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The Asymptotic Expansion of the Meijer G-Function*

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Abstract. Gamma function identities are integrated to expand the Meijer G-function in a basic set of functions, each of which is simply characterized asymptotically.

1. Introduction and Notation. In this exposition, we derive the asymptotic expansion of the Meijer G-function for large values of the variable. Although these results can be found in various places in the literature, e.g., Meijer's original papers [1], or their collection by Luke [2], they are usually obscured by a maze of special notation and the presence of a large number of results which are only of secondary interest. The following derivation seems more direct.

Throughout this work, we assume that the integers p, q, m, n and parameters a_j, b_j satisfy the hypothesis,

$$\begin{gathered} 0 \leqq m \leqq q, 0 \leqq n \leqq p, \\ a_j - b_k \neq \text{a positive integer}; \; j = 1, \cdots, p; \; k = 1, \cdots, q, \\ a_j - a_k \neq \text{an integer}; \; j, k = 1, \cdots, p; \; j \neq k. \end{gathered} \tag{1.1}$$

Extensive use will be made of the notations,

$$\begin{gathered} \Gamma_n(c_P - t) = \prod_{k=n+1}^{p} \Gamma(c_k - t), \qquad \Gamma(c_M - t) = \Gamma_0(c_M - t), \\ {}_pF_q\left(\begin{matrix} a_P \\ b_Q \end{matrix} \middle| \, w \right) = \sum_{k=0}^{\infty} \frac{\Gamma(a_P + k)\Gamma(b_Q)}{\Gamma(b_Q + k)\Gamma(a_P)} \cdot \frac{w^k}{k!}. \end{gathered} \tag{1.2}$$

The Meijer G-function is then defined by

$$\begin{aligned} G^{m,n}_{p,q}(z) &= G^{m,n}_{p,q}\left(z \, \middle| \, \begin{matrix} a_P \\ b_Q \end{matrix}\right) = G^{m,n}_{p,q}\left(z \, \middle| \, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right) \\ &= \frac{1}{2\pi i} \int_L \frac{\Gamma(b_M - t)\Gamma(1 - a_N + t) z^t}{\Gamma_m(1 - b_Q + t)\Gamma_n(a_P - t)} \, dt, \end{aligned} \tag{1.3}$$

where L is an upward oriented loop contour which separates the poles of $\Gamma(b_M - t)$ from those of $\Gamma(1 - a_N + t)$ and which begins and ends at $+\infty$ ($L = L_+$) or $-\infty$

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($L = L_-$). A simple computation shows that $G^{m,n}_{p,q}(z)$ satisfies the linear differential equation

$$\left\{\prod_{j=1}^{q} (\delta - b_j) + (-1)^{p+1-m-n} z \prod_{j=1}^{p} (\delta + 1 - a_j)\right\} y(z) = 0, \qquad \delta = z \frac{d}{dz}, \tag{1.4}$$

of order $\max(p, q)$.

If $q \leqq p$, $z = \infty$ is a regular singular point of (1.4). The behaviour of $G^{m,n}_{p,q}(z)$ for z large then follows from the residue calculus result, $L = L_-$,

THEOREM 1. *Under the conditions of* (1.1),

$$\begin{aligned} G^{m,n}_{p,q}(z) = \sum_{j=1}^{n} \frac{\Gamma^*(a_j - a_N)\Gamma(1 + b_M - a_j)}{\Gamma_n(1 + a_P - a_j)\Gamma_m(a_j - b_Q)} z^{-1+a_j} \\ \times {}_{q+1}F_p\left(\begin{matrix} 1, 1 + b_Q - a_j \\ 1 + a_P - a_j \end{matrix} \middle| \frac{(-1)^{q-m-n}}{z}\right), \\ q < p \text{ or } q = p \text{ and } |z| > 1, \end{aligned} \tag{1.5}$$

where

$$\Gamma^*(a_j - a_N) = \prod_{k=1;k\neq j}^{n} \Gamma(a_j - a_k). \tag{1.6}$$

If $n = 0$, (1.5) *reduces to* $G^{m,0}_{p,q}(z) \equiv 0$.

If $q > p$, $z = \infty$ is an irregular singular point of (1.4), and the analysis is more involved. For convenience, we set

$$\nu = q - p \geqq 1, \qquad \mu = q - m - n. \tag{1.7}$$

A special case of the above, $m = q$ and $n = 0$ or 1, was treated by Barnes [3], who obtained

THEOREM 2 (BARNES). *Under the conditions of* (1.1), $\nu \geqq 1$,

$$\begin{aligned} L_j(w) \equiv G^{q,1}_{p,q}\left(w \middle| \begin{matrix} a_j, a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right), \qquad j = 1, \cdots, p, \\ \sim \frac{\Gamma(1 + b_Q - a_j)}{\Gamma(1 + a_P - a_j)} w^{-1+a_j} {}_{q+1}F_p\left(\begin{matrix} 1, 1 + b_Q - a_j \\ 1 + a_P - a_j \end{matrix} \middle| \frac{-1}{w}\right), \\ w \to \infty, \qquad |\arg w| < \pi(\nu/2 + 1), \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} G(w) \equiv G^{q,0}_{p,q}\left(w \middle| \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right) \\ \sim \left(\frac{(2\pi)^{\nu-1}}{\nu}\right)^{1/2} \exp(-\nu w^{1/\nu}) \sum_{j=0}^{\infty} K_j w^{\gamma - j/\nu}, \\ w \to \infty, \qquad |\arg w| < \pi(\nu + \min(1, \nu/2)), \end{aligned} \tag{1.9}$$

where

$$\nu\gamma = \frac{1-\nu}{2} + B_1 - A_1, \qquad K_0 = 1,$$

$$K_1 = A_2 - B_2 + \frac{(B_1 - A_1)}{2\nu}[\nu(A_1 + B_1) + A_1 - B_1] + \frac{1-\nu^2}{24\nu}, \tag{1.10}$$

$$\prod_{j=1}^{p}(x + a_j) = \sum_{j=0}^{p} A_j x^{p-j}, \quad \prod_{j=1}^{q}(x + b_j) = \sum_{j=0}^{q} B_j x^{q-j},$$

and the remaining K_j are polynomials in A_j, B_j independent of w.

When $\nu \geqq 1$, the contour L of (1.3) is equal to L_+. In particular, if $m = 0$, $G^{0,n}_{p,q}(z) \equiv 0$. A straightforward computation shows that $L_j(ze^{i\pi(\mu+1-2r)})$ and $G(ze^{i\pi(\mu-2s)})$ are also solutions of (1.4), r, s arbitrary integers. For a given value of $\arg z$, there exists at least one pair of integers (r, s) such that

$$\begin{aligned} &|\arg z + \pi(\mu + 1 - 2r)| < \pi(\nu/2 + 1), \\ &|\arg z + \pi(\mu + 2 - 2s - 2h)| < \pi(\nu + \min(1, \nu/2)), \qquad h = 1, \cdots, \nu. \end{aligned} \tag{1.11}$$

It then follows from Theorem 2, that in a suitable sector, the q functions $L_j(ze^{i\pi(\mu+1-2r)})$, $j = 1, \cdots, p$, $G(ze^{i\pi(\mu+2-2s-2h)})$, $h = 1, \cdots, \nu$, form a basis of solutions for (1.4).

THEOREM 3 (MEIJER). *Under the conditions of* (1.1), $\nu \geqq 1$, *if the sector*

$$S_{r,s}: \quad \begin{aligned} &\pi\left(\nu - \mu - 2 + \max\left[2r - \frac{3\nu}{2}, 2s - \min\left(1, \frac{\nu}{2}\right)\right]\right) < \arg z \\ &\arg z < \pi\left(\frac{\nu}{2} - \mu + \min\left[2r, \frac{\nu}{2} + 2s + \min\left(1, \frac{\nu}{2}\right)\right]\right) \end{aligned}$$

is not empty, then there exist constants $C_j(r, s)$, $D_j(r, s)$ such that

$$G^{m,n}_{p,q}(z) = \sum_{j=1}^{p} C_j(r, s) L_j(ze^{i\pi(\mu+1-2r)}) + \sum_{h=1}^{\nu} D_h(r, s) G(ze^{i\pi(\mu+2-2s-2h)}). \tag{1.12}$$

Equation (1.12) *will be referred to as the* (r, s) *expansion for* $G^{m,n}_{p,q}(z)$.

Once the values of $C_j(r, s)$, $D_h(r, s)$ have been determined for $z \in S_{r,s}$, (1.12) remains valid for all values of $\arg z$. However, it is useful for the asymptotic evaluation of $G^{m,n}_{p,q}(z)$ only when the arguments of the L_j and G functions which are actually present satisfy the argument restrictions of (1.8) and (1.9), respectively.

Thus, the only practical difficulty in using Theorem 3 to determine the behaviour of $G^{m,n}_{p,q}(z)$ for z large lies in the determination of $C_j(r, s)$, $D_h(r, s)$ for any given value of $\arg z$. This problem is discussed in Section 2.

If the complex conjugate of the (r, s) expansion, (1.12), is taken, treating z, a_j, b_j as real, one obtains the $(\mu + 1 - r, \mu + 1 - \nu - s)$ expansion. This is particularly useful when $\nu = 1$ or 2.

2. Coefficient Determination. The practical problem of determining the $C_j(r, s)$, $D_h(r, s)$ for any given value of $\arg z$ is simplified by noticing that for ν fixed, it is sufficient to consider only certain diagonal sectors $S_{r,s}$.

PROPOSITION 1. *For $\nu \geqq 1$, let k_0 be the nonnegative integer such that*

$$\nu - 1 \leqq 4k_0 \leqq \nu + 2, \quad \textit{or} \quad 4k_0 - 2 \leqq \nu \leqq 4k_0 + 1. \tag{2.1}$$

Then

$$\begin{aligned} S_{r,r-k_0}&: \ \pi(\nu - \mu + 2r - 3 - 2k_0) < \arg z < \pi(\nu/2 - \mu + 2r); \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nu \geqq 2, k_0 \geqq 1, \\ S_{r,r}&: \ \pi(2r - \mu - \tfrac{3}{2}) < \arg z < \pi(2r - \mu + \tfrac{1}{2}); \quad \nu = 1, k_0 = 0, \\ S_{r,r-1}&: \ \pi(2r - \mu - \tfrac{5}{2}) < \arg z < \pi(2r - \mu - \tfrac{1}{2}); \quad \nu = 1, \end{aligned} \tag{2.2}$$

and the diagonal sectors,

$$\begin{aligned} \nu = 1; &\quad S_{r,r}, S_{r,r-1}, \quad r = 0, \pm 1, \pm 2, \cdots, \\ \nu \geqq 2; &\quad S_{r,r-k_0}, \quad r = 0, \pm 1, \pm 2, \cdots, \end{aligned} \tag{2.3}$$

completely cover the z-plane.

Thus, if the (r, s) expansion is known, and $S_{r,s}$ is one of the sectors in (2.3), it is sufficient to give recursion relations for the $(r \pm 1, s)$ and $(r, s \pm 1)$ expansions. These follow directly from the following L_j and G recursion relationships.

PROPOSITION 2. *Under the conditions of* (1.1), $\nu \geqq k \geqq 1$, *let the constants* $C_j(k)$, $D_h(k)$ *be chosen such that*

$$\begin{aligned} (-1)^{\nu+1}(2\pi i)^\nu e^{i\pi(B_1 - A_1)} e^{-i\pi t\nu} \frac{\Gamma(a_P - t)\Gamma(1 - a_P + t)}{\Gamma(b_Q - t)\Gamma(1 - b_Q + t)} \\ = \sum_{j=1}^{p} C_j(k) e^{i\pi(1-2k)t}\Gamma(a_j - t)\Gamma(1 - a_j + t) - 1 + \sum_{h=1}^{\nu} D_h(k) e^{-i\pi 2ht}, \end{aligned} \tag{2.4}$$

an identity which can be built up from the case $k = 1$, *by repeated use of*

$$\begin{aligned} &\Gamma(a_j - t)\Gamma(1 - a_j + t) \\ &\qquad = e^{i\pi 2a_j} e^{-i\pi 2t}\Gamma(a_j - t)\Gamma(1 - a_j + t) + (-2\pi i) e^{i\pi a_j} e^{-i\pi t}. \end{aligned} \tag{2.5}$$

In particular,

$$\begin{aligned} C_j(k) &= (-1)^{\nu+1}(2\pi i)^\nu e^{i\pi(B_1 - A_1)} e^{i\pi(2k-\nu-1)a_j} \frac{\Gamma^*(a_P - a_j)\Gamma(1 - a_P + a_j)}{\Gamma(b_Q - a_j)\Gamma(1 - b_Q + a_j)}, \\ D_h(k) &= D_h(1) + (-2\pi i) \sum_{j=1}^{p} C_j(1) e^{i\pi(2h-1)a_j}, \qquad 1 \leqq h \leqq k - 1, \\ &= D_h(1), \qquad k \leqq h \leqq \nu, \\ D_\nu(1) &= (-1)^{\nu+1} e^{i\pi 2(B_1 - A_1)}. \end{aligned} \tag{2.6}$$

Then,

$$L_j(w) = e^{i\pi 2a_j} L_j(we^{-i\pi 2}) + (-2\pi i) e^{i\pi a_j} G(we^{-i\pi}), \tag{2.7}$$

$$G(w) = \sum_{j=1}^{p} C_j(k) L_j(we^{i\pi(1-2k)}) + \sum_{h=1}^{\nu} D_h(k) G(we^{-i\pi 2h}). \tag{2.8}$$

Proof. The existence of the expansion (2.4) follows from the partial fraction decomposition

$$\frac{\prod_{j=1}^{q}(y-\beta_j)}{\prod_{j=1}^{p}(y-\alpha_j)} = \sum_{j=1}^{p}\frac{c_{j,k}y^k}{y-\alpha_j} + \sum_{h=0}^{\nu} d_{h,k}y^h,$$

$$(2.9) \qquad d_{\nu,k} = 1 \qquad d_{0,k} = (-1)^\nu \frac{\prod_{j=1}^{q}\beta_j}{\prod_{j=1}^{p}\alpha_j},$$

$$\alpha_j \neq \alpha_r, \quad j \neq r, \quad 0 < k \leqq \nu = q - p,$$

with $y = e^{-i\pi 2t}$, $\beta_j = e^{-i\pi 2b_j}$ and $\alpha_j = e^{-i\pi 2a_j}$. Multiplying (2.5), (2.4) by

$$w^t\Gamma(b_Q - t)[\Gamma(a_P - t)]^{-1},$$

and integrating along a contour L_+ which separates the poles of $\Gamma(b_Q - t)$ from those of $\Gamma(1 - a_P + t)$, we arrive at (2.7) and an expansion of $G_{p,q}^{0,p}(we^{-i\pi\nu}) \equiv 0$ which reduces to (2.8), respectively. □

Remark 1. Equations (2.7) and (2.8) can also be written in the form

$$(2.10) \qquad L_j(w) = e^{-i\pi 2a_j}L_j(we^{i\pi 2}) + (2\pi i)e^{-i\pi a_j}G(we^{i\pi}),$$

$$(2.11) \qquad -D_\nu(1)G(we^{-\pi 2\nu}) = \sum_{j=1}^{p} C_j(k)L_j(we^{i\pi(1-2k)}) + \sum_{h=1}^{\nu} D_{h-1}(k)G(we^{i\pi(2-2h)}),$$

$$1 \leqq k \leqq \nu, \qquad D_0(k) = -1.$$

Using Proposition 2, the variables of $L_j(w)$ and $G(w)$ can be changed in a systematic fashion.

PROPOSITION 3. *Under the conditions* (1.1), *let* $k = r - s$, $1 \leqq k + 1 \leqq \nu$. *Then the following recursion relations hold.*

$(r, s) \to (r + 1, s)$

$$C_j(r+1, s) = e^{i\pi 2a_j}C_j(r, s),$$

$$(2.12) \qquad D_h(r+1, s) = D_h(r, s), \qquad h \neq k+1,$$

$$= D_{k+1}(r, s) + (-2\pi i)\sum_{j=1}^{p} C_j(r, s)e^{i\pi a_j}, \qquad h = k+1,$$

$(r + 1, s) \to (r + 1, s + 1)$

$$C_j(r+1, s+1) = C_j(r+1, s) + D_1(r+1, s)C_j(k+1),$$

$$(2.13) \qquad D_h(r+1, s+1) = D_1(r+1, s)D_h(k+1) + D_{h+1}(r+1, s), \qquad 1 \leqq h \leqq \nu - 1,$$

$$= D_1(r+1, s)D_\nu(k+1), \qquad h = \nu,$$

$(r, s) \to (r, s - 1)$

$$C_j(r, s-1) = C_j(r, s) - \frac{D_\nu(r, s)}{D_\nu(1)}C_j(k+1),$$

$$(2.14) \qquad D_h(r, s-1) = \frac{D_\nu(r, s)}{D_\nu(1)}, \qquad h = 1,$$

$$= D_{h-1}(r, s) - \frac{D_\nu(r, s)}{D_\nu(1)}D_{h-1}(k+1), \qquad 2 \leqq h \leqq \nu,$$

$\underline{(r, s-1) \rightarrow (r-1, s-1)}$

$$\begin{aligned} C_j(r-1, s-1) &= e^{-i\pi 2a_j} C_j(r, s-1), \\ D_h(r-1, s-1) &= D_h(r, s-1), \qquad h \neq k+1, \\ &= D_{k+1}(r, s-1) + (2\pi i) \sum_{j=1}^{p} C_j(r, s-1) e^{-i\pi a_j}. \end{aligned} \tag{2.15}$$

These recursion formulae are valid if $k = k_0$.

Proof. Equations (2.12) and (2.15) follow from (2.7) and (2.10), respectively, whereas Eqs. (2.13) and (2.14) follow when the $h = 1$ and $h = \nu$ terms in the $(r + 1, s)$ or (r, s) expansions are replaced by the expansions given in (2.8) and (2.11), respectively. □

Suppose that an (s, s) expansion for $G_{p,q}^{m,n}(z)$ is known, s arbitrary. Then k_0 applications of (2.12) yield the $(s + k_0, s)$ expansion for $G_{p,q}^{m,n}(z)$, and Propositions 1, 3 imply the asymptotic behaviour of $G_{p,q}^{m,n}(z)$ for all values of arg z as $z \to \infty$. Hence, the only remaining practical problem is to find a particular (s, s) expansion.

PROPOSITION 4. *Under the conditions of* (1.1), $\nu \geqq 1$, *there exist constants* E_j, F_h *and* H_j *such that*

$$\begin{aligned} &\frac{\Gamma(a_N - t)\Gamma(1 - a_N + t)}{\Gamma_m(b_Q - t)\Gamma_m(1 - b_Q + t)} - \frac{\Gamma(a_P - t)\Gamma(1 - a_P + t)}{\Gamma(b_Q - t)\Gamma(1 - b_Q + t)} \sum_{j=0}^{\mu-\nu} H_j e^{i\pi(\mu-\nu-2j)t} \\ &\qquad = \sum_{j=1}^{p} E_j e^{i\pi(\mu+1-2s)t} \Gamma(a_j - t)\Gamma(1 - a_j + t) \\ &\qquad\quad + \sum_{h=1}^{\nu} F_h e^{i\pi(\mu+2-2s-2h)t}, \end{aligned} \tag{2.16}$$

$s = \max(0, 1 + \mu - \nu)$. *Then,*

$$E_j = C_j(s, s), \qquad F_h = D_h(s, s). \tag{2.17}$$

In particular, if $\mu \leqq \nu - 1$, (2.16) *reduces to*

$$\begin{aligned} &\frac{\Gamma(a_N - t)\Gamma(1 - a_N + t)}{\Gamma_m(b_Q - t)\Gamma_m(1 - b_Q + t)} \\ &\qquad = \sum_{j=1}^{n} C_j(0, 0) e^{i\pi(\mu+1)t} \Gamma(a_j - t)\Gamma(1 - a_j + t) \\ &\qquad\quad + \sum_{h=1}^{1+\mu} D_h(0, 0) e^{i\pi(\mu+2-2h)t}, \end{aligned} \tag{2.18}$$

$$C_j(0, 0) = e^{-i\pi(\mu+1)a_j} \frac{\Gamma(1 + a_j - a_N)\Gamma^*(a_N - a_j)}{\Gamma_m(1 - b_Q + a_j)\Gamma_m(b_Q - a_j)}, \qquad j = 1, \cdots, n.$$

If $\mu \leqq -1$, *the sum* $\sum_{h=1}^{1+\mu}$ *in* (2.18) *also disappears.*

Proof. We begin by noticing that (2.17) is a direct consequence of (2.16). For if (2.16) is multiplied by $z^t \Gamma(b_Q - t)[\Gamma(a_P - t)]^{-1}$, and the resulting expansion is integrated over the contour L_+ of (1.3), one obtains the (s, s) expansion of

$$G_{p,q}^{m,n}(z) - \sum_{j=0}^{\mu-\nu} H_j G_{p,q}^{0,p}(z e^{i\pi(\mu-\nu-2j)}) = G_{p,q}^{m,n}(z). \tag{2.19}$$

First, assume that $\mu \leqq \nu - 1$. Then (2.18) follows directly from the partial fraction decomposition

$$\frac{\prod_{j=m+1}^{q} (y - \beta_j)}{\prod_{j=1}^{n} (y - \alpha_j)} = \sum_{j=1}^{n} \frac{e_j}{y - \alpha_j} + \sum_{h=0}^{\mu} f_h y^h, \tag{2.20}$$

with $y = e^{-i\pi 2t}$, $\beta_j = e^{-i\pi 2b_j}$ and $\alpha_j = e^{-i\pi 2a_j}$. Similarly for the case, $\mu \geqq \nu$, we prove by induction on s, $0 \leqq s \leqq 1 + \mu - \nu$, that there exist constants $e_{j,s}$, $f_{h,s}$ and $h_{j,s}$ such that

$$\begin{aligned} &\frac{\prod_{j=m+1}^{q} (y - \beta_j)}{\prod_{j=1}^{n} (y - \alpha_j)} - \left(\sum_{j=0}^{s-1} h_{j,s} y^j\right) \frac{\prod_{j=1}^{q} (y - \beta_j)}{\prod_{j=1}^{p} (y - \alpha_j)} \\ &\qquad = \sum_{j=1}^{p} \frac{e_{j,s} y^s}{y - \alpha_j} + \sum_{h=s}^{\mu} f_{h,s} y^h, \qquad 0 \leqq s \leqq 1 + \mu - \nu. \end{aligned} \tag{2.21}$$

If $s = 0$, (2.21) reduces to (2.20). Assuming (2.21) valid for a particular value of $s \leqq \mu - \nu$, we note that systematic application of

$$1/(y - a) = y/a(y - a) - 1/a, \tag{2.22}$$

allows us to write $R(y)$, the left-hand side of (2.21), in the form

$$R(y) = \sum_{j=1}^{p} \frac{e_{j,s} y^{s+1}}{\alpha_j (y - \alpha_j)} + y^s \left\{ f_{s,s} - \sum_{j=1}^{p} \frac{e_{j,s}}{\alpha_j} \right\} + \sum_{h=s+1}^{\mu} f_{h,s} y^h. \tag{2.23}$$

Finally, if the expansion (2.9) with $k = 1$ is multiplied by $y^s h_{s,s}$,

$$d_{0,1} h_{s,s} = f_{s,s} - \sum_{j=1}^{p} (e_{j,s}/\alpha_j), \tag{2.24}$$

and the resulting expression is subtracted from (2.23), one obtains (2.21) with s replaced by $s + 1$. Equation (2.16) then follows from (2.21) with $s = 1 + \mu - \nu$, $y = e^{-i\pi 2t}$, $\beta_j = e^{-i\pi 2b_j}$ and $\alpha_j = e^{-i\pi 2a_j}$. □

Remark 2. A particular (r, s) expansion may directly imply the asymptotic expansion of $G_{p,q}^{m,n}(z)$ in a larger sector than $S_{r,s}$. For example, if $\nu \geqq 1$, $\mu \leqq -1$, it follows from Proposition 4 that

$$G_{p,q}^{m,n}(z) = \sum_{j=1}^{n} C_j(0, 0) L_j(z e^{i\pi(\mu+1)}), \tag{2.25}$$

an expansion which, though valid for all values of arg z, directly implies the asymptotic behaviour of $G_{p,q}^{m,n}(z)$ only when $-\pi(\nu/2 + \mu + 2) < \arg z < \pi(\nu/2 - \mu)$.

Remark 3. Once a particular (r, s) expansion has been derived, the conditions (1.1) can be weakened by an appeal to analytic continuation.

Remark 4. The above results are easily transformed into results for the hypergeometric functions,

$${}_pF_q\left(\begin{matrix}\alpha_P \\ \beta_Q\end{matrix} \,\middle|\, -z\right) = \frac{\Gamma(\beta_Q)}{\Gamma(\alpha_P)} G_{p,q+1}^{1,p}\left(z \,\middle|\, \begin{matrix} 1 - \alpha_P \\ 0, 1 - \beta_Q \end{matrix}\right). \tag{2.26}$$

3. Generalizations. The above expansions are derived by multiplying certain identities by the function $z^t \Gamma(b_Q - t)[\Gamma(a_P - t)]^{-1}$ and integrating the resultant identities over an appropriate contour. If these identities are multiplied by more general functions, the above expansions can be generalized.

Let c_j, $j = 1, \cdots, k$; d_j, $j = 1, \cdots, v$; f_j, $j = 1, \cdots, w$, be constants such that

$$c_j - b_u \neq \text{a positive integer}; \qquad j = 1, \cdots, k;\ u = 1, \cdots, q. \tag{3.1}$$

Then Theorem 3 can be generalized as follows.

THEOREM 4. *Let a_j, b_j, c_j satisfy the conditions* (1.1) *and* (3.1). *If the expansion* (1.12) *is derivable from Propositions* 2, 3 *and* 4, *then*

$$\begin{aligned} &G^{m,n+k}_{p+k+v,q+w}\left(z \,\middle|\, \begin{matrix} c_K, a_P, d_V \\ b_Q, f_W \end{matrix}\right) \\ &\quad = \sum_{j=1}^{p} C_j(r, s) G^{q,k+1}_{p+k+v,q+w}\left(ze^{i\pi(\mu+1-2r)} \,\middle|\, \begin{matrix} c_K, a_j, a_P^{*j}, d_V \\ b_Q, f_W \end{matrix}\right) \\ &\qquad + \sum_{h=1}^{v} D_h(r, s) G^{q,k}_{p+k+v,q+w}\left(ze^{i\pi(\mu+2-2s-2h)} \,\middle|\, \begin{matrix} c_K, a_P, d_V \\ b_Q, f_W \end{matrix}\right), \end{aligned} \tag{3.2}$$

where a_P^{*j} denotes the sequence $a_1, \cdots, a_p$ with a_j deleted, and ν, μ, $C_j(r, s)$, $D_h(r, s)$ have the same values as in (1.12).

Proof. If the expansion (1.12) can be built up from Propositions 2, 3 and 4, the expansion (3.2) can be built up from appropriate generalizations of (2.7), (2.8) and (2.19). In particular, if (2.5) and (2.4) are multiplied by

$$z^t \Gamma(b_Q - t)\Gamma(1 - c_K + t)[\Gamma(a_P - t)\Gamma(d_V - t)\Gamma(1 - f_W + t)]^{-1},$$

and integrated over a contour L_+^* which separates the poles of $\Gamma(b_Q - t)$ from those of $\Gamma(1 - a_P + t)\Gamma(1 - c_K + t)$, one obtains

$$\begin{aligned} L_j^*(z) &= e^{i\pi 2a_j} L_j^*(ze^{-i\pi 2}) + (-2\pi i)e^{i\pi a_j} G^*(ze^{-i\pi}), \\ G^*(z) &= \sum_{j=1}^{p} C_j(u) L_j^*(ze^{i\pi(1-2u)}) + \sum_{h=1}^{\nu} D_h(u) G^*(ze^{-i\pi 2h}), \end{aligned} \tag{3.3}$$

where $1 \leqq u \leqq \nu$, and

$$\begin{aligned} G^*(z) &= G^{q,k}_{p+k+v,q+w}\left(z \,\middle|\, \begin{matrix} c_K, a_P, d_V \\ b_Q, f_W \end{matrix}\right), \\ L_j^*(z) &= G^{q,k+1}_{p+k+v,q+w}\left(z \,\middle|\, \begin{matrix} c_K, a_j, a_P^{*j}, d_V \\ b_Q, f_W \end{matrix}\right). \end{aligned} \tag{3.4}$$

Treating (2.16) similarly, one obtains (3.2) with $r = s$. □

The significance of (3.2) lies in the fact that if c_K, d_V and f_W are "large", then in some restricted sense and in some restricted region,

$$\begin{aligned} &CG^*(z) \quad \text{behaves like } G(\Omega z), \\ &CL_j^*(z) \quad \text{behaves like } L_j(\Omega z), \\ &C = \frac{\Gamma(d_V)\Gamma(1 - f_W)}{\Gamma(1 - c_K)}, \qquad \Omega = \prod_{j=1}^{v} d_j \prod_{j=1}^{k} (1 - c_j) \Big/ \prod_{j=1}^{w} (1 - f_j). \end{aligned} \tag{3.5}$$

A special $k = 1$, $w = 0$, $v = 1$ case of Theorem 4 is discussed in [4].

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