

# The Numerical Computation of Two Transcendental Functions Related to the Exponential Integral\*

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**Abstract.** Algorithms for the computation of numerical values of the two transcendental functions

$$\int_0^x \frac{1}{t} [\text{Ei}(t) - \gamma - \ln |t|] dt \quad \text{and} \quad \int_0^x \frac{1}{t} [\text{Ei}(t) - \gamma - \ln |t|] \exp(-t) dt,$$

where  $\gamma$  is Euler's constant and  $\text{Ei}(t)$  is the exponential integral, are presented for all ranges of the real variable  $x$ . A table of values of these functions is also given.

**1. Introduction.** A large class of integrals involving the exponential integral times powers, exponentials, and other exponential integrals may be expressed, after integration by parts, in terms of exponential integrals and other well-known functions and two less well-known transcendental functions defined by

$$(1.1) \quad F(x) = \int_0^x \frac{1}{t} [\text{Ei}(t) - \gamma - \ln |t|] dt,$$

$$(1.2) \quad G(x) = \int_0^x \frac{1}{t} [\text{Ei}(t) - \gamma - \ln |t|] \exp(-t) dt.$$

Here, the constant  $\gamma = 0.5772156649015328 \dots$  is Euler's constant and  $\text{Ei}(t)$  is the exponential integral defined by [1]:

$$\text{Ei}(-t) = - \int_t^\infty \frac{1}{s} \exp(-s) ds \quad (t > 0),$$

$$\text{Ei}(t) = \int_{-\infty}^t \frac{1}{s} \exp(s) ds \quad (t > 0).$$

The bar through the integral sign indicates that the Cauchy principal value of the integral is to be taken. The logarithmic singularity of  $\text{Ei}(t)$  at the origin is cancelled out in the combination  $\text{Ei}(t) - \ln |t|$ , which occurs in the definitions (1.1) and (1.2).

The functions  $F(x)$  and  $G(x)$  have arisen in a problem in molecular quantum mechanics [2] and may occur in other fields, such as astrophysics and the theory of transport properties. Efficient methods for the numerical computation of these two functions are obtained here for all ranges of the real variable  $x$ .

The functions  $F(x)$  and  $G(x)$  are not completely independent because of the relation

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$$(1.3) \quad F(x) + F(-x) - G(x) - G(-x) \\ + [\text{Ei}(x) - \gamma - \ln |x|][\text{Ei}(-x) - \gamma - \ln |x|] = 0.$$

However, a serious loss of significant figures may result in some regions if this relation is used to give one of these functions in terms of the others. For this reason, the functions are treated independently here and (1.3) has been used only as a check on the final results.

2. Method of Computation. The following series may be inferred from [3]:

$$(2.1) \quad F(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2(k!)},$$

and

$$G(x) = \sum_{k=0}^{\infty} \frac{1 - e_k(x) \exp(-x)}{(k+1)^2},$$

where

$$e_k(x) = \sum_{n=0}^k x^n/n!.$$

These series are valid for all finite values of  $x$ , but only (2.1) is in a form which is convenient for numerical computation. To obtain other expressions, the range of the argument  $x$  is broken up into four regions, depending on whether  $x$  is positive or negative and whether  $|x|$  is large or small. A different expression is useful in each region.

The series (2.1) may be obtained by term-by-term integration of the series [1]:

$$\text{Ei}(t) - \gamma - \ln |t| = \sum_{k=1}^{\infty} \frac{t^k}{k(k!)}$$

in the definition (1.1). The term-by-term integration of (1.2) gives, after some algebraic manipulation, the result

$$(2.2) \quad G(x) = \exp(-x) \sum_{k=1}^{\infty} \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \right) \frac{x^k}{k!}.$$

When  $x$  is small and positive, (2.1) and (2.2) are quite convenient for computation. However, when  $x$  is negative the terms in these series alternate in sign so it may be necessary to keep more significant figures during the calculation than will be obtained in the result. When  $|x|$  is large, it would be necessary to compute many terms in these series to obtain the functions to a given accuracy. Accordingly, alternative expressions will be obtained for these other regions.

Expressions which are useful when  $|x|$  is small and  $x$  is negative may be obtained by factoring  $\exp(x)$  out of the series in (2.1) and absorbing the factor  $\exp(-x)$  into the series (2.2) to give

$$(2.3) \quad F(x) = -\exp(x) \sum_{k=1}^{\infty} \left[ 1 + \frac{1}{2} \left( 1 + \frac{1}{2} \right) + \frac{1}{3} \left( 1 + \frac{1}{2} + \frac{1}{3} \right) \right. \\ \left. + \cdots + \frac{1}{k} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) \right] \frac{(-x)^k}{k!},$$

$$(2.4) \quad G(x) = - \sum_{k=1}^{\infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} \right) \frac{(-x)^k}{k(k!)}$$

In (2.3) and (2.4) the terms in the series are all positive when  $x$  is negative, so the series may be summed without concern for the loss of significant figures from subtractions.

Asymptotic series are useful when  $|x|$  is large. Using integration by parts and the known result [4],

$$\int_0^{\infty} [\ln(t)]^2 \exp(-t) dt = \gamma^2 + \frac{\pi^2}{6},$$

the function  $F(x)$  may be written

$$(2.5) \quad F(x) = -\frac{\pi^2}{12} - \frac{1}{2}(\gamma + \ln |x|)^2 + A(x)$$

where

$$(2.6a) \quad A(-x) = - \int_x^{\infty} \frac{1}{t} \text{Ei}(-t) dt \quad (x > 0),$$

$$(2.6b) \quad A(x) = \int_{-\infty}^x \frac{1}{t} \text{Ei}(t) dt \quad (x > 0).$$

An asymptotic series for the function  $A(x)$  may be obtained by solving the first-order differential equation satisfied by  $A(x) \exp(-x)$ , assuming a series solution in inverse powers of  $x$ . The result is

$$(2.7) \quad A(x) = \frac{\exp(x)}{x} \left[ \sum_{k=1}^{N-1} \left( 1 + \frac{1}{2} + \dots + \frac{1}{k} \right) \frac{k!}{x^k} + O(|x|^{-N}) \right].$$

It can be shown that

$$(2.8) \quad \int_0^{\infty} \frac{1}{t} [\text{Ei}(t) - \gamma - \ln(t)] \exp(-t) dt = \frac{\pi^2}{6}.$$

This is easily demonstrated by solving the first-order differential equation satisfied by the function

$$I(a) = \int_0^{\infty} \frac{1}{t} [\text{Ei}(at) - \gamma - \ln(at)] \exp(-t) dt$$

at the point  $a = 1$ . This reduces the problem to the evaluation of integrals which may be found in [4].

From (2.8) and the integration of (1.2) by parts, it follows that

$$(2.9) \quad G(x) = \frac{\pi^2}{6} - (\gamma + \ln |x|) \text{Ei}(-x) + A(-x) - B(x),$$

where

$$(2.10a) \quad B(-x) = - \int_{-\infty}^x \frac{1}{t} \text{Ei}(-t) \exp(t) dt \quad (x > 0),$$

$$(2.10b) \quad B(x) = \int_x^{\infty} \frac{1}{t} \text{Ei}(t) \exp(-t) dt \quad (x > 0).$$

Term-by-term integration of the usual asymptotic expansion [1] for  $Ei(t)$ ,

$$Ei(t) = \frac{\exp(t)}{t} \left[ \sum_{k=0}^{N-1} \frac{k!}{t^k} + O(|t|^{-N}) \right]$$

gives the following asymptotic expansion for  $B(x)$ :

$$(2.11a) \quad B(-x) = \frac{\pi^2}{2} + \sum_{k=1}^{N-1} \frac{(k-1)!}{k(-x)^k} + O(|x|^{-N}) \quad (x > 0),$$

$$(2.11b) \quad B(x) = \sum_{k=1}^{N-1} \frac{(k-1)!}{kx^k} + O(|x|^{-N}) \quad (x > 0).$$

Table 1 gives the lowest value of  $|x|$  for which the asymptotic expansions (2.7) and (2.11) for  $A(x)$  and  $B(x)$  may be used to obtain a specified number of significant figures.

TABLE 1

*Lowest value of  $|x|$  for which the asymptotic expansions for  $A(x)$  and  $B(x)$  may be used to obtain  $N$  significant figures*

| $N$ | Lowest $ x $ for $A(x)$ | Lowest $ x $ for $B(x)$ |
|-----|-------------------------|-------------------------|
| 2   | 10                      | 5                       |
| 5   | 18                      | 12                      |
| 10  | 32                      | 22                      |
| 15  | 42                      | 34                      |

The asymptotic expansions given above may be used when  $x$  is positive or negative and  $|x|$  is large. However, for some negative  $x$  there may be a loss of significant figures because of the alternating signs in the series. The asymptotic series may be summed for negative  $x$  by developing the expansions into corresponding-type continued fractions with the Q-D algorithm [5] to give

$$(2.12) \quad A(x) = -\frac{\exp(x)}{x} \cfrac{1}{-x + \cfrac{a_1}{1 + \cfrac{a_2}{-x + \cfrac{a_3}{1 + \cfrac{a_4}{-x + \dots}}}}} \quad (x < 0),$$

TABLE 2

*Coefficients in the corresponding-type continued fractions for A(x) and B(x)*

| $n$ | $a_n$                   | $b_n$                  |
|-----|-------------------------|------------------------|
| 1   | .300000000000000000(1)  | .500000000000000000(0) |
| 2   | .666666666666666667(0)  | .833333333333333333(0) |
| 3   | .483333333333333333(1)  | .146666666666666667(1) |
| 4   | .879310344827586207(0)  | .178181818181818182(1) |
| 5   | .944421906693711967(1)  | .245287569573283859(1) |
| 6   | -.164139882757226602(1) | .275388922076659118(1) |
| 7   | -.919998772704958272(1) | .344528366918547414(1) |
| 8   | .191774670759561337(2)  | .373554441230166083(1) |
| 9   | .120986030342341091(1)  | .444048435146756559(1) |
| 10  | .111469799921919850(2)  | .472223141367309548(1) |
| 11  | .279777396240906360(1)  | .543719104227596573(1) |
| 12  | .126268337257797987(2)  | .571195795576397123(1) |
| 13  | .251027526733963660(1)  | .643480507817256221(1) |
| 14  | .207784190321309722(2)  | .670369243675089503(1) |
| 15  | -.386252722106876554(1) | .743300868460585226(1) |
| 16  | -.179427843772579110(2) | .769683863755687429(1) |
| 17  | .367960296816356522(2)  | .843161701520605316(1) |
| 18  | .238270459894158081(1)  | .869102404830198058(1) |
| 19  | .184838464381368151(2)  | .943051503401986103(1) |
| 20  | .581193386817538409(1)  | .968600195944079280(1) |
| 21  | .171301755871848527(2)  | .104296273304345446(2) |
| 22  | .749528822310504566(1)  | .106816014012491029(2) |
| 23  | .176038930028753514(2)  | .114289023855999487(2) |
| 24  | .852091767592453652(1)  | .116776995133427968(2) |
| 25  | .188822681725935945(2)  | .124283037995040995(2) |
| 26  | .903810728254682644(1)  | .126742053434754105(2) |
| 27  | .210079866463860708(2)  | .134278051260934166(2) |
| 28  | .882179333826835655(1)  | .136710498698025821(2) |
| 29  | .249241462089435543(2)  | .144273866932785061(2) |
| 30  | .687276011636422493(1)  | .146681796006304627(2) |
| 31  | .365564240026266328(2)  | .154270335700406548(2) |
| 32  | -.275435212745211379(1) | .156655523255827781(2) |
| 33  | -.103132018855008737(3) | .164267342245802769(2) |
| 34  | .138972432804639181(3)  | .166631342109838050(2) |
| 35  | .229010076709430513(1)  | .174264796138122932(2) |
| 36  | .356264051496208667(2)  | .176608977631153377(2) |
| 37  | .992524217946531086(1)  | .184262625505066438(2) |
| 38  | .301203732485409827(2)  | .186588203679854352(2) |
| 39  | .129332235226490197(2)  | .194260772540043659(2) |
| 40  | .293287664381389678(2)  | .196568832237394302(2) |
| 41  | .144897614896085839(2)  | .204259190252200114(2) |
| 42  | .301538505640609295(2)  | .206550705464503190(2) |
| 43  | .151578911219962032(2)  | .214257840075792051(2) |
| 44  | .322348489910877221(2)  | .216533689700270583(2) |
| 45  | .148265030354279632(2)  | .224256690085035328(2) |
| 46  | .363479808311700832(2)  | .226517670864082469(2) |
| 47  | .125782352690804964(2)  | .234255713642844070(2) |
| 48  | .468631286843471267(2)  | .236502550887597323(2) |
| 49  | .398938358825622867(1)  | .244254888365305874(2) |
| 50  | .160658399460559061(3)  | .246488244913984113(2) |

TABLE 4  
 Numerical values of the functions  $F(x)$  and  $G(x)$  defined in Eqs. (1.1) and (1.2)

| $x$  | $F(-x)$             | $F(x)$             | $G(-x)$             | $G(x)$             |
|------|---------------------|--------------------|---------------------|--------------------|
| 0.1  | -0.987682613970(-1) | 0.101268782308( 0) | -0.103854060621( 0) | 0.963497192087(-1) |
| 0.2  | -0.195144085730( 0) | 0.205152424003( 0) | -0.215850792074( 0) | 0.185781274994( 0) |
| 0.3  | -0.289229688986( 0) | 0.311717192878( 0) | -0.336685488831( 0) | 0.268833086834( 0) |
| 0.4  | -0.381217802161( 0) | 0.421253401071( 0) | -0.467115492096( 0) | 0.345999723998( 0) |
| 0.5  | -0.470911898432( 0) | 0.533738627932( 0) | -0.607966212156( 0) | 0.417735749759( 0) |
| 0.6  | -0.558686727171( 0) | 0.649365340222( 0) | -0.760137088119( 0) | 0.484458884095( 0) |
| 0.7  | -0.644528388272( 0) | 0.768288031797( 0) | -0.924608586249( 0) | 0.546553158790( 0) |
| 0.8  | -0.728514716359( 0) | 0.890668407479( 0) | -0.110244966304( 1) | 0.604371701335( 0) |
| 0.9  | -0.810719515327( 0) | 0.101667804644( 1) | -0.129482596461( 1) | 0.658239299500( 0) |
| 1.0  | -0.891212798111( 0) | 0.114649907253( 1) | -0.150300879989( 1) | 0.708454748544( 0) |
| 1.1  | -0.970061010886( 0) | 0.128032490201( 1) | -0.172838496164( 1) | 0.755292999803( 0) |
| 1.2  | -1.04732727277( 1)  | 0.148361009721( 1) | -0.197286747655( 1) | 0.799007127702( 0) |
| 1.3  | -1.12307142202( 1)  | 0.156082578080( 1) | -0.223690737327( 1) | 0.839830130804( 0) |
| 1.4  | -1.19735049969( 1)  | 0.170795141492( 1) | -0.252350565628( 1) | 0.87976581111( 0)  |
| 1.5  | -1.27021862151( 1)  | 0.185998490055( 1) | -0.283423196250( 1) | 0.913644134637( 0) |
| 1.6  | -1.34172728891( 1)  | 0.201718906288( 1) | -0.317123090824( 1) | 0.947014915103( 0) |
| 1.7  | -1.41192550983( 1)  | 0.217984369118( 1) | -0.35684810556( 1)  | 0.978256781612( 0) |
| 1.8  | -1.48085940031( 1)  | 0.234824675203( 1) | -0.393364443813( 1) | 0.100752449020( 1) |
| 1.9  | -1.5487501561( 1)   | 0.252271569513( 1) | -0.436441576577( 1) | 0.103496075831( 1) |
| 2.0  | -1.61511307720( 1)  | 0.270388885928( 1) | -0.483221815108( 1) | 0.106069724047( 1) |
| 2.2  | -1.74481773531( 1)  | 0.308601485192( 1) | -0.589257630014( 1) | 0.110754744248( 1) |
| 2.4  | -1.87027498186( 1)  | 0.349870974869( 1) | -0.714531332936( 1) | 0.114893925231( 1) |
| 2.6  | -1.99175910560( 1)  | 0.394530265401( 1) | -0.86267973564( 1)  | 0.118560986115( 1) |
| 2.8  | -2.10952043125( 1)  | 0.442993501090( 1) | -1.03812659196( 2)  | 0.121818897962( 1) |
| 3.0  | -2.22378703651( 1)  | 0.495733933622( 1) | -1.24604122812( 2)  | 0.124721520445( 1) |
| 3.2  | -2.33477078534( 1)  | 0.552931721981( 1) | -1.49269357652( 2)  | 0.127315004400( 1) |
| 3.4  | -2.44266160234( 1)  | 0.616291891737( 1) | -1.78556610700( 2)  | 0.129638917632( 1) |
| 3.6  | -2.54763686962( 1)  | 0.685442343626( 1) | -2.13361970065( 2)  | 0.13177258908( 1)  |
| 3.8  | -2.64985912312( 1)  | 0.76152967549( 1)  | -2.54759007338( 2)  | 0.133609318989( 1) |
| 4.0  | -2.74947818187( 1)  | 0.845595239381( 1) | -3.040344461757( 2) | 0.135310287162( 1) |
| 4.2  | -2.84663236079( 1)  | 0.938624230074( 1) | -3.62731179709( 2)  | 0.136851945627( 1) |
| 4.4  | -2.94144955197( 1)  | 1.04190078763( 2)  | -4.32699797667( 2)  | 0.138253122849( 1) |
| 4.6  | -3.03404819003( 1)  | 0.115686978174( 2) | -5.16160964041( 2)  | 0.139530130203( 1) |
| 4.8  | -3.12453811513( 1)  | 0.128520153999( 2) | -6.15780286645( 2)  | 0.140697123995( 1) |
| 5.0  | -3.21302134571( 1)  | 0.142882937961( 2) | -7.34758480885( 2)  | 0.141766412691( 1) |
| 5.5  | -3.42605696892( 1)  | 0.187041551265( 2) | -11.4407529622( 3)  | 0.144079230680( 1) |
| 6.0  | -3.62844180470( 1)  | 0.26869352845( 2)  | -17.8526970145( 3)  | 0.145978662754( 1) |
| 6.5  | -3.82128643935( 1)  | 0.32909296049( 2)  | -27.92706222926( 3) | 0.147560993793( 1) |
| 7.0  | -4.0053572800( 1)   | 0.44358336791( 2)  | -43.7982387797( 3)  | 0.148896699137( 1) |
| 7.5  | -4.1819954386( 1)   | 0.604904592045( 3) | -66.862702335( 3)   | 0.150037689482( 1) |
| 8.0  | -4.35137692521( 1)  | 0.834654142091( 4) | -108.529328473( 4)  | 0.151022867219( 1) |
| 8.5  | -4.51427519284( 1)  | 0.116501039909( 3) | -171.432772719( 4)  | 0.151881764169( 1) |
| 9.0  | -4.6722520615( 1)   | 0.164442587308( 3) | -271.365165986( 4)  | 0.152637000413( 1) |
| 9.5  | -4.822693563177( 1) | 0.234434969003( 3) | -430.376667177( 4)  | 0.153306286998( 1) |
| 10.0 | -4.96909288324( 1)  | 0.337479467572( 4) | -683.78684377( 4)   | 0.153903508649( 1) |
| 10.5 | -5.11078923173( 1)  | 0.490060926413( 3) | -1088.19052541( 5)  | 0.154439746743( 1) |
| 11.0 | -5.24810946982( 1)  | 0.71254361343( 3)  | -1734.38077108( 5)  | 0.154923929476( 1) |
| 11.5 | -5.38134642260( 1)  | 0.105725183230( 4) | -2768.13741128( 5)  | 0.15536336715( 1)  |
| 12.0 | -5.51076353408( 1)  | 0.156838812089( 4) | -4423.72295897( 5)  | 0.155763959829( 1) |
| 12.5 | -5.63659870708( 1)  | 0.234000262018( 4) | -7077.88210747( 5)  | 0.156130626333( 1) |
| 13.0 | -5.75906753636( 1)  | 0.35092605158( 4)  | -11336.9471672( 6)  | 0.156476428206( 1) |
| 13.5 | -5.87836604581( 1)  | 0.528725034387( 4) | -18177.4251527( 6)  | 0.156778401551( 1) |
| 14.0 | -5.999467301754( 1) | 0.799993994670( 4) | -29173.0251052( 6)  | 0.157065908169( 1) |

|      |                     |                     |                      |                     |
|------|---------------------|---------------------|----------------------|---------------------|
| 14.5 | -0.610815198367( 1) | 0.121493062518( 5)  | -0.468614604304( 6)  | 0.157332678120( 1)  |
| 15.0 | -0.621895293745( 1) | 0.185157491377( 5)  | -0.753376999366( 6)  | 0.157580885871( 1)  |
| 15.5 | -0.632721380985( 1) | 0.283071361070( 7)  | -0.121211194292( 7)  | 0.157812410869( 1)  |
| 16.0 | -0.643306174911( 1) | 0.434040044861( 5)  | -0.195160563779( 7)  | 0.158028866112( 1)  |
| 16.5 | -0.653661423396( 1) | 0.667291796655( 5)  | -0.314441383681( 7)  | 0.158231737491( 1)  |
| 17.0 | -0.663798000487( 1) | 0.102843149420( 6)  | -0.508595932118( 7)  | 0.158422215392( 1)  |
| 17.5 | -0.673726012122( 1) | 0.158865915146( 6)  | -0.817829477861( 7)  | 0.158600424262( 1)  |
| 18.0 | -0.683454830076( 1) | 0.245930711218( 6)  | -0.13201979736( 8)   | 0.158770320335( 1)  |
| 18.5 | -0.692993199096( 1) | 0.381469318953( 6)  | -0.2413201961049( 8) | 0.158929806300( 1)  |
| 19.0 | -0.702349774945( 1) | 0.592810313721( 6)  | -0.344508446347( 8)  | 0.159080619868( 1)  |
| 19.5 | -0.711530680473( 1) | 0.922849991911( 6)  | -0.506598471008( 8)  | 0.159223459527( 1)  |
| 20.0 | -0.720544351888( 1) | 0.145899861039( 7)  | -0.90084908034( 8)   | 0.159359425534( 1)  |
| 21.0 | -0.738096047543( 1) | 0.351479066801( 7)  | -0.233980555637( 9)  | 0.159610004108( 1)  |
| 22.0 | -0.755025584451( 1) | 0.863265660600( 7)  | -0.619170168983( 9)  | 0.159837628403( 1)  |
| 23.0 | -0.771457436303( 1) | 0.213084783264( 8)  | -0.162700984523(10)  | 0.160044955004( 1)  |
| 24.0 | -0.787349152440( 1) | 0.528343734785( 8)  | -0.428117994691(10)  | 0.160234585991( 1)  |
| 25.0 | -0.80276223278( 1)  | 0.131540376261( 9)  | -0.112793242739(11)  | 0.160408696396( 1)  |
| 26.0 | -0.81727688037( 1)  | 0.328719847520( 9)  | -0.297510744606(11)  | 0.160569118466( 1)  |
| 27.0 | -0.832273498700( 1) | 0.824294864480( 9)  | -0.785597151751(11)  | 0.160717406699( 1)  |
| 28.0 | -0.846423008607( 1) | 0.207353266405(10)  | -0.207647762501(12)  | 0.160854888505( 1)  |
| 29.0 | -0.860205250008( 1) | 0.523126006614(10)  | -0.549363883770(12)  | 0.160982704708( 1)  |
| 30.0 | -0.873635221796( 1) | 0.132335405257(11)  | -0.145469428961(13)  | 0.161101840380( 1)  |
| 31.0 | -0.886734126311( 1) | 0.335611800176(11)  | -0.385512180804(13)  | 0.161213151372( 1)  |
| 32.0 | -0.899519572281( 1) | 0.853130566810(11)  | -0.102243895776(14)  | 0.161317384462( 1)  |
| 33.0 | -0.912007749595( 1) | 0.217342160823(12)  | -0.271362435140(14)  | 0.161415194277( 1)  |
| 34.0 | -0.924213580501( 1) | 0.554832876823(12)  | -0.720704113598(14)  | 0.161507157092( 1)  |
| 35.0 | -0.936150850977( 1) | 0.141910749099(13)  | -0.191525266402(15)  | 0.161593782213( 1)  |
| 36.0 | -0.947832325321( 1) | 0.363624733499(13)  | -0.509320783636(15)  | 0.161675521440( 1)  |
| 37.0 | -0.959269846511( 1) | 0.933321015179(13)  | -0.135515610191(16)  | 0.161752776960( 1)  |
| 38.0 | -0.970474424387( 1) | 0.239942144127(14)  | -0.360764265628(16)  | 0.161825907982( 1)  |
| 39.0 | -0.981456313430( 1) | 0.617791152163(14)  | -0.960908539005(16)  | 0.1618952536328( 1) |
| 40.0 | -0.992225081567( 1) | 0.159294710983(15)  | -0.256067178936(17)  | 0.161961051174( 1)  |
| 42.0 | -0.101315845369( 2) | 0.1063327844415(16) | -0.182094754037(18)  | 0.162083157507( 1)  |
| 44.0 | -0.103333951237( 2) | 0.713301406456(16)  | -0.12971213010(19)   | 0.1621994027102( 1) |
| 46.0 | -0.105282552561( 2) | 0.480640529110(17)  | -0.925420871913(19)  | 0.1622295143169( 1) |
| 48.0 | -0.107166724962( 2) | 0.325189851961(18)  | -0.661177056231(20)  | 0.162387738822( 1)  |
| 50.0 | -0.108990989530( 2) | 0.220837449899(19)  | -0.473006130588(21)  | 0.162472847672( 1)  |
| 52.0 | -0.11075932274( 2)  | 0.150485913004(20)  | -0.338799479128(22)  | 0.162551342616( 1)  |
| 54.0 | -0.112475569359( 2) | 0.102870137473(21)  | -0.242944436491(23)  | 0.162623365904( 1)  |
| 56.0 | -0.114342801650( 2) | 0.705265000589(21)  | -0.174391322976(24)  | 0.162691352710( 1)  |
| 58.0 | -0.115764060284( 2) | 0.484831346703(22)  | -0.125304155998(25)  | 0.162754049768( 1)  |
| 60.0 | -0.117342045112( 2) | 0.334135744663(23)  | -0.901157824839(25)  | 0.162812530249( 1)  |
| 62.0 | -0.118879217296( 2) | 0.230820935739(24)  | -0.648643099916(26)  | 0.162867205735( 1)  |
| 64.0 | -0.120377827101( 2) | 0.159801493650(25)  | -0.467258755041(27)  | 0.162918435921( 1)  |
| 66.0 | -0.121839937716( 2) | 0.110861194545(26)  | -0.33684879796(28)   | 0.1629665266547( 1) |
| 68.0 | -0.123267445779( 2) | 0.770578009160(26)  | -0.243007400991(29)  | 0.163011785911( 1)  |
| 70.0 | -0.124662099122( 2) | 0.536589476484(27)  | -0.175425853348(30)  | 0.163054430260( 1)  |
| 72.0 | -0.126025512195( 2) | 0.374292085488(28)  | -0.126718897427(31)  | 0.163094688277( 1)  |
| 74.0 | -0.127359179532( 2) | 0.261505240407(31)  | -0.915900379714(31)  | 0.163132754831( 1)  |
| 76.0 | -0.128664487522( 2) | 0.182984006549(32)  | -0.662370042894(32)  | 0.163168804136( 1)  |
| 78.0 | -0.129942724778( 2) | 0.12825272429(31)   | -0.479276646876(33)  | 0.163202992414( 1)  |
| 80.0 | -0.131195091284( 2) | 0.899765640689(31)  | -0.346971244949(34)  | 0.163235460160( 1)  |

$$(2.13) \quad B(x) = \frac{\pi^2}{2} - \frac{1}{-x + \frac{b_1}{1 + \frac{b_2}{-x + \frac{b_3}{1 + \frac{b_4}{-x + \dots}}}}} \quad (x < 0)$$

where

$$a_1 = 3, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{29}{6}, \quad a_4 = \frac{51}{58} \dots$$

and

$$b_1 = \frac{1}{2}, \quad b_2 = \frac{5}{6}, \quad b_3 = \frac{22}{15}, \quad b_4 = \frac{98}{55} \dots$$

Unfortunately, expressions for the general coefficients have not been obtained. However, numerical values for the first fifty coefficients for each function are given in Table 2. The Q-D algorithm is somewhat unstable in this case, so the coefficients were computed in quadruple precision (39S) to insure accuracy to 18S. The continued fractions appear to be convergent for all negative  $x$ . As  $|x|$  increases, fewer approximants are needed in the continued fractions to obtain  $A(x)$  and  $B(x)$  to a given accuracy.

**3. Remarks.** A summary of the expressions which are most useful for the numerical computation of the functions  $F(x)$  and  $G(x)$  in each range of the argument  $x$  is given in Table 3. The dividing line between "small" and "large"  $x$  depends

TABLE 3

*Equation numbers of the most useful expressions for the numerical evaluation of the functions  $F(x)$  and  $G(x)$  in the various ranges of the real variable  $x$*

| Range of $x$              | $F(x)$                 | $G(x)$                         |
|---------------------------|------------------------|--------------------------------|
| $ x $ small, $x$ positive | (2.1)                  | (2.2)                          |
| $ x $ small, $x$ negative | (2.3)                  | (2.4)                          |
| $ x $ large, $x$ positive | (2.5), (2.7)           | (2.9), (2.7) or (2.12), (2.11) |
| $ x $ large, $x$ negative | (2.5), (2.7) or (2.12) | (2.9), (2.7), (2.11) or (2.13) |

on the accuracy which is desired in the results. These algorithms have been implemented for numerical computation on the Univac 1108 computer at the University of Wisconsin. This program was used to obtain the values of the functions which are



given in Table 4. The results in Table 4 have been independently checked to 8 significant figures by numerical integration, and further, they satisfy (1.3) to the accuracy with which they are given.

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