

# The Finite Element Method for Infinite Domains. I

By Ivo Babuška\*

**Abstract.** Numerical methods (finite element methods) for the approximate solution of elliptic partial differential equations on unbounded domains are considered, and error bounds, with respect to the number of unknowns which have to be determined, are proven.

**1. Introduction.** The finite element method, its theory and practice, has recently become of interest in numerical analysis, see e.g. [1]–[13], and the papers of Aubin, Birkhoff, Bramble, Ciarlett, Schatz, Schultz, Varga, etc.

The theoretical analyses of the finite element method have been concerned with bounded domains. Strang and Fix, [1], [2], have, however, analyzed the finite element method with respect to an infinite domain (the space  $R_n$ ), but their procedure requires the solution of an infinite system of linear algebraic equations.

This paper will deal with the problem of finding, by the finite element method, an approximate solution of a boundary value problem for elliptic partial differential equations on an infinite domain by solving only a finite system of linear algebraic equations.

The approach will be shown on a model problem. Our task will be to find the solution of the equation (weak solution)

$$(1.1) \quad -\Delta u + u = f$$

on  $R_n$ , where  $u \in W_2^1(R_n)$  and  $f \in W_2^0(R_n)$ ,  $k \geq 0$ .

We will show that the rate of convergence on compact sets of  $R_n$  is practically the same as the rate of convergence for boundary value problems on bounded domains. The rate of convergence will turn out to be determined by the number of unknowns in the system of linear algebraic equations.

Our approach may be easily generalized to the case of an elliptic differential equation of order  $2m$ , provided that the coefficient of the zero order term of the equation is bounded above and below by positive constants.

We will analyze only the case when  $\Omega = R_n$ . By combining the approach described above with the results concerning bounded domains (see e.g. [6]–[11]), it is easy to get the corresponding results for unbounded domains with bounded boundary.

Throughout this paper, let  $x$  denote the  $n$ -dimensional vector in  $R_n$ , i.e.  $x \equiv (x_1, \dots, x_n)$  where  $x_i \in R_1$ ,  $i = 1, 2, \dots, n$ .

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Let  $\|x\|^2 = \sum_{i=1}^n x_i^2$  and  $|x| = \sum_{i=1}^n |x_i|$ . If  $a \in R_1$ , let  $x + a_i \equiv (x_1, \dots, x_{i-1}, x_i + a, x_{i+1}, \dots, x_n)$ . Let  $k$  denote a multi-integer, i.e.  $k \equiv (k_1, \dots, k_n)$ , where  $k_i$  is an integer,  $i = 1, 2, \dots, n$ . Let  $\|k\|^2 = \sum_{i=1}^n k_i^2$  and  $|k| = \sum_{i=1}^n |k_i|$ . Define the inequality  $k \geq 0$  to mean  $k_i \geq 0$ ,  $i = 1, 2, \dots, n$ . Let  $C$  denote a generic constant which may have different values wherever it appears in the text.

**2. The Spaces.** In this section, we shall introduce the spaces which will be used in the paper.

*Definition 2.1.* Let the space  $W_{2,\mu}^l(R_n)$ , with  $l \geq 0$  an integer,  $\mu$  real, be the Banach space of all functions  $u$  such that

$$(2.1) \quad \|u\|_{W_{2,\mu}^l(R_n)}^2 = \int_{R_n} e^{2\mu|x|} \sum_{|k| \leq l; k \geq 0} (D^k u)^2 dx < \infty,$$

where  $D^k = \partial^{k_1+k_2+\dots+k_n} / \partial x_1^{k_1} \dots \partial x_n^{k_n}$ . For  $\mu = 0$ , we get the usual Sobolev space. For  $\mu \neq 0$ , we get a weighted Sobolev space. Note that  $L_2(R_n) = W_{2,0}^0(R_n)$ .

Let us now introduce the so called  $B$  splines. For  $x \in R_1$ , let

$$(2.2) \quad \begin{aligned} \varphi_1(x) &= 1, & |x| < \frac{1}{2}, \\ &= 0, & |x| \geq \frac{1}{2}. \end{aligned}$$

Let  $t$  be an integer  $\geq 2$ . Starting with  $t = 2$ , we recursively define  $\varphi_t(x)$  as

$$(2.3) \quad \varphi_t(x) = \varphi_1(x) * \varphi_{t-1}(x), \quad t \geq 2,$$

where  $*$  denotes convolution.

Now, for  $x \in R_n$ ,  $x \equiv (x_1, \dots, x_n)$ , define

$$(2.4) \quad \varphi_t(x) = \prod_{i=1}^n \varphi_t(x_i), \quad t \geq 1,$$

and

$$(2.5) \quad \varphi_{t,i}(x) = \varphi_{t-1}(x_i - \frac{1}{2}) \prod_{i=1; i \neq i}^n \varphi_t(x_i), \quad t \geq 2, i = 1, \dots, n.$$

Let us mention some of the well-known properties of these functions which will be important later on:

(1)  $\varphi_t(x) \geq 0$ ,  $\varphi_{t,i}(x) \geq 0$  for all  $x \in R_n$ ,

(2)  $\varphi_t(x)$  and  $\varphi_{t,i}(x)$  have compact support.

(3) Denoting by  $F(\varphi)(\sigma)$ , the Fourier transform of  $\varphi(x)$ , we have, with  $\sigma = (\sigma_1, \dots, \sigma_n)$ ,

$$(2.6) \quad F(\varphi_t)(\sigma) = \prod_{i=1}^n \left( \frac{\sin \frac{1}{2}\sigma_i}{\frac{1}{2}\sigma_i} \right)^t,$$

$$(2.7) \quad F(\varphi_{t,i})(\sigma) = \left( \frac{\sin \frac{1}{2}\sigma_i}{\frac{1}{2}\sigma_i} \right)^{t-1} e^{i\sigma_i/2} \prod_{i=1; i \neq i}^n \left( \frac{\sin \frac{1}{2}\sigma_i}{\frac{1}{2}\sigma_i} \right)^t,$$

$$(2.8) \quad \frac{\partial \varphi_{t,i}(x)}{\partial x_i} = -\varphi_{t,i}(x) + \varphi_{t,i}(x + 1_i).$$

In the following, let  $U$  and  $V$  denote the functions defined on the set of all multi-integers  $k$ .

*Definition 2.2.* Let the space  $Q_\mu^{l,h}$ ,  $l = 0, 1$ ,  $h > 0$ ,  $\mu$  real, be the space of all functions  $U$  such that

$$(2.9) \quad \|U\|_{Q_\mu^{l,h}}^2 = h^n \sum_k |U(k)|^2 e^{2\mu h |k|} < \infty$$

(for  $l = 0$ ) and

$$(2.10) \quad \|U\|_{Q_\mu^{l,h}}^2 = h^n \left[ \frac{1}{h^2} \sum_k \sum_{i=1}^n |U(k+1_i) - U(k)|^2 e^{2\mu h |k|} + \sum_k |U(k)|^2 e^{2\mu h |k|} \right] < \infty$$

(for  $l = 1$ ).

Let  $U \in Q_\mu^{l,h}$  for  $l = 0$  and  $l = 1$ . Define the mapping  $S_j^h$ ,  $j \geq 1$ , by

$$(2.11) \quad S_1^h U = \sum_k U(k) \varphi_1(x/h - k),$$

and the mapping  $S_j^{h,i}$ ,  $j \geq 2$ , by

$$(2.12) \quad S_j^{h,i} U = \sum_k U(k) \varphi_{j,i}(x/h - k).$$

**THEOREM 2.1.** For  $|\mu|$  sufficiently small, there exist constants  $C_1$  and  $C_2$ ,  $0 < C_1 < C_2 < \infty$ , such that

$$(2.13) \quad C_1 \|U\|_{Q_\mu^{l,h}} \leq \|S_j^h U\|_{W_{2,\mu^l}(R_n)} \leq C_2 \|U\|_{Q_\mu^{l,h}}, \quad l = 0, 1, j > l,$$

and

$$(2.14) \quad C_1 \|U\|_{Q_\mu^{l,h}} \leq \|S_j^{h,i} U\|_{W_{2,\mu^l}(R_n)} \leq C_2 \|U\|_{Q_\mu^{l,h}}.$$

*Proof.* 1. Let us first consider (2.13) for  $l = 0$  and  $\mu = 0$ . It is sufficient to prove this inequality for  $h = 1$ .

In [14] we have proved that

$$(2.15) \quad F(S_1^1 U)(\sigma) = Z_1(\sigma) \sum_k U(k) \exp(i\langle k, \sigma \rangle),$$

where

$$(2.16) \quad Z_j(\sigma) = F(\varphi_j)(\sigma)$$

and

$$(2.17) \quad \langle k, \sigma \rangle = \sum_{i=1}^n k_i \sigma_i.$$

By a well-known property of the Fourier transform, we have

$$(2.18) \quad \|F(S_1^1 U)\|_{L_2(R_n)}^2 = (2\pi)^n \|S_1^1 U\|_{L_2(R_n)}^2.$$

Hence, we may write

$$(2.19) \quad \|F(S_1^1 U)\|_{L_2(R_n)}^2 = \sum_i \int_{\Omega_0} \left| \sum_k U(k) \exp(i\langle k, \sigma \rangle) \right|^2 |Z_1(\sigma - \tau)|^2 d\sigma,$$

where  $\Omega_0 = \{x; |x_i| < \pi\}$ . Since there exists a  $C$  such that  $|Z_1(\sigma)|^2 > C > 0$  on  $\Omega_0$ , we have, from (2.19),

$$(2.20) \quad \|F(S_i^1 U)\|_{L_s(R_n)}^2 \geq C \int_{\Omega_0} \left| \sum_k U(k) \exp(i\langle k, \sigma \rangle) \right|^2 d\sigma \geq C(2\pi)^n \sum_k |U(k)|^2.$$

From (2.20), (2.18) and (2.9), we obtain

$$(2.21) \quad \|S_i^1 U\|_{L_s(R_n)} \geq C \|U\|_{Q_0 \dots 1}.$$

From (2.19), we also have

$$(2.22) \quad \|F(S_i^1 U)\|_{L_s(R_n)}^2 \leq \int_{\Omega_0} \left| \sum_k U(k) \exp(i\langle k, \sigma \rangle) \right|^2 d\sigma \sum_l Z_{i,l}^2$$

with

$$(2.23) \quad Z_{i,l} = \max_{\sigma \in \Omega_0} |Z_i(\sigma - l)|.$$

From (2.6) it is clear that  $\sum_l Z_{i,l}^2 < \infty$ . Therefore,

$$(2.24) \quad \|S_i^1 U\|_{L_s(R_n)} \leq C \|U\|_{Q_0 \dots 1}.$$

Inequalities (2.21) and (2.24) together prove (2.13) for the case  $l = 0$  and  $\mu = 0$ . Inequality (2.14) can be proved in the same manner.

2. Let us now prove (2.13) for  $l = 0$  and  $\mu \neq 0$ . Let  $U \in Q_{\mu}^{0,1}$  and  $U_{\mu}(k) = U(k)e^{\mu|k|}$ . Using (2.13) for  $\mu = 0$  and  $U_{\mu}$ , we obtain, because of (2.9),

$$(2.25) \quad \int_{R_n} \left| \sum_k |U(k)| e^{\mu|k|} \varphi_i(x - k) \right|^2 dx \leq C \|U\|_{Q_{\mu}^2 \dots 1}.$$

On the other hand, we have

$$(2.26) \quad \|S_i^1 U\|_{W_{s,\mu}(R_n)}^2 = \int_{R_n} \left| \sum_k U(k) \varphi_i(x - k) \right|^2 e^{2\mu|x|} dx.$$

Now since  $\varphi_i(x) \geq 0$  for all  $x \in R_n$  and since  $\varphi_i$  has compact support, it follows from (2.26) that

$$(2.27) \quad \begin{aligned} \|S_i^1 U\|_{W_{s,\mu}(R_n)} &\leq \int_{R_n} \left| \sum_k |U(k)| \varphi_i(x - k) \right|^2 e^{2\mu|x|} dx \\ &\leq C \int_{R_n} \left| \sum_k |U(k)| e^{\mu|k|} \varphi_i(x - k) \right|^2 dx. \end{aligned}$$

From (2.25) and (2.27) we obtain

$$\|S_i^1 U\|_{W_{s,\mu}(R_n)} \leq C \|U\|_{Q_{\mu}^2 \dots 1}.$$

Now from (2.13) for  $\mu = 0$ , we have with  $C > 0$ , that

$$\begin{aligned} C \|U\|_{Q_{\mu}^2 \dots 1} &\leq C \int_{R_n} \left| \sum_k |U(k)| e^{\mu|k|} \varphi_i(x - k) \right|^2 dx \\ &\leq \int_{R_n} \left| \sum_k U(k) e^{\mu|k|} \varphi_i(x - k) \right|^2 dx \\ &= \int_{R_n} \left| \sum_k [U(k) e^{\mu|x|} \varphi_i(x - k) + U(k)(e^{\mu|k|} - e^{\mu|x|}) \varphi_i(x - k)] \right|^2 dx \\ &\leq 2 \left[ \|S_i^1 U\|_{W_{s,\mu}(R_n)}^2 + \mu^2 \int_{R_n} \left| \sum_k |U(k)| e^{\mu|k|} Q_k(x) \varphi_i(x - k) \right|^2 dx \right], \end{aligned}$$

where  $|Q_k(x)| \leq C$ , independently of  $k$ . Hence, we have

$$(2.28) \quad C \|U\|_{Q_{\mu^{\dots}}} \leq 2[\|S_i^1 U\|_{W_{\mu^{\dots}}(R_n)}^2 + \mu^2 C \|U\|_{Q_{\mu^{\dots}}}^2].$$

Therefore,

$$(2.29) \quad \|U\|_{Q_{\mu^{\dots}}} \leq C \|S_i^1 U\|_{W_{\mu^{\dots}}(R_n)},$$

for sufficiently small  $|\mu|$ .

This proves (2.13) for the case  $l = 0$  and  $\mu \neq 0$ ,  $\mu$  sufficiently small in absolute value. Inequality (2.14) can be proved for the case  $l = 0$  and  $\mu \neq 0$ ,  $\mu$  sufficiently small in absolute value, in the same manner.

3. Let us now prove the theorem for  $l = 1$ . We have

$$(2.30) \quad \begin{aligned} \frac{\partial S_i^1 U}{\partial x_s} &= \frac{1}{h} \left[ - \sum_k U(k) \left[ \varphi_{i,s} \left( \frac{x}{h} - k \right) - \varphi_{i,s} \left( \frac{x}{h} - 1_s - k \right) \right] \right] \\ &= \frac{1}{h} \left[ \sum_k (U(k+1_s) - U(k)) \varphi_{i,s} \left( \frac{x}{h} - k \right) \right] \\ &= \sum_k V(k) \varphi_{i,s} \left( \frac{x}{h} - k \right), \end{aligned}$$

where

$$(2.31) \quad V(k) = (U(k+1_s) - U(k))/h.$$

From (2.14) with  $l = 0$ , we obtain

$$(2.32) \quad C_1 \|V\|_{Q_{\mu^{\dots}}} \leq \left\| \frac{\partial S_i^1 U}{\partial x_s} \right\|_{W_{\mu^{\dots}}(R_n)} \leq C_2 \|V\|_{Q_{\mu^{\dots}}(R_n)}.$$

Inequality (2.14) follows immediately from (2.30), (2.31) and (2.32). This completes the proof of Theorem 2.1.

**3. Some Auxiliary Theorems about a Bilinear Form.** Let  $\mathfrak{H}_{\mu} = W_{2,\mu}^1(R_n) \times W_{2,-\mu}^1(R_n)$ . Define on  $\mathfrak{H}_{\mu}$  the bilinear form  $A$  by

$$(3.1) \quad A(u, v) = \int_{R_n} \left( \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + uv \right) dx.$$

We shall now prove some important properties of this bilinear form.

**THEOREM 3.1.** For sufficiently small  $|\mu|$ ,

$$(3.2) \quad |A(u, v)| \leq C \|u\|_{W_{\mu^{\dots}}(R_n)} \|v\|_{W_{-\mu^{\dots}}(R_n)},$$

$$(3.3) \quad \sup_{\|u\|_{W_{\mu^{\dots}}(R_n)} \leq 1} |A(u, v)| \geq C \|v\|_{W_{-\mu^{\dots}}(R_n)},$$

where  $C > 0$  and

$$(3.4) \quad \sup_{\|v\|_{W_{-\mu^{\dots}}(R_n)} \leq 1} |A(u, v)| \geq C \|u\|_{W_{\mu^{\dots}}(R_n)}, \quad C > 0.$$

*Proof.* 1. Inequality (3.2) is easily proved by the following inequalities:

$$\begin{aligned}
|A(u, v)| &= \left| \int_{R_n} \left( \sum_{i=1}^n e^{\mu|x_i|} \frac{\partial u}{\partial x_i} e^{-\mu|x_i|} \frac{\partial v}{\partial x_i} + e^{\mu|x_i|} u e^{-\mu|x_i|} v \right) dx \right| \\
(3.5) \quad &\leq \sum_{i=1}^n \left[ \int_{R_n} \left( e^{\mu|x_i|} \frac{\partial u}{\partial x_i} \right)^2 dx \right]^{1/2} \left[ \int_{R_n} \left( e^{-\mu|x_i|} \frac{\partial v}{\partial x_i} \right)^2 dx \right]^{1/2} \\
&\quad + \left[ \int_{R_n} (e^{\mu|x_i|} u)^2 dx \right]^{1/2} \left[ \int_{R_n} (e^{-\mu|x_i|} v)^2 dx \right]^{1/2} \\
&\leq C \|u\|_{W_{\mu, \mu^1}(R_n)} \|v\|_{W_{\mu, -\mu^1}(R_n)}.
\end{aligned}$$

2. Let  $v \in W_{2, -\mu}^1(R_n)$  and let  $u = ve^{-2\mu|x|}$ . By direct computation, we have

$$(3.6) \quad \frac{\partial (ve^{-2\mu|x|})}{\partial x_i} = e^{-2\mu|x|} \frac{\partial v}{\partial x_i} - 2\mu e^{-2\mu|x|} v \operatorname{sgn} x_i.$$

Hence,

$$\begin{aligned}
\|u\|_{W_{\mu, \mu^1}(R_n)}^2 &\leq 2 \left[ \int_{R_n} \left[ \sum_{i=1}^n e^{-4\mu|x_i|} \left( \frac{\partial v}{\partial x_i} \right)^2 + 4n\mu^2 v^2 e^{-4\mu|x_i|} + v^2 e^{-4\mu|x_i|} \right] e^{2\mu|x_i|} dx \right] \\
(3.7) \quad &\leq C \|v\|_{W_{\mu, -\mu^1}(R_n)}^2
\end{aligned}$$

which implies that  $u \in W_{2, \mu}^1(R_n)$ . Furthermore,

$$\begin{aligned}
|A(ve^{-2\mu|x|}, v)| &= \left| \int_{R_n} \sum_{i=1}^n \left[ \frac{\partial v}{\partial x_i} - 2\mu v \operatorname{sgn} x_i \right] \left[ \frac{\partial v}{\partial x_i} \right] e^{-2\mu|x_i|} dx + \int_{R_n} v^2 e^{-2\mu|x_i|} dx \right| \\
(3.8) \quad &\geq \|v\|_{W_{\mu, -\mu^1}(R_n)}^2 - 2\mu n \|v\|_{W_{\mu, -\mu^1}(R_n)}^2 \geq C \|v\|_{W_{\mu, -\mu^1}(R_n)}^2.
\end{aligned}$$

Inequality (3.8), together with (3.7), proves (3.3).

3. Replacing  $\mu$  by  $-\mu$  in the above discussion we have inequality (3.4).

Let us prove a further theorem.

**THEOREM 3.2.** For sufficiently small  $|\mu|$ , we have

$$(3.9) \quad |A(S_i^h U, S_i^h V)| \leq C \|S_i^h U\|_{W_{\mu, \mu^1}(R_n)} \|S_i^h V\|_{W_{\mu, -\mu^1}(R_n)},$$

$$(3.10) \quad \sup_{\|S_i^h U\|_{W_{\mu, \mu^1}(R_n)} \leq 1} |A(S_i^h U, S_i^h V)| \geq C \|S_i^h V\|_{W_{\mu, -\mu^1}(R_n)}$$

and

$$(3.11) \quad \sup_{\|S_i^h V\|_{W_{\mu, -\mu^1}(R_n)} \leq 1} |A(S_i^h U, S_i^h V)| \geq C \|S_i^h U\|_{W_{\mu, \mu^1}(R_n)}.$$

*Proof.* 1. Inequality (3.9) follows immediately from (3.2).

2. To prove (3.10), take  $V \in Q_{-\mu}^{1, h}$  and let  $U(k) = V(k)e^{-2\mu|k|h}$ , then

$$\begin{aligned}
\|U\|_{Q_{\mu^1, h}}^2 &\leq 2h^n \left[ \frac{1}{h^2} \sum_k \sum_{i=1}^n |V(k+1_i) - V(k)|^2 e^{-2\mu|k|h} \right. \\
(3.12) \quad &\quad \left. + (2\mu)^2 C \sum_k |V(k)|^2 e^{-2\mu|k|h} + \sum_k |V(k)|^2 e^{-2\mu|k|h} \right]
\end{aligned}$$

which implies that

$$\|U\|_{Q_{-\mu^1, \lambda}} \leq C \|V\|_{Q_{-\mu^1, \lambda}}.$$

Hence, by Theorem 2.1, we have

$$(3.13) \quad \|S_i^\lambda U\|_{W_{\mu, \mu^1}(R_n)} \leq C \|S_i^\lambda V\|_{W_{\mu, -\mu^1}(R_n)}.$$

Furthermore,

$$(3.14) \quad \begin{aligned} & |A(S_i^\lambda U, S_i^\lambda V)| \\ &= \left| \int_{R_n} \sum_{i=1}^n \left[ \frac{\partial}{\partial x_i} \sum_k V(k) e^{-2\mu|k|\lambda} \varphi_i\left(\frac{x}{h} - k\right) \right] \left[ \frac{\partial}{\partial x_i} \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right] dx \right. \\ & \quad \left. + \int_{R_n} \left[ \sum_k V(k) e^{-2\mu|k|\lambda} \varphi_i\left(\frac{x}{h} - k\right) \right] \left[ \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right] dx \right|. \end{aligned}$$

However,

$$(3.15) \quad \begin{aligned} & \frac{\partial}{\partial x_i} \sum_k V(k) e^{-2\mu|k|\lambda} \varphi_i\left(\frac{x}{h} - k\right) \\ &= \sum_k \frac{V(k+1_i) - V(k)}{h} \varphi_{i,i}\left(\frac{x}{h} - k\right) e^{-2\mu|k|\lambda} \\ & \quad - 2\mu \sum_k V(k+1_i) \varphi_{i,i}\left(\frac{x}{h} - k\right) Q_k e^{-2\mu|k|\lambda} \end{aligned}$$

with  $Q_k$  uniformly bounded, independently of  $k$ . Now we obtain

$$(3.16) \quad |A(S_i^\lambda U, S_i^\lambda V)| \geq C [\|V\|_{Q_{-\mu^1, \lambda}}^2 - 2\mu C \|V\|_{Q_{-\mu^1, \lambda}}^2] \quad \text{with } C > 0.$$

In fact,

$$\begin{aligned} & \int_{R_n} \left[ \sum_k V(k) e^{-2\mu|k|\lambda} \varphi_i\left(\frac{x}{h} - k\right) \right] \left[ \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right] dx \\ &= \int_{R_n} \left[ e^{-2\mu|x|} \left[ \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right]^2 \right. \\ & \quad \left. + \left[ \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right] (e^{-2\mu|k|\lambda} - e^{-2\mu|x|}) \left[ \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right] \right] dx. \end{aligned}$$

Using Theorem 2.1, we obtain

$$\begin{aligned} & \int_{R_n} \left[ \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) (e^{-2\mu|k|\lambda} - e^{-2\mu|x|}) \right] \left[ \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right] dx \\ & \leq \int_{R_n} \left[ \sum_k |V(k)| \varphi_i\left(\frac{x}{h} - k\right) |Q(\mu, x, k)| 2\mu \right] \left[ \sum_k |V(k)| \varphi_i\left(\frac{x}{h} - k\right) \right] e^{-2\mu|x|} dx \\ & \leq 2\mu C \|V\|_{Q_{-\mu^1, \lambda}}^2, \end{aligned}$$

where  $Q(\mu, x, k)$  is bounded independently of  $k$  because of the compact support of  $\varphi_i(x)$ . By a similar argument, using (3.15), we obtain

$$(3.17) \quad \begin{aligned} & \int_{R_n} \left[ \frac{\partial}{\partial x_i} \sum_k V(k) e^{-2\mu|k|\lambda} \varphi_i\left(\frac{x}{h} - k\right) \right] \left[ \frac{\partial}{\partial x_i} \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right] dx \\ &= \int_{R_n} e^{-2\mu|x|} \sum_{i=1}^n \left[ \frac{\partial}{\partial x_i} \sum_k V(k) \varphi_i\left(\frac{x}{h} - k\right) \right]^2 dx + R \end{aligned}$$

with  $|R| \leq 2\mu C \|\mathcal{V}\|_{0,-\mu}^2$ . Hence, for sufficiently small  $|\mu|$ , we have

$$(3.18) \quad |A(S_1^\dagger U, S_1^\dagger V)| \geq C \|\mathcal{V}\|_{0,-\mu}^2 \geq C \|S_1^\dagger \mathcal{V}\|_{\mathcal{W}_2, -\mu^2(H_2)}^2.$$

From (3.18), using (3.13), we get inequality (3.10). Inequality (3.11) is proved by changing  $\mu$  to  $-\mu$  in the above discussion.

This completes the proof of Theorem 3.2.

**4. The Finite Element Method and its Convergence.** In [6] we proved the following two theorems.

**THEOREM 4.1.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces with scalar product  $(\cdot, \cdot)_{H_1}$  and  $(\cdot, \cdot)_{H_2}$ , respectively.*

*Let  $B(u, v)$ ,  $u \in H_1$ ,  $v \in H_2$ , be a bilinear form on  $H_1 \times H_2$  such that*

$$(4.1) \quad |B(u, v)| \leq C_1 \|u\|_{H_1} \|v\|_{H_2},$$

$$(4.2) \quad \sup_{\|u\|_{H_1} \leq 1} |B(u, v)| \geq C_2 \|v\|_{H_2},$$

and

$$(4.3) \quad \sup_{\|v\|_{H_2} \leq 1} |B(u, v)| \geq C_3 \|u\|_{H_1},$$

with  $C_1 < \infty$ ,  $C_2 > 0$ ,  $C_3 > 0$ .

*Let  $H_2'$  be the space of bounded linear functionals on  $H_2$ . Let  $f \in H_2'$ . Then there exists exactly one element  $u_0 \in H_1$  with*

$$(4.4) \quad \|u_0\|_{H_1} \leq \|f\|_{H_2'} / C_3,$$

such that

$$(4.5) \quad B(u_0, v) = f(v)$$

for all  $v \in H_2$ .

**THEOREM 4.2.** *Let the assumptions of the Theorem 4.1 be fulfilled. Further, let  $M_1$  and  $M_2$  be closed subspaces of  $H_1$  and  $H_2$ , respectively. For every  $v \in M_2$ , let*

$$(4.6) \quad \sup_{\|u\|_{H_1} \leq 1} |B(u, v)| \geq d_2(M_1, M_2) \|v\|_{H_2}$$

with  $d_2(M_1, M_2) > 0$ , and for every  $u \in M_1$  let

$$(4.7) \quad \sup_{\|v\|_{H_2} \leq 1} |B(u, v)| \geq d_3(M_1, M_2) \|u\|_{H_1}$$

with  $d_3(M_1, M_2) > 0$ .

*Let  $f \in H_2'$  be given and let  $u_0$  denote that element of  $H_1$  for which*

$$(4.8) \quad B(u_0, v) = f(v)$$

holds for all  $v \in H_2$  (such an element exists and is unique by Theorem 4.1).

*Assume there exists  $w \in M_1$  such that*

$$(4.9) \quad \|u_0 - w\|_{H_1} \leq \vartheta.$$

Furthermore, let  $\hat{u}_0 \in M_1$  such that

$$(4.10) \quad B(\hat{u}_0, v) = f(v)$$



for all  $v \in M_2$ . Then

$$(4.11) \quad \|u_0 - \hat{u}_0\|_{H_1} \leq \left[ 1 + \frac{C_1}{d_3(M_1, M_2)} \right] \vartheta.$$

We will utilize the Theorems 4.1 and 4.2 to analyze the solution of the equation

$$(1.1) \quad -\Delta u + u = f.$$

Let us first prove Lemma 4.1.

LEMMA 4.1. Let  $f \in W_{2,-\mu}^0(R_n)$ , then

$$(4.12) \quad \left| \int_{R_n} \frac{\partial f}{\partial x_i} u \, dx \right| \leq \|f\|_{W_{2,-\mu^0}(R_n)} \|u\|_{W_{2,\mu^1}(R_n)}.$$

*Proof.* For any  $f \in C$  with compact support,

$$\left| \int_{R_n} \frac{\partial f}{\partial x_i} u \, dx \right| = \left| \int_{R_n} f \frac{\partial u}{\partial x_i} \, dx \right| \leq \|f\|_{W_{2,-\mu^0}(R_n)} \|u\|_{W_{2,\mu^1}(R_n)}.$$

The functions  $f \in C^\infty$  with compact support are dense in the space  $W_{2,-\mu}^0(R_n)$ , hence, (4.12) holds for all  $f \in W_{2,-\mu}^0(R_n)$ . The following theorem will complete the preparation for the main results of the paper.

THEOREM 4.3. Let  $f \in W_{2,\mu}^1(R_n)$ ,  $l \geq 0$ ,  $|\mu|$  sufficiently small. Then there exists exactly one solution  $u$  of Eq. (1.1) in  $W_{2,\mu}^1$ , such that

$$(4.13) \quad \|u\|_{W_{2,\mu^{l+1}}(R_n)} \leq C \|f\|_{W_{2,\mu^l}(R_n)}.$$

*Proof.* Since  $f \in W_{2,\mu}^0(R_n)$ , we have

$$T_l(u) = \int_{R_n} f u \, dx, \quad \|T_l\|_{(W_{2,-\mu^l}(R_n))'} \leq \|f\|_{W_{2,\mu^0}(R_n)}.$$

Using Theorems 4.1 and 3.1, we have

$$\|u\|_{W_{2,\mu^1}(R_n)} \leq C \|f\|_{W_{2,\mu^0}(R_n)} \leq C \|f\|_{W_{2,\mu^1}(R_n)}.$$

Differentiating both sides of Eq. (1.1), using Theorems 4.1, 3.1 and Lemma 4.1, we obtain

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{W_{2,\mu^1}(R_n)} \leq C \|f\|_{W_{2,\mu^0}(R_n)} \leq C \|f\|_{W_{2,\mu^1}(R_n)}.$$

Hence,

$$\|u\|_{W_{2,\mu^2}(R_n)} \leq C \|f\|_{W_{2,\mu^1}(R_n)}.$$

By differentiating Eq. (1.1)  $l + 1$  times and using induction, we obtain (4.13) for Eq. (1.1).

Let us now describe the finite element method. Let  $f \in W_{2,0}^1(R_n)$  and let  $\psi(h)$  be a decreasing function of  $h$  defined for  $h > 0$ . Let

$$(4.14) \quad u_{h,\psi,i}(x) = \sum_{|k| \leq \psi(h)} C(k) \varphi_i(x/h - k).$$

Let us determine the coefficients  $C(k)$  such that

$$(4.15) \quad A(u_{h,\psi,j}, v) = \int_{R_n} f v \, dx$$

for all  $v$  of the form (4.14).

We can now prove the main theorem of the paper.

**THEOREM 4.4.** *Let  $f \in W_{2,0}^l(R_n)$  and let  $\psi(h) = h^{-\epsilon}$  for any given  $\epsilon > 0$ . Let  $u_0$  be the solution of (1.1) in  $W_{2,0}^l(R_n)$ .*

*Then on every compact domain  $\Omega$ ,*

$$(4.16) \quad \|u_0 - u_{h,\psi,j}\|_{W_{2,0}^l(\Omega)} \leq Ch^{l+1} \|f\|_{W_{2,0}^l(R_n)}$$

for  $j \geq l + 2$ , where

$$(4.17) \quad \|u\|_{W_{2,0}^l(\Omega)} = \int_{\Omega} \left[ \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i} \right)^2 + u^2 \right] dx.$$

*Proof.* From Theorem 4.3, we have  $u \in W_{2,0}^{l+2}(R_n)$ . Therefore,  $u_0 \in W_{2,\mu}^{l+2}(R_n)$  for  $\mu \leq 0$  also. Let us use a cut-off function  $\chi(x) \in C^\infty$  with  $\chi(x) = 1$  for  $|x| \leq 1$  and  $\chi(x) = 0$  for  $|x| \geq 2$ . Define  $\chi_{h,\psi}(x) = \chi((\psi(h))^{-1}3x)$  and  $u_{0,h}(x) = \chi_{h,\psi}(x)u_0(x)$ . Then

$$u_{0,h}(x) = 0, \quad \text{for } |x| \geq \frac{2}{3}\psi(h),$$

$$u_{0,h}(x) - u_0(x) = 0, \quad \text{for } |x| \leq \frac{1}{3}\psi(h),$$

and

$$\|u_{0,h}\|_{W_{2,0}^{l+2}(R_n)} \leq C \|f\|_{W_{2,0}^l(R_n)}.$$

In [14], we have shown that for  $j \geq l + 2$  there exist  $d(h, k)$  such that

$$(4.18) \quad \left\| u_{0,h} - \sum_k d(h, k) \varphi_i \left( \frac{x}{h} - k \right) \right\|_{W_{2,0}^l(R_n)} \leq C \|u_{0,h}\|_{W_{2,0}^{l+2}(R_n)} h^{l+1}.$$

We have also shown that the support of

$$w(x) = \sum_k d(h, k) \varphi_i \left( \frac{x}{h} - k \right)$$

lies in a  $Lh$  neighborhood of the support of  $u_{0,h}$ , i.e. the support of  $w(x)$  lies in the set  $Q_{h,\psi}^* = \{x; |x| \leq \frac{2}{3}\psi(h) + Lh\}$ . Therefore, for  $h$  small enough,

$$Q_{h,\psi} = \{x; |x| \leq \psi(h)\} \supset Q_{h,\psi}^*.$$

For  $\mu < 0$ ,  $|\mu|$  sufficiently small, we obviously have

$$(4.19) \quad \|u_0 - w\|_{W_{2,\mu}^l(R_n)} \leq \|u_{0,h} - w\|_{W_{2,\mu}^l(R_n)} + \|u_{0,h} - u\|_{W_{2,\mu}^l(R_n)} \\ \leq C[h^{l-1} \|f\|_{W_{2,0}^l(R_n)} + e^{\mu\psi(h)/3} \|f\|_{W_{2,0}^l(R_n)}],$$

since  $u_{0,h} - u = 0$  for  $|x| \leq \frac{1}{3}\psi(h)$ . However,  $e^{\mu\psi(h)/3} = e^{\mu h^{-\epsilon}/3} \leq Ch^{l+1}$ . Therefore,

$$(4.20) \quad \|u_0 - w\|_{W_{2,\mu}^l(R_n)} \leq Ch^{l+1} \|f\|_{W_{2,0}^l(R_n)},$$

and hence, from Theorems 4.2 and 3.2, we have

$$(4.21) \quad \|u_0 - u_{h,\psi,j}\|_{W_{2,\mu}^l(R_n)} \leq Ch^{l+1} \|f\|_{W_{2,0}^l(R_n)},$$

the desired result.

Let us count the number  $T$  of unknowns in (4.18) which we have to determine by solving a system of linear equations. It is clear that  $T$  is of the order  $h^{-n(1+\epsilon)}$ .

Now let the "effective"  $H$  (see [7]) be defined by

$$(4.22) \quad H = (T)^{-1/n}.$$

Thus  $H = h^{1+\epsilon}$ . Hence, instead of (4.16), we may write

$$(4.23) \quad \begin{aligned} \|u_0 - u_{h,\psi,j}\|_{W_{2,0}^1(\Omega)} &\leq C(\epsilon, \Omega) H^{l+1-\epsilon} \|f\|_{W_{2,0}^1(\mathbb{R}^n)} \\ &= C(\epsilon, \Omega) T^{-(l+1)/n+\epsilon} \|f\|_{W_{2,0}^1(\mathbb{R}^n)}. \end{aligned}$$

Now, by the same manner as in [7], we can show that the rate of convergence indicated in (4.23) is the highest possible rate of convergence on every compact domain, provided that we neglect the  $\epsilon$ .

Institute for Fluid Dynamics  
University of Maryland  
College Park, Maryland 20742

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