

Inversion of Isoclinal Matrices

By Leroy J. Derr

Abstract. Triangular matrices (over the reals), whose elements satisfy $a_{i,j} = a_{i+1,j+1}$, can be mapped algebraically into the rational functions to facilitate determination of inverses and factors of the matrices. It is further shown in this note that these methods can be extended to the nontriangular forms to obtain inverses or demonstrate singular cases. Applications are made to certain matrices of Hessenberg type.

1. Introduction. Certain matrices of Toeplitz form [5] will be considered as having arbitrary but finite dimension $n \times n$. The matrices will have real elements and the property that $a_{i,j} = a_{i+1,j+1}$ (isoclinal). The purpose of this note is to extend to the nontriangular case methods which have proved useful in obtaining decompositions and inverses for the lower triangular case. Several examples are given, including the inversion of a certain class of Hessenberg matrices.

2. Mathematical Description. Let us first consider the class V_n of lower triangular isoclinal (L.T.I.) $n \times n$ matrices. A matrix $A \in V_n$ can be denoted by its first column, $[a_0, a_1, \dots, a_{n-1}]$, which defines it completely. The space V_n has been shown in [2] to be a commutative vector algebra where the product AB of two elements is given simply by $\sum_{p=0}^k a_p b_{k-p}$, $k = 0, 1, \dots, n - 1$. Hence, if $A \in V_n$ is invertible ($a_0 \neq 0$), then $A^{-1} = B$ is easily computed from the recursive system:

$$(1) \quad a_0 b_0 = 1, \quad \sum_{p=0}^k a_p b_{k-p} = 0, \quad k = 1, 2, \dots, n - 1.$$

While the above, Eq. (1), are sufficient for the inversion of a given L.T.I. matrix, the problem can frequently arise to seek 'closed' formulas for inverses of such matrices with arbitrary n ; for example, the first column of the matrix may consist of the first n elements of a given infinite sequence. Before proceeding to the nontriangular case, the following algebraic mapping theorem is given as an aid in such circumstances. Its use will be later illustrated in the examples.

2.1. THEOREM. *Let R be the ring of rational functions $\{P(y)/Q(y), Q(0) \neq 0\}$ and let $A \in V_n$ with first column $[a_0, a_1, \dots, a_{n-1}]$. Define $e_n: V_n \rightarrow R$ by $e_n[a_0, a_1, \dots, a_{n-1}] = \sum_{p=0}^{n-1} a_p y^p$; let $p_n: R \rightarrow R/\langle y^n \rangle$ be the unique polynomial representation modulo the principal ideal $\langle y^n \rangle$ of $P(y)/Q(y) \in R$ and let $j_n: R/\langle y^n \rangle \rightarrow V_n$ be $j_n(\sum_{p=0}^{n-1} c_p y^p) = [c_0, c_1, \dots, c_{n-1}]$. Then, for nonsingular A , $1/e_n(A) \in R$ and $j_n p_n\{1/e_n(A)\} = A^{-1}$.*

Proof. Simply note that A is nonsingular if and only if $a_0 \neq 0$, so $1/e_n(A) \in R$. The mapping $j_n p_n$ is a ring homomorphism onto and $j_n p_n e_n$ is the identity, so $j_n p_n\{e_n(A)/e_n(A)\} = A \cdot j_n p_n\{1/e_n(A)\} = 1$.

Such mappings can also be used to factor matrices as shown in [2].

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Now, turning to the nontriangular case, we first observe that such an isoclinal matrix can be extended to an L.T.I. matrix by suitable bordering.

2.2. LEMMA. *Let H be an $n \times n$ isoclinal matrix such that the k th supra-diagonal is not zero, $k < n$ (and suppose all succeeding supra-diagonals are zero). Then, H may be bordered above and on the right by k rows and columns, by extension of the isoclinal property to form an $(n + k) \times (n + k)$ nonsingular L.T.I. matrix T .*

Clearly, T is nonsingular since its diagonal is nonzero. Now, multiply T by a suitable permutation matrix P , so that H becomes the principal $n \times n$ submatrix of PT . The inverse, $(PT)^{-1} = T^{-1}P^T$, may now be calculated and the following standard result on matrix partitioning may be applied.

2.3. LEMMA. *Let $T^{-1}P^T$ be partitioned into submatrices $A_{11}, A_{12}, A_{21}, A_{22}$, where A_{11} is the principal $n \times n$ submatrix and A_{22} is $k \times k$. Then, H^{-1} exists if and only if A_{22}^{-1} exists and in this case $H^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.*

A proof may be found in [3].

3. Application. *Example 1.* We first illustrate the procedure with the simple band matrix H of arbitrary dimension $n \geq 2$, given by $a_{i,j} = -1, 1$, when $i - j = 0, -1$, respectively, and zero otherwise. This has one nonzero supra-diagonal, and its companion L.T.I. matrix for Lemma 2.2 is given by its first column as $T = [1, -1, 0 \dots]$. By Theorem 2.1, T is mapped to $1 - y$ and $1/(1 - y)$ maps back to give $T^{-1} = [1, 1, 1 \dots]$. The required permutation matrix P merely makes the first row of T last, and $T^{-1}P^T$ consists of "1" below the main diagonal and in the last column, and zero otherwise. By computation from Lemma 2.3, if the elements of H^{-1} are $c_{i,i}$, then $c_{i,i} = -1$ if $i - j \leq 0$, and zero otherwise.

Example 2. Many matrices of Hessenberg type [4] become singular beyond a certain dimension n . A simple example of this is afforded by considering the L.T.I. matrix T whose first column is given by the negative binomial coefficients for $(1 + y)^{-3}$, namely, $(-1)^p \binom{2+p}{p}$, $p = 0, 1, \dots$. An isoclinal matrix of Hessenberg type H is obtained by deleting the first row and last column of T . By applying Theorem 2.1 to T , one obtains the first column of T^{-1} as the positive binomial coefficients $[1, 3, 3, 1, 0 \dots]$. But if T is $n \times n$, then A_{22} from Lemma 2.3 is the 1×1 matrix whose element is the n th coefficient of this sequence. Hence, H is singular when it exceeds 3×3 in size. Analogous results may be obtained for matrices with more than one nonzero supra-diagonal.

Example 3. In [1], Chow gives formulas for the inverses of a certain class of isoclinal Hessenberg matrices which we shall also do as a further illustration of the above methods. If α and δ are real numbers, $\alpha \neq 0$, we construct the L.T.I. matrix T whose elements are $a_{i,j} = 0$ if $i - j < 0$, 1 if $i - j = 0$, $\alpha + \delta$ if $i - j = 1$ and α^{i-j} if $i - j > 1$. The corresponding Hessenberg matrix H is again obtained from T by deleting the first row and last column.

The rational function image of T is $\sum_{p=0}^{\infty} \alpha^p y^p + \delta y = 1/(1 - \alpha y) + \delta y$, and the reciprocal of this function may be simplified to the form:

$$(I) \quad (1 - \alpha y) / \{(1 - ay)(1 - by)\},$$

where $a = D - \delta/2$, $b = -D - \delta/2$ and $D = (1/2)(\delta^2 + 4\alpha\delta)^{1/2}$. We shall treat the case $a \neq b$; the other case involves analogous techniques. The fraction in (I) may be re-expressed by geometric series and multiplied out to give $1 + \sum_{q=1}^{\infty} y^q \{c_q - \alpha c_{q-1}\}$,

where $c_q = \sum_{p=0}^q a^p b^{q-p} = (a^{q+1} - b^{q+1})/(a - b)$, $q = 0, 1, \dots$. The final reduction gives

$$(II) \quad \{1/(a - b)\} \sum_{q=0}^{\infty} y^q \{a^q(a - \alpha) - b^q(b - \alpha)\}.$$

Hence, the first column of T^{-1} is given by $\phi_q = \{a^q(a - \alpha) - b^q(b - \alpha)\}/(a - b)$, $q = 0, 1, \dots$. If T is $n \times n$, the 1×1 matrix $A_{2,2}$ of Lemma 2.3 is the element ϕ_{n-1} , which is not zero in the case we are treating, and H^{-1} may be constructed. If the elements of H^{-1} are $c_{i,k}$ and ϕ_q is defined to be zero for $q < 0$, the result is $c_{i,k} = \{-\phi_i \cdot \phi_{n-k-1}/\phi_n\} + \phi_{i-k-1}$. This result was verified on a rainy afternoon by direct multiplication of HH^{-1} .

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Mathematics Department
Louisiana State University in New Orleans
New Orleans, Louisiana 70122

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