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On Thabit ibn Kurrah's Formula for Amicable Numbers

By Walter Borho

Abstract. This note describes some methods for deriving explicit formulas for amicable numbers which are similar to Thabit's famous rule. The search for new amicable pairs by such rules consists merely in primality tests for certain big numbers. Only two new pairs are actually given, demonstrating the usefulness of the results and simultaneously illustrating the quite strange form of pairs obtained by this new method. It is proposed to use a computer in order to get more solutions.

1. The numbers 220, 284 seem to have been known already to Pythagoras as the smallest pair of amicable numbers. They represent the case $n = 2$ of the following formula due to Thabit ibn Kurrah, an Arabian mathematician of the 9th century: *If* $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, *and* $r = 9 \cdot 2^{2n-1} - 1$ *are primes and* $n \geqq 2$, *then* $2^n pq$ *and* $2^n r$ *are amicable numbers* [4].

This theorem was rediscovered by Fermat (1636) and Descartes (1638) and generalized by Euler: $2^n pq$ *and* $2^n r$ *are amicable numbers, if the three integers* $p = 2^{n-m}f - 1$ $q = 2^n f - 1$, *and* $r = 2^{2n-m}f^2 - 1$ *are primes, where* $n > m \geqq 1$ *and* $f = 2^m + 1$.

Thabit's rule yields amicable numbers for $n = 2, 4, 7$, but for no other value $n < 200$. With Euler's generalization, there is still one more solution: $m = 7$, $n = 8$ (Legendre, Tchebychev). Whether there are any further solutions is not known.* In particular, it is an open question, whether there is an infinity of amicable pairs of numbers, each having at most three distinct prime divisors (but cf. [6]).

Apparently it has not been noticed up to now, that various rules can be found which are quite analogous to Thabit's. They arise naturally on considering the question whether there exists an infinity of amicable pairs with a given number of distinct prime divisors (cf. also [2]).

Notation. $\sigma(n)$ denotes the sum of all, $\tau(n)$ the sum of all proper divisors of n, i.e., $\sigma(n) = n + \tau(n)$.

2. By a "*Thabit-ibn-Kurrah-rule*" or "*Thabit-rule*" $T(b_1, b_2, p; F_1, F_2)$, with given natural numbers b_1, b_2, a prime p, and polynomials $F_1(X), F_2(X) \in \mathbf{Z}[X]$, we mean a statement of the form:

$p^n b_1 F_1(p^n)$ and $p^n b_2 F_2(p^n)$ are amicable numbers, if
$F_i(p^n)$ is prime and prime to $b_i p$ for $i = 1, 2$.

The following Lemma yields the key for the consideration of such "rules":

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* In a letter, E. J. Lee pointed out that it can be seen from H. Riesel's tables in *Math. Comp.*, v. 23, 1969, pp. 869–875, that there is no further solution for $200 \leqq n \leqq 1000$.

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LEMMA. *Let b_1, b_2 be positive integers, p a prime not dividing b_1, b_2. Let $\mathfrak{M}$ denote the set of amicable pairs (m_1, m_2) of the form $m_i = p^n b_i q_i$ $(i = 1, 2)$, with q_1, q_2 prime and n natural. Then, a necessary condition for $\mathfrak{M}$ to be infinite is*

$$\frac{p}{p-1} = \frac{b_1}{\sigma(b_1)} + \frac{b_2}{\sigma(b_2)}. \tag{1}$$

Proof. First, we note that, for $(m_1, m_2) \in \mathfrak{M}$, from

$$\begin{aligned} \sigma(m_i) &= m_1 + m_2 = p^n(b_1q_1 + b_2q_2), \\ \sigma(m_i) &= \sigma(p^n)\sigma(b_iq_i) \quad \text{if } q_i \neq p, \\ &= \sigma(p^{n+1})\sigma(b_i) \quad \text{if } q_i = p, \end{aligned}$$

it follows that always

$$\sigma(b_iq_i) \equiv 0 \pmod{p^n} \tag{2}$$

for $i = 1, 2$.

We claim that for all but at most finitely many pairs $(m_1, m_2) \in \mathfrak{M}$, $q_i \nmid b_ip$ for $i = 1, 2$. Indeed, if q_1 or q_2 is one of the (finitely many) prime divisors of b_1b_2p, then the possible values for n are bounded by (2), and one of the q_i, together with n, determines the pair (m_1, m_2).

Now, we have, for almost all $(m_1, m_2) \in \mathfrak{M}$,

$$\sigma(m_i) = \sigma(p^nb_i)(q_i + 1) = p^nb_1q_1 + p^nb_2q_2 = m_1 + m_2 \qquad (i = 1, 2), \tag{3}$$

which, for given n, is a nonsingular linear equation system determining q_1, q_2 uniquely. From (2), it follows that $q_i(n) \to \infty$ if $n \to \infty$. Now, the equation

$$1 = \frac{m_1}{\sigma(m_1)} + \frac{m_2}{\sigma(m_2)} = \sum_{i=1,2} \frac{b_i}{\sigma(b_i)} \frac{p^n}{\sigma(p^n)} \frac{q_i}{q_i + 1}$$

yields (1), if we take the limits $n \to \infty$ on the right side. Q.E.D.

Conversely, we now suppose that b_1, b_2 and p are any numbers given as in the Lemma and satisfying condition (1). We ask for the amicable pairs in $\mathfrak{M}$ with $q_i \nmid b_ip$ $(i = 1, 2)$. They can be found by merely solving the system (3) for q_1, q_2. This yields

THEOREM 1. *Let b_1, b_2 be natural numbers and $p \nmid b_1b_2$ a prime with property* (1). *If for some $n \geqq 1$, both numbers*

$$q_i = p^n(p-1)(b_1 + b_2)/\sigma(b_i) - 1, \qquad i = 1, 2, \tag{4}$$

are prime integers not dividing b_ip, then $m_i = p^nb_iq_i$ $(i = 1, 2)$ are amicables.

Remark. Let b_1, b_2 be given as in Theorem 1, and suppose that the theorem yields any amicable pair with these given values of b_1, b_2. *Then, b_1 and b_2 must contain the prime* 2 *in the same power.* This is easy to prove (cf. [2] for a more general statement).

3. The following examples will demonstrate how we can deduce Thabit-rules from this theorem.

In this section, we start from a pair b_1, b_2 of the special form $b_1 = au$, $b_2 = a$ with $(a, u) = 1$. We have to choose a and u such that (1) determines a prime, p. Eqs. (1) and (4) now read:

$$\frac{p}{p-1} = \frac{a}{\sigma(a)}\frac{u+\sigma(u)}{\sigma(u)}, \tag{5}$$

$$q_1 = p^n(p-1)\frac{a}{\sigma(a)}\frac{u+1}{\sigma(u)} - 1 = p^{n+1}\frac{u+1}{u+\sigma(u)} - 1.$$

If, for instance, $(u + 1, u + \sigma(u)) = 1$, then, q_1 is an integer if and only if $p = u + \sigma(u)$. Hence, by (5), $\sigma(a)/a = (p - 1)/\sigma(u)$, and

$$q_1 = p^n(u+1) - 1, \qquad q_2 = p^n\sigma(u)(u+1) - 1.$$

So we arrive at

THEOREM 2. *Choose a positive number u such that*

$$p = u + \sigma(u) \tag{6}$$

becomes a prime. Determine a positive number a prime to up as a solution of

$$\sigma(a)/a = (p-1)/\sigma(u). \tag{7}$$

Then, $T(au, a, p, (u + 1)X - 1, (u + 1)\sigma(u)X - 1)$ *is a Thabit-rule.*

Now, we must find solutions of the system (6), (7). Of course, we can do this by testing small values of u. With $u = 1$, we have $p = 2$, $a = 1$, and the theorem becomes the well-known rule of Euclid for even perfect numbers.

Observe that *Theorem 2 cannot yield any amicable pair, if u is even, or a square* >1, *or prime, or three times a prime, or if* $(u, \sigma(u)) \neq 1$. That u cannot be even follows from the remark in Section 2. If u is a square, then both u and $\sigma(u)$ are odd. Then, p must be even by (6). Hence, $p = 2$ and $u = 1$. If u is a prime, then it follows from (6) and (7) that $\sigma(au) = \sigma(a)(u + 1) = 2au$. This means that au is a perfect number. If now $m_1 = aup^nq_1$ and $m_2 = ap^nq_2$ are amicable and $(pq_1, au) = 1$, then au evidently divides m_2. Hence, both m_1 and m_2 are abundant numbers, which is a contradiction, since they are amicable.—If $3 \mid u$, then necessarily $\sigma(u) \equiv 1(3)$, since otherwise one of p, q_1, q_2 becomes divisible by 3 and hence is not prime. If, furthermore, $u = 3r$, r a prime >3, we deduce from $\sigma(u) \equiv 1(3)$ the contradiction $3 \mid r$.—Finally, $(u, \sigma(u)) = 1$ holds, since $p = u + \sigma(u)$ is prime.—According to these observations, the least values of u which we have to test are $u = 5\cdot 7 = 35$ and $u = 5\cdot 11 = 55$.

Example. Putting $u = 5\cdot 11$, we have $p = 5\cdot 11 + 6\cdot 12 = 127$ and $\sigma(a)/a = 7/4$ by (6) and (7). The equation for a has the (unique) solution $a = 2^2$. Hence, the rule $T(2^2\cdot 5\cdot 11, 2^2, 127, 56X - 1, 56\cdot 72X - 1)$ holds. In other words, $2^2\cdot 127^n\cdot 5\cdot 11\cdot q_1$ and $2^2\cdot 127^n\cdot q_2$ are amicable, if $q_1 = 56\cdot 127^n - 1$ and $q_2 = 56\cdot 72\cdot 127^n - 1$ are prime. —With $n = 2$, we obtain *the new amicable pair*

$$2^2\cdot 127^2\cdot 5\cdot 11\cdot 903223 \quad \textit{and} \quad 2^2\cdot 127^2\cdot 65032127. \tag{8}$$

Note that one of these numbers is divisible by 220, the smaller member of Pythagoras' pair mentioned in the introduction. As we shall see soon, this is by no means an accident.

4. Now, we shall show how to solve Eqs. (6) and (7) of Theorem 2 in a more elegant way than by trial and error. Eliminating p and putting $s = \sigma(u) - 1$, we obtain

$$\sigma(au) = \sigma(a)\sigma(u) = \sigma(a)(s+1) = au + as. \tag{9}$$

If we suppose that s is a prime not dividing a, then (9) is just the statement that au and as are amicable numbers. So we make the striking observation that we can gain Thabit-rules starting from certain already known amicable numbers.

THEOREM 3. *Let au, as be an amicable pair with $(us, a) = 1$ and s a prime. If $p = u + \sigma(u)$ is a prime not in a, then the following Thabit-rule holds*:

$$m_1 = aup^n q_1 \text{ and } m_2 = ap^n q_2 \text{ are amicable if } q_1 = p^n(u+1) - 1$$
$$\text{and } q_2 = p^n(u+1)\sigma(u) - 1 \text{ are primes not dividing } a.$$

If we take, for au, as, Pythagoras' pair 220, 284, the theorem is applicable and we reobtain the example of Section 3.

There are at least 64 known amicable pairs au, as of the form needed in Theorem 3. An inspection of the smaller of these pairs produced primes $p = u + \sigma(u) = u + s + 1$ in 15 cases and hence 15 Thabit-rules. They are listed in Table 1. The last

TABLE 1

Thabit-ibn-Kurrah-rules $T(au, a, p, (u+1)X - 1, (u+1)\sigma(u)X - 1)$ obtained by applying Theorem 3

No.	*a*	*u*	*p*	$\sigma(u)$	*rule is obtained from pair No.*
1	2^2	$5\cdot 11$	127	72	(1) of [4]
2	$3^2\cdot 7\cdot 13$	$5\cdot 17$	193	108	(10) of [4]
3	$3^2\cdot 5\cdot 13$	$11\cdot 19$	449	240	(7) of [4]
4	$3^2\cdot 7^2\cdot 13$	$5\cdot 41$	457	252	(13) of [4]
5	$3^2\cdot 7^2\cdot 13\cdot 97$	$5\cdot 193$	2129	1164	(14) of [4]
6	$3^4\cdot 5\cdot 11$	$29\cdot 89$	5281	2700	(15) of [4]
7	$3^2\cdot 7\cdot 13\cdot 41\cdot 163$	$5\cdot 977$	10753	5868	(11) of [4]
8	$3^2\cdot 5\cdot 19\cdot 37$	$7\cdot 887$	13313	7104	(9) of [4]
9	$3^4\cdot 7\cdot 11\cdot 29$	$13\cdot 521$	14081	7308	(3) of [1], Wulf
10	$3^2\cdot 5\cdot 13\cdot 19$	$29\cdot 569$	33601	17100	(8) of [4]
11	$3^2\cdot 7^2\cdot 13$	$5\cdot 53\cdot 97$	57457	31752	(37) of [4]
12	$3^2\cdot 5^2\cdot 13.31$	$149\cdot 449$	134401	67500	(21) of [4]
13	$3^3\cdot 5^3\cdot 13$	$149\cdot 449$	134401	67500	(22) of [4]
14	$2\cdot 7^2\cdot 19\cdot 23$	$11\cdot 13523$	311041	162288	(3) of [5]
15	$3^4\cdot 5\cdot 11\cdot 59$	$89\cdot 5309$	950401	477900	(23) of [4]

column of the table gives the number of the pair au, as in the numeration of Escott [4] or others.—Theorem 3 is applicable to any amicable pair resulting from its application. But this is of purely theoretical interest, since the arising values of q_1, q_2 become too large for the available primality tests.

Now we return to Eq. (9) and note that it is actually irrelevant for our purposes whether s is prime (and hence au and as amicable) or not. Should we restrict ourselves to Theorem 3, then we should not get those Thabit-rules of Theorem 2 for which only $p = u + \sigma(u)$ and not $s = \sigma(u) - 1$ is a prime.

But these cases may yield to some of the known methods for constructing amicable pairs. For instance, put $u = vr_1r_2$, assume a and v are given, and ask for suitable values for the primes r_1, r_2. Then, apply E. J. Lee's bilinear diophantine equation method (BDE method for short) [7], which may be stated as follows: *With given naturals* a_1, a_2, *form*

$$F := \tau(a_2)\sigma(a_1), \qquad G := a_2\sigma(a_2),$$

$$D := a_1a_2 - \tau(a_1)\tau(a_2),$$

and take any decomposition of $(F + D)F + DG = D_1D_2$ *into two natural factors* D_1, D_2. *If then*

$$r_i = (D_i + F)/D \qquad (i = 1, 2)$$

and

$$s = \frac{\sigma(a_1)}{\sigma(a_2)}(r_1 + 1)(r_2 + 1) - 1$$

are primes and $(r_1, r_2) = (r_1r_2, a_1) = (s, a_2) = 1$, *then the numbers* $a_1r_1r_2$ *and* a_2s *are amicable.*

Now, for our purposes, we put $a_1 = av$ and $a_2 = a$, and check only whether two numbers r_1, r_2 formed as in the above procedure are prime, distinct, and prime to a_1. Regardless of s being prime or not, we test whether

$$p = u + \sigma(u) = vr_1r_2 + \sigma(v)(r_1 + 1)(r_2 + 1)$$

is a prime and $p \nmid a$. If such is the case, then we have found a Thabit-rule of the form treated in Theorem 2.

As an application of the method just described, we put $v = 1$, that is $u = r_1r_2$, $r_1 \neq r_2$. (We have seen in Section 3, that this is the simplest possibility for the prime decomposition of u.) It then follows that Eq. (7) of Theorem 2 is satisfied if and only if

$$(10) \qquad r_i = (d_i + \tau(a))/(a - \tau(a)) \qquad (i = 1, 2),$$

where $d_1d_2 = a^2$. So we get the following:

THEOREM 4. *Choose any natural a and a factorization* $d_1d_2 = a^2$ *of* a^2, *such that* r_1, r_2, *determined by* (10), *are distinct primes not dividing a. If* $p = 2r_1r_2 + r_1 + r_2 + 1$ *is a prime,* $p \nmid a$, *then the following rule holds:*

$$m_1 = ap^n r_1r_2q_1 \quad \textit{and} \quad m_2 = ap^n q_2$$

are amicable if

$$q_1 = (r_1r_2 + 1)p^n - 1 \quad \textit{and} \quad q_2 = (r_1r_2 + 1)(r_1 + 1)(r_2 + 1)p^n - 1$$

are primes not dividing a.

Putting $a = 2^m$, we obtain the special case:

THEOREM 5. *Let k, m, n be naturals,* $k < m$, *and put* $f = 2^k + 1$, $g = 2^{m-k}f^2$. *If all the following five numbers*

$$r_1 = f\cdot 2^{m-k} - 1, \qquad r_2 = f\cdot 2^m - 1,$$

$$p = g(2^{m+1} - 1) + 1, \qquad q_1 = p^n[(2^m - 1)g + 2] - 1,$$

$$q_2 = 2^m p^n g[(2^m - 1)g + 2] - 1$$

are primes, then

$$m_1 = 2^m p^n r_1 r_2 q_1 \quad \text{and} \quad m_2 = 2^m p^n q_2$$

are amicable numbers.

With $k = 1, m = 2, n = 2$, the above five numbers are indeed primes. The resulting amicable numbers are pair (8) again. With $k = 1, m = 2$ and with $k = 1, m = 3$, Theorem 5 yields Thabit-rules. The first is of the type considered in Theorem 3, the second is not (since $s = 287$ is not a prime).

5. Returning to Theorem 1, we now leave the restrictions made at the beginning of Section 3, and put, more generally, $b_1 = au, b_2 = av$ with $(a, uv) = 1$. Then condition (1) can be given the form

$$\frac{\sigma(a)}{a} \cdot \frac{p}{p-1} = \frac{u}{\sigma(u)} + \frac{v}{\sigma(v)}. \tag{11}$$

To solve this equation by trial and error, it is convenient to proceed as follows: Choose special small values for u and v. Then, since $p \nmid a$, p must be one of the prime divisors of the numerator on the right side of (11). Take one of these values for p, and, if possible, solve the remaining equation for a. Finally, look whether the solution of (1) thus found yields a Thabit-rule with Theorem 1.

Three of the rules in Table 2 have been found in this way.

TABLE 2

Thabit-ibn-Kurrah-rules $T(b_1, b_2, p, F_1, F_2)$ as defined in Section 2, which are not obtained by Theorem 3

No.	a	b_1	b_2	p	F_1	F_2	*obtained by*
16	$3^2\cdot5\cdot13$	$a\cdot29$	$a\cdot11$	113	$80X-1$	$200X-1$	Section 5
17	$2\cdot7$	$a\cdot5\cdot13$	$a\cdot17$	433	$246X-1$	$1148X-1$	Section 5
18	$2\cdot5$	$a\cdot23\cdot29$	$a\cdot7$	1297	$674X-1$	$60660X-1$	Section 5
19	2^3	$a\cdot11\cdot23$	a	541	$254X-1$	$73152X-1$	Theorem 5
20	$3^2\cdot5\cdot13\cdot19$	$a\cdot37\cdot113$	a	8513	$4182X-1$	$18116424X-1$	Theorem 4

One remark is in order: If one has found any Thabit-rule $T(b_1, b_2, p, F_1, F_2)$, then it may happen that one of the congruences

$$F_i(p^n) \equiv 0 \pmod{r},$$

for $i = 1$ or 2, is satisfied identically for a prime, r. Such rules have, of course, been deleted from our list, since it is evident that they cannot produce more than (at most) one amicable pair.

6. In the last two sections, we leave the Thabit-rules in the narrow sense of Section 2 and demonstrate how to use the above ideas for two other purposes. In Section 6, we shall combine the Thabit-rule method with Lee's BDE method to get a new method for the constructive search for amicable pairs, while, in Section 7, we

shall deal with analogues of Thabit-rules in the case of amicable 4-cycles. For shortness, we only calculate representative examples and omit the details.

We ask for amicable pairs of the form $m_1 = p^n b_1 q$, $m_2 = p^n b_2 r_1 r_2$ with b_1, b_2 given, n variable, and p, q, r_1, r_2 unknown primes. This means that we replace the prime q_2 of our former considerations by the product of two primes r_1, r_2. We demand that b_1, b_2 and p satisfy condition (1). If we put $b_1 = au$, $b_2 = av$ with $(a, uv) = 1$, then (1) becomes (11).

Example 1. Take $u = 53$, $v = 5$. Then,

$$\frac{\sigma(a)}{a} = \frac{p-1}{p} \cdot \frac{7^2}{3^3}.$$

It follows that $p = 7$, and then that $a = 3^2 \cdot 13$. So we look for amicables

$$m_1 = 3^2 \cdot 7^n \cdot 13 \cdot 53 \cdot q, \qquad m_2 = 3^2 \cdot 7^n \cdot 13 \cdot 5 \cdot r_1 r_2 \qquad (n = 1, 2, 3, \cdots).$$

Applying the BDE method as stated in Section 4, we find that m_1, m_2 are amicable if (and only if)

$$r_i = \tfrac{1}{2}(d_i + 45 \cdot 7^{n-1}) - 1 \qquad (i = 1, 2)$$

and

$$q = \tfrac{1}{9}(r_1 + 1)(r_2 + 1) - 1$$

are prime integers, where

$$d_1 d_2 = 3^3 \cdot 7^{n-1}(75 \cdot 7^{n-1} + 32).$$

With $n = 2$, *this yields the new amicable pair*

$$m_1 = 3^2 \cdot 7^2 \cdot 13 \cdot 53 \cdot 49727, \qquad m_2 = 3^2 \cdot 7^2 \cdot 13 \cdot 5 \cdot 167 \cdot 2663.$$

Example 2. $m_1 = 2 \cdot 5^n \cdot 7 \cdot q$ and $m_2 = 2 \cdot 5^n \cdot r_1 r_2$ are amicable, if n is natural, the three numbers

$$r_i = \tfrac{1}{3}(8 \cdot 5^n + d_i) - 1 \qquad (i = 1, 2),$$
$$q = \tfrac{1}{8}(r_1 + 1)(r_2 + 1) - 1$$

are integer and prime, and

$$d_1 d_2 = 2^4 \cdot 5^n(4 \cdot 5^n + 9).$$

7. Let $\tau^{(k)}$ denote the k-fold iteration of τ. If $\tau^{(k)}(n) = n$ and $\tau^{(i)}(n) \neq n$ for $1 \leqq i < k$, then $\tau(n), \cdots, \tau^{(k)}(n)$ is called an amicable (or "sociable") k-cycle. This is an old generalization of amicable numbers ($k = 2$) recently treated again in [1] and [3]. In [1], the following method for constructing amicable 4-cycles is given:

THEOREM. *Let* a_i, d_i $(i = 1, 2)$ *be given naturals with* $a_1 \neq a_2$, $a_1 a_2 = d_1 d_2$. *Let* $D = a_1 a_2 - \tau(a_1)\tau(a_2)$,

$$p_{ji} = D^{-1}[\tau(a_{i+1})\sigma(a_i) + d_j \sigma(a_{i+1})],$$
$$r_i = a_i^{-1} \tau(a_i p_{1i} p_{2i}),$$

where $i, j = 1, 2$ *and* $a_3 := a_1$. *If then the six numbers* p_{ji}, r_i *are primes and* $p_{ji} \nmid a_i$, $r_i \nmid a_i$, $p_{1i} \neq p_{2i}$ *for* $i, j = 1, 2$, *then the four numbers*

$$n_{2i-1} = a_i p_{1i} p_{2i}, \qquad n_{2i} = a_i r_i \qquad (i = 1, 2)$$

form an amicable 4-cycle. For the p_{ji} *to be integer, a necessary condition is*

$$(a_1 - a_2)(\sigma(a_1), \sigma(a_2)) \equiv 0 \pmod{D}. \tag{12}$$

For our present intention, we take a_i in the form $a_i = p^n b_i$ $(i = 1, 2)$ with a triple b_1, b_2, p (p prime and prime to $b_1 b_2$), solving Eq. (1) with n variable. We check condition (12) and then try to apply the theorem. ((12) is automatically satisfied if, for instance, $b_1 = au$, $b_2 = a$, with a, u as in Theorem 2.)

Example. $a_1 = 2\cdot 5\cdot 7^n$, $a_2 = 2\cdot 11\cdot 7^n$ satisfy all of our conditions. The theorem yields: *The four numbers*

$$n_1 = 2\cdot 5\cdot 7^n\cdot p_{11}p_{21}, \qquad n_2 = 2\cdot 5\cdot 7^n\cdot r_1,$$
$$n_3 = 2\cdot 11\cdot 7^n\cdot p_{21}p_{22}, \qquad n_4 = 2\cdot 11\cdot 7^n\cdot r_2$$

form an amicable 4-cycle, if the six numbers

$$p_{11} = \frac{2}{3}(7^\lambda + 5\cdot 7^n) - 1, \qquad p_{21} = \frac{10}{3}(11\cdot 7^{2n-\lambda} + 7^n) - 1,$$

$$p_{12} = \frac{1}{3}(7^\lambda + 11\cdot 7^n) - 1, \qquad p_{22} = \frac{11}{3}(5\cdot 7^{2n-\lambda} + 7^n) - 1,$$

$$r_i = \frac{1}{a_i}\tau(n_{2i-1}) \qquad (i = 1, 2)$$

are primes. Here, λ is an integer with $0 \leqq \lambda \leqq 2n$. The author does not know, whether this example actually yields a new amicable 4-cycle.

Note that the point of our method is that the denominators D of Sections 6–7 can be made independent of p^n by use of condition (1). Many examples similar to those given in Sections 6 and 7 can be found easily.

For a Thabit-rule like formula for amicable 3-cycles, the interested reader is referred to [1].

Added in proof. E. J. Lee tested by computer several of the rules given in Tables 1 and 2 for some small values of the exponent n. In a letter dated August 20, 1971, he announced primality of q_1, q_2 in the case $n = 1$ of rule no. 6. This means discovery of the *new amicable pair*

$$3^4\cdot 5\cdot 11\cdot 5281\cdot 29\cdot 89\cdot 13635541 \quad\text{and}\quad 3^4\cdot 5\cdot 11\cdot 5281\cdot 36815963399.$$

Essener Str. 13f
2 Hamburg 62
West Germany

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