

Product Integration for the Generalized Abel Equation

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Abstract. The solution of the generalized Abel integral equation

$$g(t) = \int_0^t \{k(t, s)/(t - s)^\alpha\} f(s) ds, \quad 0 < \alpha < 1,$$

where $k(t, s)$ is continuous, by the product integration analogue of the trapezoidal method is examined. It is shown that this method has order two convergence for $\alpha \in [\alpha_1, 1)$ with $\alpha_1 \doteq 0.2117$. This interval contains the important case $\alpha = \frac{1}{2}$. Convergence of order two for $\alpha \in (0, \alpha_1)$ is discussed and illustrated numerically. The possibility of constructing higher order methods is illustrated with an example.

1. Introduction. The generalized Abel integral equation

$$(1.1) \quad g(t) = \int_0^t \frac{k(t, s)}{(t - s)^\alpha} f(s) ds, \quad 0 < \alpha < 1, 0 \leq t \leq T < \infty,$$

where $k(t, s)$ is continuous for $0 \leq s \leq t \leq T$ and

$$(1.2) \quad k(t, t) \neq 0,$$

is of considerable importance in many fields. For example, consider the equation

$$(1.3) \quad Y(y) = 2 \int_y^1 \frac{R(r)r}{(r^2 - y^2)^{1/2}} dr, \quad 0 \leq y \leq 1,$$

which occurs frequently in mathematical physics. By a change of variables, (1.3) can be transformed to

$$(1.4) \quad g(t) = \int_0^t \frac{f(s)}{(t - s)^{1/2}} ds, \quad 0 \leq t \leq 1,$$

which is an important special case of (1.1). Various numerical methods for performing the inversion of (1.3) as well as (1.4) have been developed. A bibliography can be found in Minerbo and Levy [1].

As a second example, consider equations of the type

$$(1.5) \quad \begin{aligned} & (2\pi t)^{-1/2} \exp[-\frac{1}{2}\{a(t)\}^2/t] \\ &= \int_0^t \{2\pi(t - s)^{-1/2}\} \exp[-\frac{1}{2}\{a(t) - a(s)\}^2/(t - s)] f(s) ds, \end{aligned}$$

where $a(t)$ is a differentiable function with $a(0) > 0$. Such equations and systems of

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similar structure appear in the analysis of diffusion processes {see Fortet [2]}. A method for solving (1.5) has been proposed by Durbin [3].

However, none of the methods suggested for (1.4) as well as (1.5) can be applied to the general case (1.1). Only methods based on the concept of product integration {see Young [4]} appear to be applicable. Such methods have been suggested on the basis of rigorous numerical experimentation by Linz [5] and Noble [6]. Independent support for this can be found in [3], though the product integration type method developed there is not generally applicable.

A theoretical justification for the use of product integration methods can be found in Weiss and Anderssen [7]. They show that the midpoint product integration method is convergent of order one if $f(s)$ and $k(t, s)$ are suitably smooth and convergent of order at least $(1 - \alpha)$ if weaker smoothness conditions apply.

In this paper, we extend the result of [7] by examining the applicability of other product integration methods to (1.1). Initially, we show that the use of trapezoidal product integration leads to a method with second order convergence. The result again depends on the smoothness of $f(t)$ and $k(t, s)$. In addition, the possibility of determining higher order methods is discussed.

The existence and uniqueness of solutions of (1.1) is considered in Section 2. In Section 3, we give a basic definition and a lemma required in the subsequent analysis of Section 4, where we define the trapezoidal method and investigate its convergence. In Section 5, we examine the effect of computational errors, and a numerical example is presented in Section 6. A third order method is considered in Section 7.

2. Analytic Solution of (1.1). The existence and uniqueness of solutions of (1.1) is guaranteed under the following conditions:

THEOREM 2.1 {KOWALEWSKI [8, Section 1, pp. 80–82]}. *The integral equation (1.1) has a unique continuous solution $f(t)$, $t \in [0, T]$, if $k(t, s)$ and $g(t)$ satisfy*

- (i) $k(t, s)$ and $\partial k(t, s)/\partial t$ are continuous for $0 \leq s \leq t \leq T$,
- (ii) $k(t, t) \neq 0$ for $t \in [0, T]$,
- (iii) $G(t) = d/dt \int_0^t (g(s)/(t - s)^{1-\alpha}) ds$ is continuous for $t \in [0, T]$.

3. Preliminaries. To solve (1.1) we introduce the grid $\{t_i = i\Delta t, i = 0, 1, \dots, I = T/\Delta t\}$ with gridspacing Δt . An approximate solution, defined with respect to these grid points, will be denoted by \tilde{f}_i .

Definition 3.1. Let $\tilde{f}_i(\Delta t)$ ($i = 0, 1, \dots$) denote the approximation to $f(t_i)$ obtained by a finite difference method with gridspacing Δt . Then, the method is said to be convergent if and only if

$$\max_{0 \leq i \leq I} |\tilde{f}_i(\Delta t) - f(t_i)| \rightarrow 0$$

as $\Delta t \rightarrow 0, I \rightarrow \infty$ such that $I\Delta t = T$. Furthermore, the method is said to be *convergent of order p* , if p is the largest value for which there exists a finite constant C and a Δt_0 such that

$$\max_{0 \leq i \leq I} |\tilde{f}_i(\Delta t) - f(t_i)| \leq C\Delta t^p$$

for all $0 < \Delta t \leq \Delta t_0$.

The convergence proof developed in Section 4 is based on the following lemma which is a consequence of the standard results for regular infinite systems of algebraic equations {see Kantorovich and Krylov [9, Section 2, p. 27]}.

LEMMA 3.1. *Let the real numbers x_i ($i = 1, 2, \dots$) satisfy*

$$|x_1| \leq K, \quad |x_2| \leq K, \quad K \text{ is const,}$$

$$|x_{i+1}| \leq \sum_{j=1}^i |a_{i,j}| |x_j| + |b_i| \quad (i = 2, 3, \dots),$$

with

$$\rho_i = 1 - \sum_{j=1}^i |a_{i,j}| > 0,$$

$$|b_i| \leq K\rho_i \quad (i = 2, 3, \dots),$$

then $|x_i| \leq K$ ($i = 1, 2, \dots$).

Throughout the paper, we use the notation

$$g_i = g(t_i), \quad f_i = f(t_i), \quad k_{i,j} = k(t_i, t_j), \quad \epsilon_i = \bar{f}_i - f_i.$$

In addition, the subscripted capitals C_1, C_2, \dots will always denote positive real constants.

4. The Trapezoidal Product Integration Method. We replace $k(t_i, s)f(s)$ in

$$g_i = \int_0^{t_i} \frac{k(t_i, s)}{(t_i - s)^\alpha} f(s) ds \quad (i = 1, \dots, I)$$

by the piecewise linear function

$$M_{\Delta i}(t_i, s) = [(t_{i+1} - s)k_{i,i}\bar{f}_i + (s - t_i)k_{i,i+1}\bar{f}_{i+1}]/\Delta t$$

$$\{t_j \leq s \leq t_{j+1} \ (j = 0, \dots, i - 1)\},$$

and thus obtain the following lower triangular system for \bar{f}_i ($i = 1, \dots, I$):

$$(4.1) \quad \sum_{j=1}^i \bar{f}_j k_{i,j} W_{i-j} = g_i / (\Delta t^{1-\alpha} \phi(\alpha)) - \bar{f}_0 k_{i,0} \bar{W}_i \quad (i = 1, \dots, I),$$

where

$$\phi(\alpha) = \frac{1}{1 - \alpha} - \frac{1}{2 - \alpha},$$

$$W_0 = \frac{1}{\phi(\alpha) \Delta t^{2-\alpha}} \int_0^{\Delta t} \frac{\Delta t - t}{t^\alpha} dt = 1,$$

$$W_i = \frac{1}{\phi(\alpha) \Delta t^{2-\alpha}} \left(\int_{(i-1)\Delta t}^{i\Delta t} \frac{t - (i-1)\Delta t}{t^\alpha} dt + \int_{i\Delta t}^{(i+1)\Delta t} \frac{(i+1)\Delta t - t}{t^\alpha} dt \right)$$

$$= (i+1)^{2-\alpha} - 2i^{2-\alpha} + (i-1)^{2-\alpha} \quad (i = 1, \dots, I-1),$$

$$\begin{aligned} \mathbb{W}_l &= \frac{1}{\phi(\alpha) \Delta t^{2-\alpha}} \int_{(l-1)\Delta t}^{l\Delta t} \frac{t - (l-1)\Delta t}{t^\alpha} dt \\ &= \frac{1}{\phi(\alpha)} \left\{ \frac{1}{2-\alpha} (l^{2-\alpha} - (l-1)^{2-\alpha}) - \frac{(l-1)}{1-\alpha} (l^{1-\alpha} - (l-1)^{1-\alpha}) \right\} \end{aligned}$$

($l = 1, \dots, I$).

For the starting value \tilde{f}_0 , we take

$$f_0 = f_0 = \lim_{t \rightarrow 0} (g(t)t^{\alpha-1})(1-\alpha)/k_{0,0},$$

which exists, whenever (1.1) has a continuous solution. It follows from (1.2) that (4.1) is nonsingular.

The numerical process (4.1) defines the product integration analogue of the trapezoidal rule applied to nonsingular first kind Volterra equations {see Linz [10]}. We shall refer to it as the *trapezoidal product integration method*.

As in [7], we first investigate the convergence of (4.1) for the case $k(t, s) \equiv 1$.

THEOREM 4.1. *If $f(t)$ is three times differentiable and $f'''(t)$ is bounded for $0 \leq t \leq T$, then, for $\alpha \in [\alpha_1, 1)$ with $\alpha_1 \doteq 0.2117$, (4.1) is convergent of order 2.*

Let $f_{\Delta t}(s)$ denote the piecewise linear function interpolating $f(s)$ at the grid points,

$$f_{\Delta t}(s) = [(t_{j+1} - s)f_j + (s - t_j)f_{j+1}]/\Delta t \quad \{t_j \leq s \leq t_{j+1} \ (j = 0, \dots, I - 1)\},$$

and let

$$e_{\Delta t}(s) = f_{\Delta t}(s) - f(s).$$

By subtracting

$$\frac{1}{\Delta t^{1-\alpha}\phi(\alpha)} \left(\int_0^{t_i} \frac{f_{\Delta t}(s)}{(t_i - s)^\alpha} ds - \int_0^{t_i} \frac{e_{\Delta t}(s)}{(t_i - s)^\alpha} ds - g_i \right) = 0$$

from (4.1), we obtain

$$(4.2) \quad \sum_{j=1}^i \epsilon_j \mathbb{W}_{i-j} = -\frac{1}{\Delta t^{1-\alpha}\phi(\alpha)} \int_0^{t_i} \frac{e_{\Delta t}(s)}{(t_i - s)^\alpha} ds - \epsilon_0 \mathbb{W}_i \quad (i = 1, \dots, I).$$

Finally, subtraction of (4.2) from (4.2), with i replaced by $i + 1$, yields the basic error equation

$$(4.3) \quad \epsilon_{i+1} = \sum_{j=1}^i \epsilon_j a_{i-j} + b_i \quad (i = 1, \dots, I - 1),$$

with

$$\begin{aligned} a_0 &= W_0 - W_1 = 3 - 2^{2-\alpha}, \quad a_l = W_l - W_{l+1} \quad (l = 1, \dots, I - 1), \\ b_i &= -\frac{1}{\phi(\alpha) \Delta t^{1-\alpha}} \beta_i^1 + \beta_i^2, \\ (4.4) \quad \beta_i^1 &= \int_{t_i}^{t_{i+1}} \frac{e_{\Delta t}(s)}{(t_{i+1} - s)^\alpha} ds - \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left(\frac{1}{(t_i - s)^\alpha} - \frac{1}{(t_{i+1} - s)^\alpha} \right) e_{\Delta t}(s) ds, \\ \beta_i^2 &= \epsilon_0 (\mathbb{W}_i - \mathbb{W}_{i+1}). \end{aligned}$$

Before we give a proof of the theorem, it is first necessary to obtain some preliminary results.

LEMMA 4.1.

$$|\beta_i^1| \leq C_1 \Delta t^{3-\alpha} i^{-\alpha} \quad (i = 1, \dots, I - 1),$$

$$|\beta_i^2| \leq C_2 \Delta t^2 i^{1-\alpha} \quad (i = 1, \dots, I - 1), \text{ if } |\epsilon_0| \leq C_3 \Delta t^2.$$

Proof. By a standard theorem on Lagrange interpolation

$$(4.5) \quad e_{\Delta t}(s) = \frac{1}{2}(s - t_i)(t_{i+1} - s)f''(t_i) + r_i(s) \quad \{t_i \leq s \leq t_{i+1} \ (j = 0, \dots, I - 1)\},$$

with

$$(4.6) \quad |r_i(s)| \leq \frac{1}{8} F_3 \Delta t^3, \quad F_3 = \max_{t \in [t_0, T]} f'''(t).$$

By substituting (4.5) into (4.4), we obtain

$$\beta_i^1 = \Delta t^{3-\alpha} \left(f''(t_i) \gamma - \sum_{j=0}^{i-1} f''(t_j) \theta_{i-j-1} \right) + \delta_i,$$

where

$$\gamma = \frac{1}{2 \Delta t^{3-\alpha}} \int_0^{\Delta t} s^{1-\alpha} (\Delta t - s) ds,$$

$$\theta_l = \frac{1}{2 \Delta t^{3-\alpha}} \int_0^{\Delta t} s (\Delta t - s) \left(\frac{1}{(l \Delta t + s)^\alpha} - \frac{1}{((l + 1) \Delta t + s)^\alpha} \right) ds,$$

$$\delta_i = \int_{t_i}^{t_{i+1}} \frac{r_i(s)}{(t_{i+1} - s)^\alpha} ds - \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left(\frac{1}{(t_i - s)^\alpha} - \frac{1}{(t_{i+1} - s)^\alpha} \right) r_j(s) ds.$$

It can easily be verified that, by (4.6),

$$(4.7) \quad |\delta_i| \leq C_4 \Delta t^{4-\alpha} \quad (i = 1, \dots, I - 1),$$

and that

$$(4.8) \quad 0 < \theta_0, 0 < \theta_l < C_5 l^{1-\alpha}, \quad l \geq 1,$$

$$(4.9) \quad \left| \gamma - \sum_{l=0}^{i-1} \theta_l \right| \leq C_6 i^{-\alpha}, \quad i \geq 1.$$

By partial summation,

$$\sum_{j=0}^{i-1} f''(t_j) \theta_{i-j-1} = f''(t_{i-1}) \sum_{l=0}^{i-1} \theta_l + \sum_{r=1}^{i-1} (f''(t_{i-r-1}) - f''(t_{i-r})) \times \sum_{l=r}^{i-1} \theta_l$$

(i = 2, \dots, I - 1)

and, using (4.8),

$$(4.10) \quad \left| \sum_{r=1}^{i-1} (f''(t_{i-r-1}) - f''(t_{i-r})) \times \sum_{l=r}^{i-1} \theta_l \right| \leq C_7 \Delta t i^{1-\alpha} \quad (i = 2, \dots, I - 1).$$

Combining (4.9) and (4.10), we obtain

$$\begin{aligned}
 & \left| f''(t_i)\gamma - \sum_{j=0}^{i-1} f''(t_j)\theta_{i-j-1} \right| \\
 (4.11) \quad & \cong \left| f''(t_i)\left(\gamma - \sum_{i=0}^{i-1} \theta_i\right) \right| + |f''(t_i) - f''(t_{i-1})| \times \sum_{i=0}^{i-1} \theta_i + C_7 \Delta t^{1-\alpha} \\
 & \cong F_2 C_6 i^{-\alpha} + \gamma F_3 \Delta t + C_7 \Delta t^{1-\alpha} \\
 & \qquad \qquad \qquad (i = 1, \dots, I - 1), F_2 = \max_{t \in [0, T]} f''(t).
 \end{aligned}$$

Since

$$\Delta t = T/I \leq T/i \quad (i = 1, \dots, I),$$

it follows from (4.7) and (4.11) that

$$|\beta_i^1| \leq C_1 \Delta t^{3-\alpha} i^{-\alpha} \quad (i = 1, \dots, I - 1).$$

The second inequality of Lemma 4.1 is an immediate consequence of the asymptotic expansion of $(W_i - W_{i+1})$. \square

Proof of Theorem 4.1. Using Lemma 4.1 it is clear that

$$(4.12) \quad |b_i| \leq C_3 \Delta t^2 i^{-\alpha} \quad (i = 1, \dots, I - 1).$$

Returning to (4.3), we note that

$$(4.13) \quad \sum_{i=0}^{i-1} a_i = 1 - W_i, \quad i \geq 1,$$

$$(4.14) \quad a_i > 0, \quad i \geq 1,$$

$$0 \leq a_0 < 1, \quad \alpha \in [\alpha_0, 1),$$

$$(4.15) \quad -1 < a_0 < 0, \quad \alpha \in (0, \alpha_0), \quad \alpha_0 = 2 - \frac{\ln 3}{\ln 2} \doteq 0.4150.$$

Substituting for ϵ_i in (4.3), we obtain

$$(4.16) \quad \epsilon_{i+1} = \sum_{j=1}^i \epsilon_j a_{i-j}^1 + b_i + a_0 b_{i-1} \quad (i = 2, \dots, I - 1),$$

with

$$a_0^1 = 0, \quad a_i^1 = a_i + a_0 a_{i-1}, \quad i \geq 1,$$

and

$$\begin{aligned}
 & a_i^1 > 0, \quad \alpha \in (0, 1), \\
 (4.17) \quad & a_i^1 \geq 0, \quad i \geq 2, \quad \alpha \in [\alpha_1, 1), \alpha_1 \doteq 0.2117, \\
 & a_2^1 < 0, \quad \alpha \in (0, \alpha_1).
 \end{aligned}$$

The latter is a consequence of the structure of $\{a_i(\alpha), i \geq 0\}$. From (4.13),

$$\sum_{i=0}^{i-1} a_i^1 = 1 - (W_i + a_0 W_{i-1}), \quad i \geq 2,$$

and, since

$$W_i = a_i + W_{i+1} = \sum_{v=i}^{\infty} a_v, \quad i \geq 0,$$

it follows from (4.17) that

$$W_i + a_0 W_{i-1} = \sum_{v=i}^{\infty} (a_v + a_0 a_{v-1}) = \sum_{v=i}^{\infty} a_v^1 > 0, \quad i \geq 2, \alpha \in (\alpha_1, 1).$$

Consequently,

$$(4.18) \quad \sum_{i=0}^{i-1} a_i^1 \leq 1 - C_9 i^{-\alpha} \quad (i = 2, \dots, I - 1), \alpha \in [\alpha_1, 1),$$

since W_i behaves asymptotically like $i^{-\alpha}$.

It follows from (4.2) that

$$(4.19) \quad |\epsilon_1| \leq C_{10} \Delta t^2, \quad |\epsilon_2| \leq C_{10} \Delta t^2.$$

We are now in a position to apply Lemma 3.1 to (4.16). On the basis of (4.19) together with (4.17), (4.18) and (4.12), the conditions of the lemma are satisfied and the required result

$$|\epsilon_i| \leq C_{11} \Delta t^2 \quad (i = 0, \dots, I), \alpha \in [\alpha_1, 1)$$

follows. \square

Note 4.1. Using Lemma 3.1 and (4.13)–(4.15), we can prove second order convergence for (4.1) with $\alpha \in [\alpha_0, 1)$ without making a substitution of the type {see (4.16)} introduced in the proof of Theorem 4.1. On the other hand, if we substitute for ϵ_{i-2} instead of ϵ_i in (4.16) using (4.3), we have

$$(4.20) \quad \epsilon_{i+1} = \sum_{j=1}^i \epsilon_j a_{i-j}^2 + b_i + a_0 b_{i-1} + a_2^1 b_{i-3} \quad (i = 4, \dots, I - 1)$$

with

$$a_0^2 = 0, \quad a_1^2 = a_1^1, \quad a_2^2 = 0, \quad a_l^2 = a_l^1 + a_2^1 a_{l-3}, \quad l \geq 3.$$

It can be shown numerically that (4.20) satisfies the conditions of Lemma 3.1 for $\alpha \in [\alpha_2, 1)$ with $\alpha_2 \doteq 0.1292$. It follows that, if the proof of order 2 convergence for all $\alpha, 0 < \alpha < 1$, by means of Lemma 3.1 is possible, it will involve a complex inductive argument. However, because our proof already covers the case $\alpha = \frac{1}{2}$, which is the most important one in application, an extension of the result of Theorem 4.1 will not be pursued further in this paper. Numerical experimentation indicates that second order convergence holds for all $\alpha \in (0, 1)$. This is illustrated in Section 6.

Since the extension of Theorem 4.1 to a general $k(t, s)$ can be established using a technique analogous to that of [7] for midpoint product integration, we only state the result.

THEOREM 4.2. *If*

- (i) $f(t)$ satisfies the conditions of Theorem 4.1,
- (ii) $k(t, s)$ is twice continuously differentiable with respect to t and s for $0 \leq s \leq t \leq T$, and

(iii) $\partial^3 k(t, s)/\partial s^3$ and $\partial^3 k(t, s)/\partial t \partial s^2$ are bounded for $0 \leq s \leq t \leq T$, then, for $\alpha \in [\alpha_1, 1)$, (4.1) is convergent of order 2.

5. Computational Errors. During the actual application of algorithm (4.1), the values $\{\bar{f}_0, \dots, \bar{f}_T\}$ instead of $\{f_0, \dots, f_T\}$, which satisfy $\bar{f}_0 = f_0 + \eta_0$,

$$\sum_{i=1}^i \bar{f}_i k_{i,i} W_{i-i} = g_i / (\Delta t^{1-\alpha} \phi(\alpha)) - \bar{f}_0 k_{i,0} W_i + \eta_i \quad (i = 1, \dots, T),$$

are determined, where the perturbations $\{\eta_i, i = 0, \dots, T\}$ represent the effect of rounding and truncation errors. It follows from Lemma 3.1 that

$$(5.1) \quad |\bar{f}_i - f_i| \leq K_1 \eta (1 + i^\alpha) \quad (i = 0, \dots, T, \alpha \in [\alpha_1, 1),$$

where $K_1 = \text{const} > 0$ and $|\eta_i| \leq \eta$ ($i = 0, \dots, T$). However, since the application of Lemma 3.1 cannot yield a sharp estimate in this case, (5.1) represents a conservative bound. Numerical experimentation indicates that

$$\max_{0 \leq i \leq T} |\bar{f}_i - f_i| \leq K_2 \eta,$$

where K_2 is a positive constant depending on T . This is the usual behaviour associated with computational errors in the numerical solution of Volterra integral equations of the second kind; equations of the first kind tend to be less well conditioned.

6. A Numerical Example. The trapezoidal product integration method was applied to the following equation:

$$(6.1) \quad \frac{t^{1-\alpha}}{(1-\alpha)2i} [{}_1F_1(1; 2-\alpha; it) - {}_1F_1(1; 2-\alpha; -it)] = \int_0^t \frac{f(s)}{(t-s)^\alpha} ds, \quad 0 \leq t \leq \pi,$$

where ${}_1F_1$ is Kummer's hypergeometric function {see Erdélyi [11, Section 13, p. 189]}. (i is the imaginary unit.)

The solution is $f(t) = \sin t$.

The errors obtained for $\alpha = 0.5$, for which convergence is guaranteed by Theorem 4.1, and for $\alpha = 0.05$ are listed in Table 6.1. In both cases, the error satisfies the predicted Δt^2 dependence.

Remark. All numerical results presented in this paper were computed in double precision arithmetic on the IBM 360/50 computer at the Australian National University.

7. A Third Order Method. The existence of higher order methods for (1.1) is not immediately obvious. So far, convergent methods of order greater than 2 have not been found for the first kind Volterra equation

$$(7.1) \quad g(t) = \int_0^t k(t, s) f(s) ds,$$

where $k(t, s)$ satisfies (1.2) {see [10]}. Equation (7.1) can be considered as a special case of (1.1) with $\alpha = 0$. Linz [5] has shown numerically that fourth order methods can be constructed for the equation

Table 6.1

$$(\Delta t = \frac{\pi}{20}, \Delta T = \frac{\pi}{10})$$

$\alpha = 0.5$

$\alpha = 0.05$

	Δt	$\Delta t/2$	$\Delta t/4$	$\Delta t/8$	Δt	$\Delta t/2$	$\Delta t/4$	$\Delta t/8$
ΔT	5.1743 E-4	1.3775 E-4	3.6036 E-5	9.2842 E-6	6.3831 E-4	1.5902 E-4	3.9708 E-5	9.9240 E-6
$2\Delta T$	1.0559 E-3	2.7529 E-4	7.0811 E-5	1.8051 E-5	1.2128 E-3	3.0232 E-4	7.5522 E-5	1.8879 E-5
$3\Delta T$	1.5057 E-3	3.8781 E-4	9.8995 E-5	2.5112 E-5	1.6691 E-3	4.1611 E-4	1.0396 E-4	2.5988 E-5
$4\Delta T$	1.8141 E-3	4.6337 E-4	1.1767 E-4	2.9747 E-5	1.9624 E-3	4.8923 E-4	1.2223 E-4	3.0554 E-5
$5\Delta T$	1.9484 E-3	4.9420 E-4	1.2494 E-4	3.1491 E-5	2.0640 E-3	5.1450 E-4	1.2853 E-4	3.2129 E-5
$6\Delta T$	1.8945 E-3	4.7712 E-4	1.2006 E-4	3.0166 E-5	1.9638 E-3	4.8943 E-4	1.2226 E-4	3.0558 E-5
$7\Delta T$	1.6571 E-3	4.1368 E-4	1.0349 E-4	2.5900 E-5	1.6718 E-3	4.1647 E-4	1.0402 E-4	2.5997 E-5
$8\Delta T$	1.2590 E-3	3.1003 E-4	7.6844 E-5	1.9108 E-5	1.2163 E-3	3.0277 E-4	7.5597 E-5	1.8891 E-5
$9\Delta T$	7.3890 E-4	1.7625 E-4	4.2714 E-5	1.0452 E-5	6.4191 E-4	1.5943 E-4	3.9776 E-5	9.9357 E-6
$10\Delta T$	1.4758 E-4	2.5414 E-5	4.4363 E-6	7.7940 E-7	4.8825 E-6	5.0150 E-7	6.0974 E-8	7.8146 E-8

Table 7.1

$$(\Delta t = \frac{\pi}{20}, \Delta T = \frac{\pi}{10})$$

$\alpha = 0.9$

$\alpha = 0.5$

	Δt	$\Delta t/2$	$\Delta t/4$	$\Delta t/8$	Δt	$\Delta t/2$	$\Delta t/4$	$\Delta t/8$
ΔT	4.6719 E-5	6.0994 E-6	7.7343 E-7	9.7490 E-8	2.9057 E-4	4.3943 E-5	5.7782 E-6	7.7325 E-7
$2\Delta T$	4.4306 E-5	5.4699 E-6	6.7833 E-7	8.4533 E-8	3.2510 E-4	4.1064 E-5	5.0030 E-6	6.1507 E-7
$3\Delta T$	3.5593 E-5	4.2153 E-6	5.1065 E-7	6.2811 E-8	2.7777 E-4	3.2081 E-5	3.7755 E-6	4.5597 E-7
$4\Delta T$	2.3024 E-5	2.5238 E-6	2.9112 E-7	3.4782 E-8	1.9225 E-4	1.9828 E-5	2.1757 E-6	2.5204 E-7
$5\Delta T$	8.0473 E-6	5.7424 E-7	4.2227 E-8	3.2715 E-9	8.5406 E-5	5.6124 E-6	3.6174 E-7	2.3337 E-8
$6\Delta T$	-7.8015 E-6	-1.4376 E-6	-2.1130 E-7	-2.8605 E-8	-3.0411 E-5	-9.1613 E-6	-1.4883 E-6	-2.0770 E-7
$7\Delta T$	-2.2939 E-5	-3.3127 E-6	-4.4446 E-7	-5.7712 E-8	-1.4343 E-4	-2.3043 E-5	-3.1930 E-6	-4.1843 E-7
$8\Delta T$	-3.5868 E-5	-4.8662 E-6	-6.3435 E-7	-8.1190 E-8	-2.4249 E-4	-3.4673 E-5	-4.5854 E-6	-5.8823 E-7
$9\Delta T$	-4.5312 E-5	-5.9454 E-6	-7.6230 E-7	-9.6737 E-8	-3.1784 E-4	-4.2911 E-5	-5.5292 E-6	-7.0046 E-7
$10\Delta T$	-5.0342 E-5	-6.4441 E-6	-8.1577 E-7	-1.0283 E-7	-3.6211 E-4	-4.6950 E-5	-5.9318 E-6	-7.4414 E-7

Table 7.1 (continued)

$\alpha = 0.1$

	Δt	$\Delta t/2$	$\Delta t/4$	$\Delta t/8$
ΔT	5.6746 E-4	1.2526 E-4	2.4850 E-5	4.1704 E-6
$2\Delta T$	9.4946 E-4	1.8666 E-4	3.0877 E-5	4.2002 E-6
$3\Delta T$	1.1366 E-3	1.9912 E-4	2.8161 E-5	3.3660 E-6
$4\Delta T$	1.1343 E-3	1.7401 E-4	2.0700 E-5	2.1474 E-6
$5\Delta T$	9.6077 E-4	1.2120 E-4	1.0516 E-5	7.1198 E-7
$6\Delta T$	6.4704 E-4	5.0222 E-5	-9.3701 E-7	-7.9496 E-7
$7\Delta T$	2.3450 E-4	-2.9383 E-5	-1.2382 E-5	-2.2223 E-6
$8\Delta T$	-2.2825 E-4	-1.0830 E-4	-2.2646 E-5	-3.4331 E-6
$9\Delta T$	-6.8959 E-4	-1.7791 E-4	-3.0700 E-5	-4.3078 E-6
$10\Delta T$	-1.0995 E-3	-2.3087 E-4	-3.5755 E-5	-4.7610 E-6

$$g(t) = \int_0^t \frac{f(s)}{(t^2 - s^2)^{1/2}} ds, \quad 0 \leq t \leq T < \infty.$$

However, it is not possible to infer the existence of higher order methods for (1.1) on the basis of this existence, since the integral operator

$$Af(t) = \int_0^t \frac{f(s)}{(t - s)^\alpha} ds, \quad \alpha < 1,$$

is a compact operator from $C[0, T]$ to $C[0, T]$, while the operator

$$Bf(t) = \int_0^t \frac{f(s)}{(t^2 - s^2)^{1/2}} ds$$

is not.

We verify numerically the existence of higher order methods for (1.1). In fact, we construct a third order one. For simplicity, we take $k(t, s) \equiv 1$. The generalization is straightforward. We assume $I = T\Delta t$ is even and, generalizing the technique used in the development of the trapezoidal method, we approximate $f(s)$ by a function $M_{\Delta t}(s)$ which is a quadratic polynomial in each of the intervals $[t_{2j}, t_{2j+2}]$ ($j = 0, \dots, I/2 - 1$):

$$M_{\Delta t}(s) = \bar{f}_{2j} + \frac{s - t_{2j}}{\Delta t} (\bar{f}_{2j+1} - \bar{f}_{2j}) + \frac{(s - t_{2j})(s - t_{2j+1})}{2 \Delta t^2} (\bar{f}_{2j+2} - 2\bar{f}_{2j+1} + \bar{f}_{2j})$$

$$\{t_{2j} \leq s \leq t_{2j+2} \ (j = 0, \dots, I/2 - 1)\}.$$

By substituting $M_{\Delta t}(s)$ for $f(s)$ in (1.1), we obtain

$$(7.2) \quad g_{2j+1} = \int_0^{t_{2j+1}} \frac{M_{\Delta t}(s)}{(t_{2j+1} - s)^\alpha} ds \quad (j = 0, \dots, I/2 - 1)$$

$$g_{2j+2} = \int_0^{t_{2j+2}} \frac{M_{\Delta t}(s)}{(t_{2j+2} - s)^\alpha} ds$$

which, for each j , is a nonsingular two by two system for \bar{f}_{2j+1} and \bar{f}_{2j+2} . Thus, (7.2) is a block by block method in the sense of [10]. Evaluation of the integrals in (7.2) and division by $\Delta t^{1-\alpha}$ yields

$$E_{-1}\bar{f}_2 + F_{-1}\bar{f}_1 + G_{-1}\bar{f}_0 = g_1/\Delta t^{1-\alpha},$$

$$E_0\bar{f}_2 + F_0\bar{f}_1 + G_0\bar{f}_0 = g_2/\Delta t^{1-\alpha},$$

$$E_{-1}\bar{f}_{2i+2} + F_{-1}\bar{f}_{2i+1} + \sum_{j=1}^{2i} \bar{f}_j V_{2i+1-j} + \bar{f}_0 G_{2i-1} = g_{2i+1}/\Delta t^{1-\alpha}$$

$$(i = 1, \dots, I/2 - 1),$$

$$E_0\bar{f}_{2i+2} + F_0\bar{f}_{2i+1} + \sum_{j=1}^{2i} \bar{f}_j W_{2i+1-j} + \bar{f}_0 G_{2i} = g_{2i+2}/\Delta t^{1-\alpha}$$

$$\bar{f}_0 = \lim_{t \rightarrow 0} (g(t)t^{\alpha-1})(1 - \alpha),$$

$$V_{2l} = F_{2l-1} \quad (l \geq 1); \quad V_{2l+1} = G_{2l-1} + E_{2l+1} \quad (l \geq 0);$$

$$W_{2l} = F_{2l} \quad (l \geq 1); \quad W_{2l+1} = G_{2l} + E_{2l+2} \quad (l \geq 0);$$

$$E_{-1} = \frac{1}{2} \left(\frac{1}{3-\alpha} - \frac{1}{2-\alpha} \right), \quad F_{-1} = \left(\frac{1}{1-\alpha} - \frac{1}{3-\alpha} \right),$$

$$G_{-1} = \frac{1}{2} \left(\frac{1}{3-\alpha} + \frac{1}{2-\alpha} \right);$$

$$E_j = L_1(j, \alpha)(1 + \frac{3}{2}j + \frac{1}{2}j^2) - L_2(j, \alpha)(\frac{3}{2} + j) + \frac{1}{2}L_3(j, \alpha) \quad (j \geq 0);$$

$$F_j = -L_1(j, \alpha)(2j + j^2) + L_2(j, \alpha)(2 + 2j) - L_3(j, \alpha) \quad (j \geq 0);$$

$$G_j = L_1(j, \alpha)(\frac{1}{2}j + \frac{1}{2}j^2) - L_2(j, \alpha)(\frac{1}{2} + j) + \frac{1}{2}L_3(j, \alpha) \quad (j \geq 0);$$

$$L_1(j, \alpha) = ((j+2)^{1-\alpha} - j^{1-\alpha})/(1-\alpha);$$

$$L_2(j, \alpha) = ((j+2)^{2-\alpha} - j^{2-\alpha})/(2-\alpha);$$

$$L_3(j, \alpha) = ((j+2)^{3-\alpha} - j^{3-\alpha})/(3-\alpha).$$

This method was applied to (6.1). The errors for the cases $\alpha = 0.9$, $\alpha = 0.5$ and $\alpha = 0.1$ are tabulated in Table 7.1. Third order convergence holds for $\alpha = 0.9$ and $\alpha = 0.5$. For $\alpha = 0.1$, a strict Δt^3 dependence is no longer apparent. However, the process is still convergent—no explosion in the errors is observed. The numerical approximations oscillate about the true solution, and the oscillations increase as α tends to zero. The simplest interpretation that can apply to such behaviour is one of increasing potential instability as α decreases.

A rigorous analysis of the convergence properties of this method cannot be given here. The problem is the lack of accurate estimates for the solution of the lower triangular system of equations which determines the behaviour of the error.

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1. GERALD N. MINERBO & MAURICE E. LEVY, "Inversion of Abel's integral equation by means of orthogonal polynomials," *SIAM J. Numer. Anal.*, v. 6, 1969, pp. 598-616. MR 41 #6426.
2. R. M. FORTET, "Les fonctions aléatoires du type de Markoff associées à certaines équations linéaires aux dérivées partielles du type parabolique," *J. Math. Pures Appl.*, v. 22, 1943, pp. 177-243. MR 7, 19.
3. J. DURBIN, "Boundary-crossing probabilities for the Brownian motion and Poisson processes and techniques for computing the power of the Kolmogorov-Smirnov test," *J. Appl. Probability* (To appear.)
4. A. YOUNG, "The application of approximate product-integration to the numerical solution of integral equations," *Proc. Roy. Soc. London Ser. A*, v. 224, 1954, pp. 561-573. MR 16, 179.
5. P. LINZ, *Applications of Abel Transforms to the Numerical Solution of Problems in Electrostatics and Elasticity*, MRC Technical Summary Report #826, University of Wisconsin, Madison, Wis., 1967.
6. B. NOBLE, *Lecture Notes*, 1970.
7. R. WEISS & R. S. ANDERSSSEN, "A product integration method for a class of singular first kind Volterra equations," *Numer. Math.* (To appear.)
8. G. KOWALEWSKI, *Integralgleichungen*, de Gruyter, Berlin und Leipzig, 1930.

9. L. V. KANTOROVIČ & V. I. KRYLOV, *Approximate Methods of Higher Analysis*, 3rd ed., GITTL, Moscow, 1950; English transl., Interscience, New York; Noordhoff, Groningen, 1958. MR 13, 77; MR 21 #5268.

10. P. LINZ, *The numerical solution of Volterra integral equations by finite difference methods*, MRC Technical Summary Report #825, University of Wisconsin, Madison, Wis., 1967.

11. A. ERDÉLYI, ET AL., *Tables of Integral Transforms*. Vol. II, McGraw-Hill, New York, 1954. MR 16, 468.