MATHEMATICS OF COMPUTATION, VOLUME 26, NUMBER 119, JULY, 1972

Density Bounds for Euler's Function*

By Charles R. Wall

Abstract. Let φ be Euler's function. Upper and lower bounds are presented for $D(x)$, the density of the integers n for which $\varphi(n)/n \leqq x$. The bounds, for $x = 0(.01)1$, have an average spread of less than 0.0203.

1. Introduction. We denote by δX the density, if it exists, of a subset X of the positive integers.

Let φ be Euler's function,

$$\varphi(n) = n \prod_{p \mid n} (1 - p^{-1}).$$

It is known (see, for example, Kac [2]) that the function

$$D(x) = \delta\{n: \varphi(n)/n \leqq x\} \tag{1}$$

exists and is continuous for all real x; $D(x)$ is clearly constant for $x \leqq 0$ and for $x \geqq 1$. In this paper, we present upper and lower bounds for $D(x)$ for $0 \leqq x \leqq 1$. The bounds, obtained in a CDC 6600 demonstration run in 70 seconds, were computed for $x = 0(.001)1$, but, for the sake of brevity, we present here only the bounds for $x = 0(.01)1$. The average spread between the upper and lower bounds presented is less that 0.0203, although near $x = 1$ and $x = \frac{1}{2}$ the spread is much larger.

2. Estimation Procedure. Let

$$M\{f\} = \lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} f(n)$$

denote the mean, if it exists, of an arithmetic function f.

Define the character function χ_k by

$$\begin{aligned} \chi_k(n) &= 1 \quad \text{if } (n, k) = 1, \\ &= 0 \quad \text{if } (n, k) > 1, \end{aligned} \tag{2}$$

where (x, y) denotes as usual the greatest common divisor of integers x and y. Note that in (2) we may as well require that k be squarefree. It is easy to prove that

$$M\{\chi_k(n)n/\varphi(n)\} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p \mid k} \frac{p^2 - p}{p^2 - p + 1}, \tag{3}$$

where ζ is Riemann's function.

Received November 29, 1971.

AMS 1970 *subject classifications*. Primary 10K20, 10K04; Secondary 10L10.

Key words and phrases. Density, Euler function.

* Supported by an East Texas State University Research Grant.

In the generalization of (1), let

$$D(x, j, k) = \delta\{n: j \mid n, (n/j, k) = 1, \varphi(n)/n \leqq x\}$$

with $D(x, 1, 1) = D(x)$. Although it is not our purpose here to prove the existence of the $D(x, j, k)$, such a result may be obtained by a slight modification of the proof presented by Kac [2] of the existence of $D(x)$.

We define

$$F(t, j, k) = \delta\{n: j \mid n, (n/j, k) = 1, n/\varphi(n) \geqq t\}.$$

It is clear that $F(t, j, k) = D(1/t, j, k)$ for all $t > 0$. Then by a modification of the author's technique [4] for bounding the density function associated with the sum of divisors, which in turn was a modification of Behrend's procedure [1] for bounding the density of the abundant numbers, we have

$$F(t, j, k) \leqq \varphi(k)/jk \tag{4}$$

with equality if $t \leqq j/\varphi(j)$, and

$$F(t, j, k) \leqq \frac{j/\varphi(j)}{t - j/\varphi(j)} \cdot \frac{\varphi(k)}{k} \frac{M-1}{j} \tag{5}$$

if $t > j/\varphi(j)$, where M is the mean from (3). If we substitute $t = 1/x$ into (4) and (5), we have

$$D(x, j, k) \leqq \varphi(k)/jk,$$

with equality if $x \geqq \varphi(j)/j$, and

$$D(x, j, k) \leqq \frac{x}{\varphi(j)/j - x} \cdot \frac{\varphi(k)}{k} \frac{M-1}{j} \qquad (x < \varphi(j)/j)$$

respectively.

It is then an easy matter to show that

$$D(x, j, k/j) \leqq \varphi(k/j)/k, \tag{6}$$

with equality if $x \geqq \varphi(j)/j$, and

$$D(x, j, k/j) \leqq \frac{x}{\varphi(j)/j - x} \cdot \frac{\varphi(k/j)}{k} (M-1) \qquad (x < \varphi(j)/j). \tag{7}$$

We used (6) and (7) with $k = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23$ and

$$D(x) = \sum_{j \mid k} D(x, j, k/j)$$

to obtain our preliminary bounds for $D(x)$.

3. Refinements. We improved our lower bounds by increasing k to

$$2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\cdot31\cdot37\cdot41.$$

Consider now Dedekind's function,

$$\psi(n) = n \prod_{p|n} (1 + p^{-1}).$$

It is clear that ψ, like φ, is a multiplicative function. If σ is the sum of divisors function, then

$$\psi(n)/n \leqq \sigma(n)/n \leqq n/\varphi(n) \tag{8}$$

for all n.

Using the observation that, if q is the largest prime dividing m, then $m/\varphi(m) \leqq q$, it is easy to prove that

$$2\psi(n)/n \geqq 1 + n/\varphi(n). \tag{9}$$

The author has investigated [3] the functions

$$B(x, j, k) = \delta\{n: j \mid n, (n/j, k) = 1, \psi(n)/n \geqq x\}.$$

It is an immediate consequence of (8) and (9) that

$$B(x) \leqq F(x, j, k) \leqq B((x + 1)/2, j, k)$$

for all x, j and k. This observation was used with the author's bounds for $B(x, j, k)$ to improve the preliminary bounds for $D(x)$ with x close to 1.

4. Bounds. Our upper and lower bounds for $D(x)$ are presented in Table I and are illustrated by Fig. 1.

TABLE I. D(x) UPPER/LOWER BOUNDS

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0002	.0004	.0006	.0008	.0010	.0012	.0014	.0017	.0020
	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.1	.0022	.0025	.0029	.0032	.0036	.0040	.0045	.0050	.0056	.0063
	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.2	.0073	.0090	.0110	.0158	.0198	.0262	.0348	.0530	.0601	.0730
	.0001	.0005	.0014	.0053	.0062	.0105	.0169	.0376	.0416	.0553
.3	.0841	.0994	.1157	.1712	.1906	.1937	.1997	.2073	.2164	.2287
	.0580	.0735	.0866	.0986	.1709	.1778	.1792	.1871	.1922	.2005
.4	.2617	.2678	.2787	.3008	.3083	.3266	.3479	.3700	.3881	.4225
	.2406	.2449	.2505	.2757	.2802	.2843	.3050	.3189	.3427	.3668
.5	.5241	.5273	.5325	.5481	.5506	.5530	.5575	.5677	.5703	.5737
	.5105	.5129	.5169	.5191	.5376	.5390	.5416	.5418	.5553	.5567
.6	.5812	.5898	.5963	.6055	.6124	.6242	.6677	.6815	.6869	.6883
	.5580	.5664	.5735	.5788	.5866	.5946	.5986	.6705	.6709	.6772
.7	.6897	.6916	.6955	.6993	.7028	.7074	.7116	.7162	.7234	.7401
	.6778	.6785	.6792	.6833	.6871	.6877	.6922	.6942	.7005	.7027
.8	.7560	.7587	.7614	.7652	.7714	.7896	.7925	.7956	.7990	.8042
	.7404	.7420	.7447	.7464	.7499	.7501	.7748	.7771	.7788	.7806
.9	.8155	.8243	.8341	.8423	.8548	.8633	.8704	.8821	.9016	.9220
	.7822	.7981	.7997	.8126	.8132	.8299	.8364	.8460	.8582	.8684

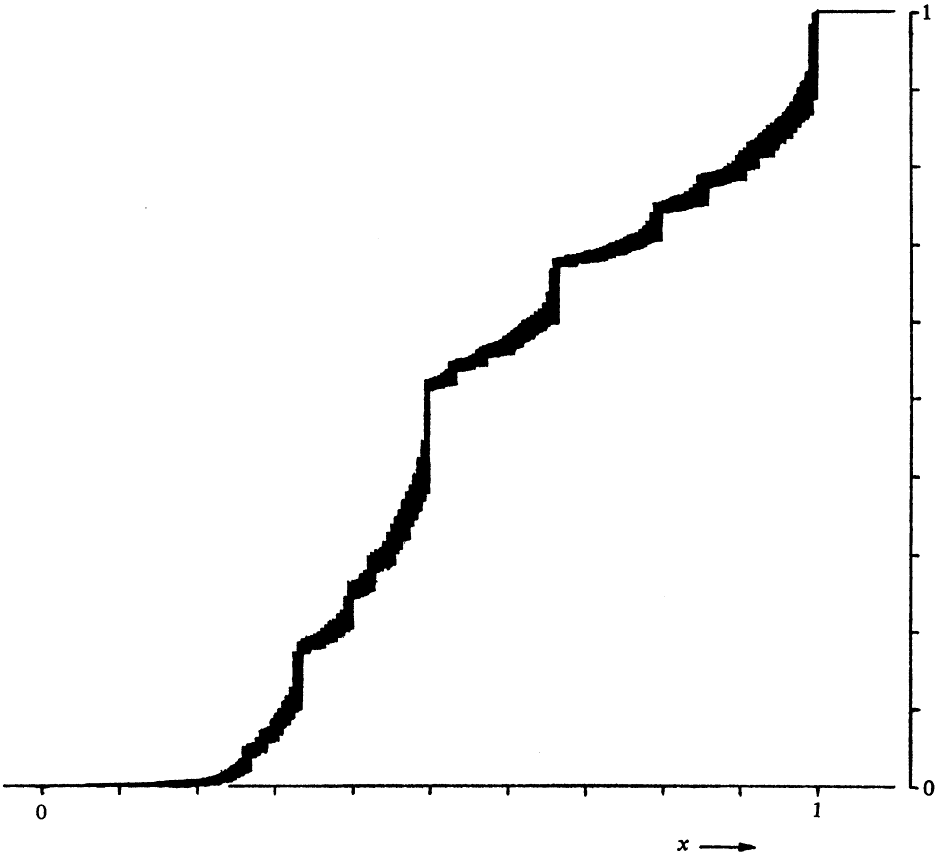

FIGURE 1. $D(x)$ *lies in the shaded area.*

5. Remarks. Similar to the situation in [4], and for much the same reasons, the dramatic changes in $D(x)$ occur near values x for which there is a relatively small integer n such that $x = \varphi(n)/n$. One may easily show that

$$D(x, j, k) = j^{-1}D(xj/\varphi(j), 1, k). \tag{10}$$

But if $\varphi(k)/k < x \leqq 1$,

$$D(x) = D(x, 1, k) + 1 - \varphi(k)/k.$$

Since $D(x)$ increases sharply as x increases to 1, we should expect, in view of (10), that $D(x)$ would also increase sharply as x increases toward $\varphi(j)/j$, the increase being more noticeable the smaller j is.

We expect from (7) that $D(x) = O(x)$ for x small and positive; our bounds bear this out and indeed support the conjectures that $D(x) \leqq x/50$ for $0 < x < .07$, and $D(x) \leqq x/25$ for $0 < x < .2$.

Department of Mathematics
East Texas State University
Commerce, Texas 75428

1. F. BEHREND, "Über 'numeri abundantes.' II," *Preuss. Akad. Wiss. Sitzungber*, v. 6, 1933, pp. 280–293.

2. M. KAC, *Statistical Independence in Probability, Analysis and Number Theory*, Carus Math. Monographs, no. 12, Math. Assoc. Amer.; distributed by Wiley, New York, 1959. MR **22** #996.

3. C. WALL, *Topics Related to the Sum of Unitary Divisors of an Integer*, Ph.D. Dissertation, University of Tennessee, Knoxville, Tenn., 1970.

4. C. WALL, P. CREWS & D. JOHNSON, "Density bounds for the sum of divisors function," *Math. Comp.*, v. 26, 1972, pp. 773–777.