1} U_{\tilde{x}_1 x_i} + f U_{x_1 \tilde{x}_1}\} = 0.$$

Proceeding as in the development of (9), we get

$$h^2 \sum_{\Omega_h'} \{\tilde{a}_{i,p_j} U_{\tilde{x}_1 x_j} U_{\tilde{x}_1 x_i} + \tilde{a}_{i,u} U_{\tilde{x}_1} U_{\tilde{x}_1 x_i} + \tilde{a}_{i,\tilde{x}_1} U_{\tilde{x}_1 x_i} + f U_{x_1 \tilde{x}_1}\} = 0$$

and hence, for $\epsilon_i > 0$ with $i = 1, 2$, we have

$$C_2 h^2 \sum (1 + |\nabla_h U|^2)^{\overline{m}/2} |\nabla_h U_{\pm_i}|^2$$

$$\leq h^2 \sum (1 + |\nabla_h U|^2)^{\overline{m}/2} \{C_6 |\nabla_h U| + C_5\} |\nabla_h U_{\pm_i}|$$

$$+ h^2 \sum C_4 (1 + |\nabla_h U|^2)^{m'/2} |\nabla_h U_{x_i}|$$

$$\leq ((\epsilon_1 + \epsilon_2)/2) h^2 \sum (1 + |\nabla_h U|^2)^{\overline{m}/2} |\nabla_h U_{\pm_i}|^2$$

$$+ (C_4^2/2\epsilon_2 + C_5^2/\epsilon_1 + C_6^2/\epsilon_1) h^2 \sum (1 + |\nabla_h U|^2)^{m/2}.$$

Now we shall prove the convergence of solutions of the difference equations to weak solutions of the differential equation. Our proof will make essential use of our interior estimates and the fact that our equation has *two* independent variables. A different convergence proof, with less stringent hypotheses, is to be found in Frehse [3, p. 331].

Let Ω be a domain with the $\partial \Omega$ in C^1. Let D, D' be domains with ∂D, $\partial D'$ in C^1 and such that $\bar{D}' \subset D$ and $\bar{D} \subset \Omega$. Let h_n be a sequence of positive numbers tending monotonically to zero such that $\Omega_{h_{n+1}} \supset \bar{\Omega}_{h_n}$. Let $h' > 0$ be such that for $n \geq N(h')$ we have that $D_{h_n} \supset D'$ and $\Omega \supset \bar{D}_{h_n}$.

Assume the appropriate—we have not yet made an assumption on m—hypotheses of Theorem 1 are satisfied so that a solution, $U_n(P) \in \mathfrak{a}_0(\Omega_h)$, exists to the difference equation

$$h_n^2 \sum_{P \in \Omega_{h_n}} \{a_i(P, U_n(P), \nabla_h U_n(P)) \zeta_{x_i}(P) + f(P, U_n(P), \nabla_h U_n(P)) \zeta(P)\} = 0$$

for all $\zeta(P) \in \mathfrak{a}_0(\Omega_h)$. Let $\mathfrak{U}_n(P)$ be the "filling-in" function given in Stummel [15, p. 180]; i.e.

$$(14) \qquad \mathfrak{U}_n(P) = h_n^2 \sum_Q S_h(P - Q) U_n(Q),$$

where Q runs over all mesh points of the plane and

$$S_h(P - Q) = [\sin(\pi(x_1 - \xi_1))/\pi(x_1 - \xi_1)][\sin(\pi(x_2 - \xi_2))/\pi(x_2 - \xi_2)]$$

with $P = (x_1, x_2)$ and $Q = (\xi_1, \xi_2)$.

By Theorem 3, we have that $U_n(P) \in H_2^2(D_{h_n})$ or we have $U_n(P) \in H_m^2(D_h)$ depending on the size of m. Let us assume $m \geq 2$. Then there exists a constant independent of h_n such that $||U_n(P)||_2^2 \leq J_4'$ for each $n \geq N(h')$ and this norm is taken over D_{h_n}. Hence, there exists a constant J_4'', independent of h_n, such that $||\mathfrak{U}_n(P)||_2^2 \leq J_4''$ where this norm is over D'; see Stummel [15, p. 181]. Applying the Variant of the Calderon Extension Theorem [13, p. 74], we have new functions $\mathfrak{V}_n(P) \in \mathfrak{K}_{2,0}^2(\Omega)$ such that $\mathfrak{V}_n(P) = \mathfrak{U}_n(P)$ in D'; these functions are also uniformly bounded over Ω in the $\mathfrak{K}_{2,0}^2(\Omega)$ norm. Hence, a subsequence of $\mathfrak{V}_n$, which we still call $\mathfrak{V}_n$, converges weakly to some $\mathfrak{V}_0(P) \in \mathfrak{K}_{2,0}^2(\Omega)$. Using Theorem 3.2.3 in [12, p. 70] and Theorem 10.2 in [4, p. 28], we conclude that a subsequence of $\mathfrak{V}_n$ converges strongly to $\mathfrak{V}_0$ in $\mathfrak{K}_{m,0}^1(\Omega)$. Since $\mathfrak{K}_{m,0}^1(\Omega)$ and $H_{m,0}^1(\Omega_h)$ are conditionally compact with respect to weak convergence, the above analysis shows that if a subsequence of $\mathfrak{U}_n$, which we still call $\mathfrak{U}_n$, converges weakly to an element $\mathfrak{U}_0 \in \mathfrak{K}_{m,0}^1(\Omega)$, then for any set $D' \subset \Omega$ satisfying the conditions above, we have that $\mathfrak{U}_n$ converges strongly to $\mathfrak{U}_0$ in $\mathfrak{K}_m^1(D')$.

We now claim that, for all $\zeta \in C_c^1(D')$,

$$(15) \quad \iint_{\Omega} \{a_i(P, \mathcal{V}_0(P), \nabla\mathcal{V}_0(P))\zeta_{x_i}(P) + f(P, \mathcal{V}_0(P), \nabla_h\mathcal{V}_0(P))\zeta(P)\} \, dx_1 \, dx_2 = 0.$$

To see this we first observe that $\mathcal{U}_n(P) = U_n(P)$ for $P \in \Omega_{h_n}$ and that, using the methods of proof in [15, pp. 186–187], derivatives and difference quotients of $\mathcal{U}_n(P)$ converge strongly to $\mathcal{U}_0(P)$ in $\mathcal{3C}_m^1(D')$. Now, to each $P \in \Omega_{h_n}$ with $n > N(h')$ let us associate the rectangular region $\Delta_n(P)$ determined by the vertices (x_1, x_2), $(x_1 + h, x_2)$, $(x_1 + h, x_2 + h)$, $(x_1, x_2 + h)$. Over $\Delta_n(P)$ let us define $\bar{U}_n(Q) = U_n(P)$ and $\nabla_h\bar{U}_n(Q) = \nabla_h U_n(P)$ for all $Q \in \Delta_n(P)$. Now we observe that there exists $\epsilon_i(n)$ for $i = 1, 2$ such that $\epsilon_i(n) \to 0$ as $n \to \infty$ and

$$\iint_{\Delta_n(P)} a_i(Q, \bar{U}_n, \nabla_h U_n)\zeta_{x_i}(Q) \, dQ$$

$$= \{\zeta_{x_i}(P) + \epsilon_i(n)\}h_n^2 a_i(P, U_n, \nabla_h U_n) + O(h_n^3)(1 + \epsilon_i(n)).$$

Also, from the strong convergence derived above and the fact that $\mathcal{U}_n(P) = U_n(P)$ for $P \in \Omega_h$ and the appropriate conditions in (A) to (E), we deduce the result that

$$\iint_{\Omega} \{a_i(Q, U_n, \nabla_h U_n) - a_i(Q, \mathcal{U}_n(Q), \nabla\mathcal{U}_n(Q))\}\zeta_{x_i}(Q) \, dQ \to 0 \quad \text{as } n \to \infty.$$

In a similar manner, we have that

$$\iint_{\Delta_n} f(Q, U_n, \nabla_h U_n)\zeta(Q) \, dQ = h_n^2 f(P, U_n, \nabla_h U_n)\zeta(P) + \epsilon(n)$$

and

$$\iint_{\Omega} \{f(Q, U_n, \nabla_h U_n) - f(Q, \mathcal{U}_n(Q), \nabla\mathcal{U}_n(Q))\}\zeta(Q) \, dQ \to 0 \quad \text{as } n \to \infty;$$

here $\epsilon(n) \to 0$ as $n \to \infty$ and comes from Condition (E) and $\zeta \in C_c^1(D')$. Using the fact that $U_n(P)$ solves the difference equation, the additivity of the integral, and the linearity of ζ and $\nabla\zeta$ in the integral, we conclude that $\mathcal{U}_0(P)$ is a weak solution of (1). The case $m \in [4/3, 2]$ proceeds along similar, but simpler, lines. Therefore, we have proved the next result.

THEOREM 5. *Let h_n be a monotonically decreasing sequence which converges to zero and such that $\Omega_{h_{n+1}} \supset \Omega_{h_n}$. Let the $\partial\Omega$ be in C^1. Let the appropriate hypotheses of Theorem 1 and Theorem 3 be satisfied. Let $U_n(P)$ be the discrete solutions to (3) with $h = h_n$ and let $\mathcal{U}_n(P)$ be as in (14). Then, there exists a subsequence of $\mathcal{U}_n(P)$ and an element $\mathcal{U}_0(P) \in \mathcal{3C}_{m,0}^1(\Omega)$ such that $\mathcal{U}_n(P)$ converges weakly to $\mathcal{U}_0(P)$ in $\mathcal{3C}_{m,0}^1(\Omega)$, $\mathcal{U}_n(P)$ converges strongly to $\mathcal{U}_0(P)$ in $\mathcal{3C}_m^1(D)$ with ∂D in C^1 and $\bar{D} \subset \Omega$ and the function $\mathcal{U}_0(P)$ is a weak solution to the differential equation (1); i.e. (15) holds for every $\zeta(P) \in \mathcal{3C}_{m,0}^1(\Omega)$.*

As an immediate consequence of the last result and that in Corollary 2 we have the following.

COROLLARY 3. *Let Ω be a rectangular region and let h_n tend monotonically to zero with $h_n > 0$, $\Omega_{h_{n+1}} \supset \Omega_{h_n}$ and $\partial\Omega_{h_{n+1}} \supset \partial\Omega_{h_n}$. Let $U_n(P)$ be a solution to (3) and let $\mathcal{U}_n(P)$ be as given in (14). Let the hypotheses of Corollary 2 be satisfied. Then there exists an element $\mathcal{U}_0(P) \in \mathcal{3C}_{2,0}^2(\Omega)$ such that some subsequence of $\mathcal{U}_n(P)$ converges*

weakly in $\mathfrak{IC}^2_{2,0}(\Omega)$ *to* $\mathfrak{U}_0(P)$ *and a further subsequence of* $\mathfrak{U}_n(P)$ *converges strongly to* $\mathfrak{U}_0(P)$ *in* $\mathfrak{IC}^1_{m,0}(\Omega)$. *This function* $\mathfrak{U}_0(P)$ *satisfies* (15).

Remark 4. From results in [12, pp. 78–81], [1] and [13], we may use estimates on norms of second-order difference quotients over all of Ω_h to prove the pointwise convergence of solutions of (***) to weak solutions of the differential equation. These same techniques also work using estimates of the $H^1_{m,0}(\Omega_h)$ norm of solutions to (3) provided m is sufficiently large relative to the number of independent variables; see [12, p. 83].

(II). *The Case that* $Q(P) \not\equiv 0$ *for* $P \in \Omega_h$. We shall assume that there exists a function $\tilde{q}(P) \in C^2(\bar{\Omega})$ such that $Q(P) = \tilde{q}(P)$ for $P \in \bar{\Omega}_h$ and $\tilde{q}(P) = q(P)$ for $P \in \partial\Omega$.

Now we reformulate (14) slightly. We seek a mesh function $U(P) \in \mathcal{Q}_0(\Omega_h)$ such that for all $\zeta(P) \in \mathcal{Q}_0(\Omega_h)$ we have

$$(16) \quad h^2 \sum_{\Omega_h} \{a_i(P, U + Q, \nabla_h(U + Q))\zeta_{x_i} + f(P, U + Q, \nabla_h(U + Q))\zeta\} = 0.$$

It is clear that all of the estimates we have obtained in (1) go through for (16) with slight modification. These new bounds will clearly depend on discrete l_m-norms of $Q(P)$ and its difference quotients.

We may extend our data to the case that $q(P)$ has a continuation $\tilde{q}(P)$ into Ω such that $\tilde{q}(P) \in \mathfrak{IC}^2_2(\Omega)$. Our analysis in this case would follow that of our proof of convergence.

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