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# A Note Concerning the Two-Step Lax-Wendroff Method in Three Dimensions

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**Abstract.** The two-step Lax-Wendroff method in three spatial dimensions is discussed and, dealing with its linear stability in the hydrodynamic case, the sufficiency of the von Neumann condition is proved.

In their paper [1], Rubin and Preiser suggest a difference scheme for the conservation law:

$$W_t + \partial f_i/\partial x_i = 0. \tag{1}$$

Their scheme is an extension of Richtmyer's two-step method to three spatial dimensions. In order to deal with the linear stability they take the linearized equation: $W_t + A_i \cdot \partial w/\partial x_i$ (where $A_i = \partial f_i/\partial W$ are taken locally constant), and, in order to get a stability criterion, they compute the amplification matrix:

$$G = I - \tfrac{2}{3} i\lambda[\text{Cos } \xi_1 + \text{Cos } \xi_2 + \text{Cos } \xi_3]M - 2\lambda^2 M^2. \tag{2}$$

Here $\lambda = \Delta t/\Delta x_1 = \Delta t/\Delta x_2 = \Delta t/\Delta x_3$, $\xi_i = k_i \Delta x_i$ (where $k_i$ are the dual variables) and $M = A_1 \text{ Sin } \xi_1 + A_2 \text{ Sin } \xi_2 + A_3 \text{ Sin } \xi_3$.

To prove sufficiency of the von Neumann condition, Rubin and Preiser use a theorem due to Kreiss [3] where the dissipativity of the scheme is assumed. However, it is easy to verify that their scheme is not dissipative because for $\xi = (\pi, 0, 0)$ (so that $|\xi| = \pi$), for example, $M$ is the null matrix and $G = I$ so that its eigenvalues are on the unit circle, as is the case for two dimensions (see [4]).

We give a different proof for the sufficiency of the von Neumann condition but only for the hydrodynamic case. (This is an extension of Richtmyer's proof in [2] to three spatial dimensions.)

In this case, $W = (\rho, \rho u, \rho v, \rho w, E)$, where $\rho$, $E$ and $\mathbf{V} = (u, v, w)$ are the density, total energy per unit volume and the velocity vector, respectively. We shall make use of the following sufficiency theorem (see [2]): "If $G$ has a complete set of eigenvectors and there exists a constant $\delta$ such that $\Delta \geqq \delta > 0$, where $\Delta^2$ is the Gram determinant of the normalized eigenvectors, then the von Neumann condition is sufficient as well as necessary for stability".

Instead of calculating the eigenvectors of $G$, we shall consider another matrix $G'$ obtained from $G$ by a similarity transformation. We introduce $W' = (\rho, u, v, w, p)$, where $p$ is the pressure, and the transformation is

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$$(2.a) \qquad dW = P \cdot dW',$$

$$(2.b) \qquad A_i = PA_i'P^{-1},$$

and so $M = PM'P^{-1}$ and $G = PG'P^{-1}$. This is done because the original $A_i$ are too complicated.

If we compute $P$ from (2.a), we get that

$$(3) \qquad \det P = \rho^4 \, \partial e/\partial p,$$

where $e$ is the internal energy per unit mass. It turns out that $P$ is triangular so that $\det(P^{-1}) = (\det P)^{-1}$.

Let $y_i$ be the normalized eigenvectors of $G$ (and $M$), then $\alpha_i P^{-1} y_i$ are the normalized eigenvectors of $G'$ (and $M'$), where $\alpha_i > 0$ are the normalizing factors.

If we define

$$(4) \qquad \Delta_1 = |\det(y_1, y_2, \cdots, y_n)|,$$

$$(5) \qquad \Delta_2 = |\det(\alpha_1 P^{-1} y_1, \alpha_2 P^{-1} y_2, \cdots, \alpha_n P^{-1} y_n)|,$$

and $\alpha = \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n > 0$, then

$$\begin{aligned} \Delta_2 &= |\det[\alpha P^{-1}(y_1, y_2, \cdots, y_n)]| \\ &= \alpha\, |\det P^{-1}| \cdot |\det(y_1, y_2, \cdots, y_n)| \\ &= \alpha\, |\det P^{-1}|\, \Delta_1 = \frac{\alpha}{\rho^4\, \partial e/\partial p} \Delta_1. \end{aligned}$$

This is the case because $\rho$ is always bounded away from zero and in the usual fluids the same holds for $\partial e/\partial p$ and consequently for $\det P^{-1}$. Therefore, $\Delta_1$ is bounded away from zero if and only if $\Delta_2$ is.

Rubin and Preiser found $M'$ to be

$$(6) \qquad M' = L \begin{bmatrix} u' & \rho \operatorname{Cos} r & \rho \operatorname{Cos} s & \rho \operatorname{Cos} t & 0 \\ 0 & u' & 0 & 0 & 1/\rho \operatorname{Cos} r \\ 0 & 0 & u' & 0 & 1/\rho \operatorname{Cos} s \\ 0 & 0 & 0 & u' & 1/\rho \operatorname{Cos} t \\ 0 & \rho C^2 \operatorname{Cos} r & \rho C^2 \operatorname{Cos} s & \rho C^2 \operatorname{Cos} t & u' \end{bmatrix}$$

where $L = (\operatorname{Sin}^2 \xi_1 + \operatorname{Sin}^2 \xi_2 + \operatorname{Sin}^2 \xi_3)^{1/2}$, $\operatorname{Cos} r = (\operatorname{Sin} \xi_1)/L$, $\operatorname{Cos} s = (\operatorname{Sin} \xi_2)/L$, $\operatorname{Cos} t = (\operatorname{Sin} \xi_3)/L$ and $u' = u \operatorname{Cos} r + v \operatorname{Cos} s + w \operatorname{Cos} t$.

A direct computation of its eigenvectors shows

$$(7) \qquad \Delta_2 = \left| \det \begin{bmatrix} 1 & 0 & 0 & -k\rho/C \cdot \operatorname{Cos} r & k\rho/C \cdot \operatorname{Cos} r \\ 0 & -\operatorname{Cos} r \cdot \operatorname{Ctg} s & -\operatorname{Cos} t/\operatorname{Sin} s & k & k \\ 0 & \operatorname{Sin} s & 0 & k \operatorname{Cos} s/\operatorname{Cos} r & k \operatorname{Cos} s/\operatorname{Cos} r \\ 0 & -\operatorname{Cos} t \cdot \operatorname{Ctg} s & \operatorname{Cos} r/\operatorname{Sin} s & k \operatorname{Cos} t/\operatorname{Cos} r & k \operatorname{Cos} t/\operatorname{Cos} r \\ 0 & 0 & 0 & -k\rho C/\operatorname{Cos} r & k\rho C/\operatorname{Cos} r \end{bmatrix} \right| = \frac{2\rho C k^2}{\operatorname{Cos}^2 r}.$$

$k^2 = (\text{Cos}^2 r)/(\rho^2 C^2 + 1 + \rho^2/C^2)$ is a normalizing factor so that finally

$$\Delta_2 = 2\rho C/(\rho^2 C^2 + 1 + \rho^2/C^2). \tag{8}$$

We see that $\Delta_2$, and so $\Delta_1$, is bounded away from zero. Hence, by the sufficiency theorem quoted above, the von Neumann condition, namely $\Delta t/\Delta x \leqq 1/\sqrt{3}(|\mathbf{V}| + C)$, implies linear stability.

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